Math. Proc. Camb. Phil. Soc. (1975), **77**, 71
MPCPS 77–6
Printed in Great Britain

Inextensible Riesz spaces

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(*Received* 18 *October* 1973)

My aim in this paper is to give an abstract characterization of the C^∞ spaces described in (6) or (9), and to develop some of the remarkable special properties of these spaces. Although the subject is in some ways highly specialized, inextensible and sequentially inextensible spaces seem common enough (they include all spaces of the forms $\mathbf{R}^X$ and L^0) to be worth studying, and I have already employed them in the proof of more general results (1).

In the first section I set out those properties that can be described in simple Riesz space terms; much of this work has already been published in slightly different forms. In the second part I go on to questions that arise when we impose a topology on an inextensible Riesz space. Finally, in the third section, I discuss some problems, arising from the work before, which are related to the famous measurable cardinal problem.

Preliminaries. In general my notation will follow (2). Particular points include the following. If E is a Riesz space (= vector lattice), a set $A \subseteq E^+$ is *disjoint* if $x \wedge y = 0$ whenever x, y are distinct members of A; a sequence (x_n) in E^+ is disjoint if $x_n \wedge x_m = 0$ whenever $m \neq n$. A *weak order unit* of E is an $e \in E^+$ such that the band generated by e is E itself. A Riesz subspace F of E (i.e. a linear sublattice) is *order-dense* if

$$x = \sup\{y : y \in F,\ 0 \leqslant y \leqslant x\}$$

for every x in E^+. F is *super-order-dense* if, for every $x \in E^+$, there is a countable set $C \subseteq F$ such that $x = \sup C$. Any order-dense Riesz subspace F of E is *regularly embedded* in E; i.e. if a subset of F has a supremum or infimum in F, it has the same supremum or infimum in E. (Another way of expressing this is to say that the embedding of F in E is order-continuous.) An element x of a Riesz space E is *discrete* (6) if the solid linear subspace of E generated by x is one-dimensional.

A Riesz space E has the *countable sup property* if whenever $A \subseteq E$ has a least upper bound in E, there is a countable $C \subseteq A$ such that $\sup C = \sup A$. For an Archimedean Riesz space E, this is so iff every order-bounded disjoint set in E^+ is countable ((2), Theorem 18D, or (6), Theorem 29·3).

If E is a Riesz space, a linear space topology $\mathfrak{T}$ on E is *locally solid* if 0 has a neighbourhood basis consisting of solid sets; this is so iff $\mathfrak{T}$ is defined by some family of *Riesz pseudo-norms*, i.e. functionals $\rho : E \to \mathbf{R}^+$ such that

$$\rho(x+y) \leqslant \rho(x) + \rho(y) \ \forall\, x, y \in E,$$
$$\rho(x) \leqslant \rho(y) \quad \text{whenever} \quad |x| \leqslant |y|,$$
$$\lim_{\alpha \to 0} \rho(\alpha x) = 0 \ \forall\, x \in E$$

https://doi.org/10.1017/S0305004100049422 Published online by Cambridge University Press

((2), Proposition 22C). A *Fatou* topology is a locally solid linear space topology $\mathfrak{T}$ such that 0 has a neighbourhood basis consisting of order-closed sets; this is so iff $\mathfrak{T}$ is defined by some family of *Fatou pseudo-norms*, i.e. Riesz pseudo-norms ρ such that

$$\text{if} \quad \varnothing \subset A \subseteq E^+, \quad A \uparrow x_0 \quad \text{then} \quad \rho(x_0) = \sup_{x \in A} \rho(x)$$

((2), Theorem 23B). A linear space topology on a Riesz space E is *Lebesgue* if

$$\text{whenever} \quad \varnothing \subset A \subseteq E^+, \quad A \downarrow 0 \quad \text{then} \quad 0 \in \bar{A}.$$

Finally, a linear space topology $\mathfrak{T}$ on a Riesz space E is *Levi* if whenever $A \subseteq E^+$ is directed upwards and bounded for $\mathfrak{T}$, it is bounded above.

1. Inextensible Riesz Spaces

The fundamental results of this section are Theorem 1·4, giving the equivalence of two possible definitions of inextensible Riesz space, and Theorem 1·6A, which is a basic universal mapping property, and also characterizes inextensible spaces. It turns out that many of the same ideas can be applied to a rather wider class, the 'sequentially inextensible' spaces; it seems worth while to follow these through in parallel, especially as the interesting results 1·12–1·15 apply to all sequentially inextensible spaces.

1·1. *Definitions.* (*a*) A Riesz space M is *inextensible* if it is Dedekind complete and 'universally complete' in the sense of (6), i.e. if every non-empty disjoint set in M^+ has a least upper bound.

(*b*) A Riesz space M is *sequentially inextensible* if it is Dedekind σ-complete and every countable non-empty disjoint set in M^+ has a least upper bound.

(*c*) If E is any Archimedean Riesz space, a set $A \subseteq E^+$ is *dominable* if, for every $x > 0$ in E, there is a $k \in \mathbf{N}$ and a $y > 0$ such that

$$(kx - z)^+ \geqslant y \; \forall \, z \in A.$$

(Observe that because E is Archimedean, finite sets in E^+ are dominable.)

Remark. In (9), chap. V, part 5, a sequentially inextensible Riesz space with a weak order unit is called 'extended'.

1·2. I begin with characterizations of inextensible and sequentially inextensible spaces.

LEMMA. *If E is an Archimedean Riesz space and $A \subseteq E^+$ is disjoint, A is dominable.*

Proof. Let $x > 0$ in E. If $x \wedge z = 0$ for every $z \in A$, then

$$(x - z)^+ = x > 0 \; \forall \, z \in A.$$

If, for some $z_1 \in A$, $x_1 = x \wedge z_1 > 0$, then let $k \in \mathbf{N}$ be such that $y = (kx_1 - z_1)^+ > 0$; we now have

$$(kx - z)^+ \geqslant (kx_1 - z)^+ = kx_1 \geqslant y \quad \text{if} \quad z \in A, \, z \neq z_1,$$

$$(kx - z_1)^+ \geqslant (kx_1 - z_1)^+ = y.$$

Putting these together, we see that the condition of 1·1 (*c*) is satisfied for every $x > 0$, and that A is dominable.

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1·3. THEOREM. *Let M be a Dedekind σ-complete Riesz space. Then M is sequentially inextensible iff every countable dominable set in M^+ is bounded above.*

Proof (*a*). If every countable dominable set in M^+ is bounded above, and $A \subseteq M^+$ is any non-empty countable disjoint set, then A is dominable (1·2), so bounded above; as M is Dedekind σ-complete, $\sup A$ exists in M. Thus M is sequentially inextensible.

(*b*) Suppose that M is sequentially inextensible, and that $A = \{x_n : n \in \mathbf{N}\}$ is a countable dominable set in M^+. Observe first that there is an $e \in M^+$ such that $x_n = \sup_{m \in \mathbf{N}} x_n \wedge me$ for every $n \in \mathbf{N}$. For set

$$y_0 = x_0, \quad y_n = x_n - \sup_{m \in \mathbf{N}}(x_n \wedge m \sup_{i<n} y_i)$$

for each $n \geqslant 1$. Then it is easy to verify, by induction on n, that

$$x_n = \sup_{m \in \mathbf{N}} x_n \wedge m \sup_{i \leqslant n} y_i,$$

$$y_n \wedge y_i = 0 \quad \text{for} \quad i < n.$$

As M is sequentially inextensible, $e = \sup_{i \in \mathbf{N}} y_i$ exists in M, and has the required property.

Now, for each $m \in \mathbf{N}$, set $z_m = \sup_{n \in \mathbf{N}} x_n \wedge me$. Then $\{z_m : m \in \mathbf{N}\}$ will have the same upper bounds (if any) as A. We see that

$$z_m = z_{m+1} \wedge me \; \forall \, m \in \mathbf{N},$$

so that, for $m \geqslant 1$,

$$u_m = m^{-1} z_m \geqslant (m+1)^{-1} z_{m+1}.$$

Let $z = \inf_{m \geqslant 1} u_m$; then $z = 0$. For, given any $k \in \mathbf{N}$,

$$\inf_{n \in \mathbf{N}} (kz - x_n)^+ \leqslant \inf_{n \in \mathbf{N}} (z_k - x_n)^+ = 0.$$

As $\{x_n : n \in \mathbf{N}\}$ is supposed to be dominable, z must be 0.

Set $u_0 = e$; we have $u_n \downarrow 0$. Set

$$v_m = e - \sup_{k \in \mathbf{N}} e \wedge k(e - u_m) \leqslant u_m \; \forall \, m \in \mathbf{N};$$

then each v_m is a component of e (i.e. $v_m \wedge (e - v_m) = 0$), and $v_m \downarrow 0$. We know that

$$(z_{m+1} - z_m) \wedge (me - z_m) = (z_{m+1} - me)^+ \wedge (me - z_{m+1})^+ = 0,$$

so

$$(z_{m+1} - z_m) \wedge (e - u_m) = 0,$$

and $z_{m+1} - z_m \leqslant v_m$ for each $m \in \mathbf{N}$. So put $w_m = v_m - v_{m+1}$ for each $m \in \mathbf{N}$; then

$$z_m = \Sigma_{i<m}(z_{i+1} - z_i) \leqslant \Sigma_{i<m} v_i = \Sigma_{i<m}(i+1) w_i + m v_m.$$

On the other hand, $\{(i+1) w_i : i \in \mathbf{N}\}$ is disjoint; as M is sequentially inextensible,

$$w = \sup_{i \in \mathbf{N}} (i+1) w_i$$

exists in M, and

$$z_m \leqslant \Sigma_{i<m}(i+1) w_i + m v_m = \sup_{i<m} (i+1) w_i + m v_m \leqslant w + m v_m,$$

for every $m \in \mathbf{N}$. As certainly $z_m \uparrow$,

$$z_m \leqslant z_k \leqslant w + k v_k \; \forall \, k > m;$$

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but also $z_m \leqslant me$, so

$$z_m \leqslant w + \inf_{k>m} kv_k \wedge me = w + \inf_{k>m} mv_k = w.$$

Thus w is an upper bound for $\{z_m : m \in \mathbf{N}\}$ and therefore for A. As A was arbitrary, $\boldsymbol{M}$ has the stated property.

1·4. THEOREM (cf. (9), Theorem V, 5·2). *Let $\boldsymbol{M}$ be a Dedekind complete Riesz space. Then these are equivalent*:

(i) *$\boldsymbol{M}$ is inextensible*;

(ii) *every dominable set in $\boldsymbol{M}^+$ is bounded above*;

(iii) *$\boldsymbol{M}$ is sequentially inextensible and has a weak order unit.*

Proof (*a*) (*i*) $\Rightarrow$ (*iii*). Of course, if $\boldsymbol{M}$ is inextensible, it is sequentially inextensible. On the other hand, let $A \subseteq \boldsymbol{M}^+$ be a maximal disjoint set. Then A has an upper bound e, and $e \wedge x > 0$ whenever $x > 0$ in $\boldsymbol{M}$. As $\boldsymbol{M}$ is Archimedean, e is a weak order unit.

(*b*) (*iii*) $\Rightarrow$ (*ii*). Let $A \subseteq \boldsymbol{M}^+$ be a non-empty dominable set. Take a weak order unit e of $\boldsymbol{M}$, and set

$$z_m = \sup_{z \in A} z \wedge me \; \forall \, m \in \mathbf{N}$$

(using the fact that $\boldsymbol{M}$ is Dedekind complete). Set $B = \{z_m : m \in \mathbf{N}\}$. Then B is dominable. For if $x > 0$ in $\boldsymbol{M}$, there is a $k \in \mathbf{N}$ such that

$$y = \inf_{z \in A} (kx - z)^+ > 0;$$

now
$$kx \wedge z_m \leqslant \sup_{z \in A} kx \wedge z \leqslant kx - y \; \forall \, m \in \mathbf{N},$$

so
$$\inf_{m \in \mathbf{N}} (kx - z_m)^+ \geqslant y > 0.$$

By Theorem 1·3, B is bounded above; but any upper bound of B will be an upper bound of A.

(*c*) (*ii*) $\Rightarrow$ (*i*). Follows immediately from Lemma 1·2, as in part (*a*) of the proof of Theorem 1·3.

1·5. *Examples* (*a*). *In* (6), *Theorem* 47·4, *it is shown that $C^\infty(X)$, the space of continuous functions $x : X \to [-\infty, \infty]$ such that $\{t : -\infty < x(t) < \infty\}$ is dense, is inextensible for any extremally disconnected topological space X.*

(*b*) *In* (9), *Theorem V*. 5·1, *it is shown that $\boldsymbol{C}^\infty(X)$ is sequentially inextensible whenever X is compact and quasi-extremally disconnected. In the Corollary following this theorem* (*loc. cit.*) *we find that any sequentially inextensible Riesz space with a weak order unit is of this type.*

(*c*) *It is plain that, if X is any measure space, the space $\boldsymbol{M}(X)$ of real-valued measurable functions on X is sequentially inextensible, and so is the space $\boldsymbol{L}^0(X)$ of equivalence classes in $\boldsymbol{M}(X)$* (*cf.* 1·14 *below*). *As $\boldsymbol{L}^0(X)$ certainly has a weak order unit it will be inextensible iff it is Dedekind complete* (1·4); *this will be so iff X has a Dedekind complete measure algebra* (*see* (2), *Lemma* 64*C*).

(*d*) *Let X be an uncountable set. Let*

$$\boldsymbol{M} = \{x : x \in \mathbf{R}^X, \exists \, \alpha \in \mathbf{R}, \{t : x(t) \neq \alpha\} \text{ is countable}\}.$$

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Then $\mathbf{M}$ is a Riesz subspace of $\mathbf{R}^X$, closed under countable suprema and infima; so is sequentially inextensible.

(*e*) *Any product of (sequentially) inextensible Riesz spaces is (sequentially) inextensible; in particular, $\mathbf{R}^X$ is inextensible, for any set X. Any band in a (sequentially) inextensible Riesz space is (sequentially) inextensible. See also* 1·14 *below.*

1·6. I come now to the fundamental theorems which make inextensible spaces important.

A. THEOREM. *Let $\mathbf{E}$ be any Archimedean Riesz space, $\mathbf{F}$ an order-dense Riesz subspace of $\mathbf{E}$, and $\mathbf{M}$ an inextensible Riesz space. If $T: \mathbf{F} \to \mathbf{M}$ is an order-continuous Riesz homomorphism, then it has a unique extension to an order-continuous Riesz homomorphism from $\mathbf{E}$ to $\mathbf{M}$.*

Proof. Use the argument of (2), Theorem 64D.

B. THEOREM. *Let $\mathbf{E}$ be any Archimedean Riesz space, $\mathbf{F}$ a super-order-dense Riesz subspace of $\mathbf{E}$, and $\mathbf{M}$ a sequentially inextensible Riesz space. If $T: \mathbf{F} \to \mathbf{M}$ is a sequentially order-continuous Riesz homomorphism, then it has a unique extension to a sequentially order-continuous Riesz homomorphism from $\mathbf{E}$ to $\mathbf{M}$.*

Proof. See (2), Theorem 62K and Exercise 62Me.

1·7A. COROLLARY. *Let $\mathbf{M}$ and $\mathbf{N}$ be inextensible Riesz spaces, with order-dense Riesz subspaces $\mathbf{E}$, $\mathbf{F}$ such that $\mathbf{E} \cong \mathbf{F}$. Then any isomorphism between $\mathbf{E}$ and $\mathbf{F}$ extends uniquely to an isomorphism between $\mathbf{M}$ and $\mathbf{N}$.*

Proof. An isomorphism $T: \mathbf{E} \to \mathbf{F}$ can be thought of as a Riesz homomorphism $T: \mathbf{E} \to \mathbf{N}$; because $\mathbf{F}$ is order-dense, $T: \mathbf{E} \to \mathbf{N}$ is order-continuous ((2), Lemma 17E), so has a unique extension to an order-continuous Riesz homomorphism $\tilde{T}: \mathbf{M} \to \mathbf{N}$. Similarly, $S = T^{-1}: \mathbf{F} \to \mathbf{E}$ extends to $\tilde{S}: \mathbf{N} \to \mathbf{M}$. As $\tilde{S}\tilde{T}: \mathbf{M} \to \mathbf{M}$ is order-continuous and agrees with the identity on the order-dense Riesz subspace $\mathbf{E}$, $\tilde{S}\tilde{T}$ is the identity on $\mathbf{M}$. Similarly, $\tilde{T}\tilde{S}$ is the identity on $\mathbf{N}$, and $\tilde{S}$ and $\tilde{T}$ are the two halves of the required isomorphism between $\mathbf{M}$ and $\mathbf{N}$.

B. COROLLARY. *Let $\mathbf{M}$ and $\mathbf{N}$ be sequentially inextensible Riesz spaces with super-order-dense Riesz subspaces $\mathbf{E}$, $\mathbf{F}$ such that $\mathbf{E} \cong \mathbf{F}$. Then any isomorphism between $\mathbf{E}$ and $\mathbf{F}$ extends uniquely to an isomorphism between $\mathbf{M}$ and $\mathbf{N}$.*

Proof. As (A) above.

1·8. The following corollaries justify the word 'inextensible'.

A. COROLLARY. *Let $\mathbf{M}$ be an Archimedean Riesz space, and $\mathbf{N}$ an order-dense Riesz subspace of $\mathbf{M}$ which is, in itself, inextensible. Then $\mathbf{N} = \mathbf{M}$.*

Proof. Apply 1·6A with $\mathbf{E} = \mathbf{N} \subseteq \mathbf{M}$, T the identity map $\mathbf{N} \to \mathbf{M}$.

B. COROLLARY. *Let $\mathbf{M}$ be an Archimedean Riesz space, and $\mathbf{N}$ a super-order-dense Riesz subspace of $\mathbf{M}$ which is, in itself, sequentially inextensible. Then $\mathbf{N} = \mathbf{M}$.*

Proof. As (A) above.

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1·9A. THEOREM. *Let E be any Archimedean Riesz space. Then E can be embedded, in an essentially unique way, as an order-dense Riesz subspace of an inextensible Riesz space.*

Proof. In (6), Theorem 50·8, or (9), Theorem V. 7·1, it is shown that E can be embedded as an order-dense Riesz subspace of a space of the form $C^\infty(X)$, which is inextensible. By 1·7A above, the embedding is essentially unique.

B. PROPOSITION. *Let E be a Dedekind σ-complete Riesz space. Then E can be embedded, in an essentially unique way, as a super-order-dense Riesz subspace of a sequentially inextensible Riesz space.*

Proof. Embed E as an order-dense Riesz subspace of an inextensible Riesz space M, as in (A) above. Let P be

$$\{x: x \in M^+, \exists \text{ countable } C \subseteq E, x = \sup C \text{ in } M\},$$

and set $N = P - P \subseteq M$. It is easy to see that P is a cone in M, so N is a linear subspace of M; also, $E^+ \subseteq P$, so $E \subseteq N$. If $x, y \in P$, then $x \vee y \in P$, so

$$(x-y)^+ = x \vee y - y \in N;$$

this shows that N is a Riesz subspace of M. Now in fact $N^+ = P$. For if $u \in N^+$, express u as $x-y$, where $x, y \in P$, and find sequences (x_n) and (y_n) in E such that $x = \sup_{n\in\mathbf{N}} x_n$, $y = \sup_{n\in\mathbf{N}} y_n$. Set

$$z_n = x_n \wedge y = \sup_{m\in\mathbf{N}} x_n \wedge y_m \in E$$

(this is where we use the fact that E is Dedekind σ-complete; as E is regularly embedded in M, we may take the supremum either in E or in M). Then

$$\begin{aligned} u = x-y = (x-y)^+ &= \sup_{n\in\mathbf{N}} (x_n - y)^+ \\ &= \sup_{n\in\mathbf{N}} (x_n - z_n) \in P. \end{aligned}$$

This last property of N shows both that E is super-order-dense in N and that N is sequentially inextensible. For if $A \subseteq N^+$ is countable and bounded above in M, $A \subseteq P$, so $\sup A$ (taken in M) is the supremum of a countable set in E, and belongs to N. In particular, if $A \subseteq N^+$ is countable and disjoint, then $\sup A$ exists in M and belongs to N.

Remark. We observe that the simple system of correspondences between inextensible and sequentially inextensible spaces, developed in 1·6–1·8, breaks down here. For example, let X be an uncountable set, and let E be the Riesz subspace of $\mathbf{R}^X$

$$\{x: x \in \mathbf{R}^X, \exists \alpha \in \mathbf{R}, \{t: x(t) \neq \alpha\} \text{ is finite}\}.$$

Then it is not hard to see that there is no Dedekind σ-complete Riesz space in which E can be embedded as a super-order-dense Riesz subspace.

The condition 'Dedekind σ-complete' of (B) above is fairly strong. Other spaces will have the same property; e.g. any Archimedean Riesz space with the countable sup property can be substituted for E.

1·10. *Inextensible extensions.* In view of 1·9A, it is tempting to speak at once of 'the' inextensible extension of an Archimedean Riesz space, as 'that inextensible Riesz space

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in which it is embedded as an order-dense Riesz subspace.' But this is bad logic, as the 'uniqueness' in 1·9 A is uniqueness up to isomorphism, not true uniqueness. I propose the following argument.

Let $\boldsymbol{E}$ be an Archimedean Riesz space. Let $\mathscr{P}$ be the set of those non-empty sets $A \subseteq \boldsymbol{E}^+$ such that (i) if $x \in \boldsymbol{E}^+$ is such that $x = \sup_{z \in A} x \wedge z$, then $x \in A$ (ii) for each $x > 0$ in $\boldsymbol{E}$, there is a $k \in \boldsymbol{N}$ such that $kx \notin A$. Let $\boldsymbol{N}$ be the set of those pairs (A, B) of members of $\mathscr{P}$ such that $A \cap B = \{0\}$. Now fix on any order-dense embedding of $\boldsymbol{E}$ in an inextensible Riesz space $\boldsymbol{M}$. Then it is clear that there is a natural one-to-one correspondence $u \mapsto A(u)$: $\boldsymbol{M}^+ \to \mathscr{P}$, given by

$$A(u) = \{x: x \in \boldsymbol{E}^+, 0 \leqslant x \leqslant u \text{ in } \boldsymbol{M}\},$$

and therefore a one-to-one correspondence between $\boldsymbol{M}$ and $\boldsymbol{N}$, given by

$$u \mapsto (A(u^+), A(u^-)).$$

This imposes an inextensible Riesz space structure on $\boldsymbol{N}$, and an embedding of $\boldsymbol{E}$ as an order-dense Riesz subspace of $\boldsymbol{N}$. At this point we observe that 1·7 A implies that the structure of $\boldsymbol{N}$ does not depend on which inextensible space $\boldsymbol{M}$, and embedding of $\boldsymbol{E}$ in $\boldsymbol{M}$, we choose. So $\boldsymbol{N}$ may legitimately be called the inextensible extension of $\boldsymbol{E}$.

In fact it is possible, with some difficulty, to define the Riesz space structure on $\boldsymbol{N}$ directly, without reference to 1·9, in such a way as to prove that it is inextensible. This gives an 'elementary' proof of 1·9, not employing the representation theory for Archimedean Riesz spaces.

1·11. We can classify inextensible Riesz spaces up to isomorphism in the following way.

PROPOSITION. *Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be inextensible Riesz spaces. Then they are isomorphic iff they have isomorphic lattices of bands. Furthermore, a lattice is isomorphic to the lattice of bands of some inextensible Riesz space iff it is a Dedekind complete Boolean algebra.*

Proof. If $\boldsymbol{M}$ is any inextensible Riesz space, it is embeddable as an order-dense Riesz subspace of $\boldsymbol{C}^\infty(X)$, for some compact extremally disconnected Hausdorff space X ((9), Theorem V. 4·1); by 1·8 A, $\boldsymbol{M} = \boldsymbol{C}^\infty(X)$. Now the lattice of bands of $\boldsymbol{C}^\infty(X)$ is naturally isomorphic to the lattice of compact open sets in X (see (9), remarks following Theorem V. 4·2), and this determines the topological space X up to homeomorphism, and therefore determines the Riesz space $\boldsymbol{C}^\infty(X)$ up to isomorphism. Thus $\boldsymbol{M}$ is determined up to isomorphism by its lattice of bands.

Of course, the lattice of bands of any Archimedean Riesz space is a Dedekind complete Boolean algebra ((6), Theorems 22·7 and 22·8). On the other hand, if $\mathfrak{A}$ is any Dedekind complete Boolean algebra, the Stone space X of $\mathfrak{A}$ is a compact extremally disconnected Hausdorff space, and $\boldsymbol{C}^\infty(X)$ is an inextensible Riesz space with lattice of bands isomorphic to $\mathfrak{A}$.

Remark. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are inextensible Riesz spaces with lattices $\mathfrak{A}$ and $\mathfrak{B}$ of bands, and if $\mathfrak{A} \cong \mathfrak{B}$, then $\boldsymbol{M} \cong \boldsymbol{N}$. But, given an isomorphism π: $\mathfrak{A} \to \mathfrak{B}$, there is no canonical corresponding isomorphism T: $\boldsymbol{M} \to \boldsymbol{N}$; in order to specify T, we have to fix on weak

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order units e_1, e_2 of $\boldsymbol{M}$ and $\boldsymbol{N}$; there will now be a unique isomorphism $T\colon \boldsymbol{M} \to \boldsymbol{N}$ such that $Te_1 = e_2$ and $T[F] = \pi F$ for every $F \in \mathfrak{U}$.

1·12. In the following paragraphs I give some further useful and interesting properties of sequentially inextensible spaces.

LEMMA. *Let $\boldsymbol{M}$ be a sequentially inextensible Riesz space. Suppose that $x_n \downarrow 0$ in $\boldsymbol{M}$. Then*

$$A = \{\alpha_n(x_n - \epsilon x_0)^+ \colon n \in \boldsymbol{N}\}$$

is bounded above for any sequence (α_n) in $\mathbf{R}^+$ and any $\epsilon > 0$.

Proof. I show that A is dominable. Let $x > 0$ in $\boldsymbol{M}$. If $u = x \wedge \epsilon x_0 = 0$, then $x \wedge \alpha_n x_n = 0$ for every $n \in \boldsymbol{N}$, so

$$(x - z)^+ = x > 0 \,\forall\, z \in A.$$

If, on the other hand, $u > 0$, there is an $m \in \mathbf{N}$ such that $v = (u - x_m)^+ > 0$. Now

$$v \wedge (x_m - \epsilon x_0)^+ \leqslant (\epsilon x_0 - x_m)^+ \wedge (x_m - \epsilon x_0)^+ = 0,$$

so

$$v \wedge \alpha_r(x_r - \epsilon x_0)^+ = 0 \quad (\forall\, r \geqslant m).$$

Let

$$w = \sup_{r \leqslant m} \alpha_r(x_r - \epsilon x_0)^+;$$

let $k \in \mathbf{N}$ be such that $y = (kv - w)^+ > 0$. Then

$$w_r = (kx - \alpha_r(x_r - \epsilon x_0)^+)^+ \geqslant (kv - w)^+ \geqslant y$$

for every $r \leqslant m$, while

$$w_r \geqslant kv \geqslant y \,\forall\, r \geqslant m.$$

Thus

$$\inf_{r \in \mathbf{N}}(kx - \alpha_r(x_r - \epsilon x_0)^+)^+ > 0.$$

As x is arbitrary, A is dominable; as $\boldsymbol{M}$ is sequentially inextensible, A is bounded above.

1·13. COROLLARY. *Let $\boldsymbol{M}$ be a sequentially inextensible Riesz space and $\boldsymbol{E}$ any Archimedean Riesz space. Then any increasing linear map $T\colon \boldsymbol{M} \to \boldsymbol{E}$ is sequentially order-continuous.*

Proof. Suppose that $x_n \downarrow 0$ in $\boldsymbol{M}$. Let y be any lower bound for $\{Tx_n \colon n \in \boldsymbol{N}\}$ in $\boldsymbol{E}$, and let $\epsilon > 0$. Now

$$x = \sup_{n \in \mathbf{N}} n(x_n - \epsilon x_0)^+$$

exists in $\boldsymbol{M}$, by 1·12. So

$$Tx \geqslant n(Tx_n - \epsilon Tx_0) \geqslant n(y - \epsilon Tx_0)$$

for every $n \in \mathbf{N}$. As $\boldsymbol{E}$ is Archimedean, $y - \epsilon Tx_0 \leqslant 0$. But ϵ is arbitrary, so (again because $\boldsymbol{E}$ is Archimedean) $y \leqslant 0$. As y is arbitrary, $Tx_n \downarrow 0$, as required.

Remark. It is *not* the case that increasing linear maps from inextensible Riesz spaces to Archimedean Riesz spaces have to be order-continuous. For example, let X be an uncountable set, $\boldsymbol{M} = \mathbf{R}^X$; let

$$\boldsymbol{N} = \{x \colon x \in \boldsymbol{M}, \{t \colon x(t) \neq 0\} \text{ is countable}\}.$$

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If $E = M/N$ and $T: M \to E$ is the canonical homomorphism, E is Archimedean, but T is not order-continuous.

1·14. PROPOSITION. *An Archimedean quotient space of a sequentially inextensible Riesz space is sequentially inextensible.*

Proof. Let M be a sequentially inextensible Riesz space and E a solid linear subspace of M such that the quotient $N = M/E$ is Archimedean. Then the canonical homomorphism $\phi: M \to N$ is sequentially order-continuous (1·13).

Observe first that N is Dedekind σ-complete. For let (u_n) be any sequence in N bounded above by $u \in N$. Choose $x, x_n \in M$ such that $\phi(x) = u$, $\phi(x_n) = u_n$ for each $n \in \mathbf{N}$. Set

$$y = \sup_{n\in\mathbf{N}} x \wedge x_n \in M.$$

As ϕ is a sequentially order-continuous Riesz homomorphism,

$$\phi(y) = \sup_{n\in\mathbf{N}} \phi(x) \wedge \phi(x_n) = \sup_{n\in\mathbf{N}} u_n$$

in N.

Now suppose that (u_n) is a disjoint sequence in N^+. For each $n \in N$, choose an $x_n \in M^+$ such that $\phi(x_n) = u_n$. Set

$$y_n = x_n - \sup_{m\in\mathbf{N}}(x_n \wedge m\textstyle\sum_{i<n} x_i) \in M.$$

Then (y_n) is disjoint in M^+ and

$$\phi(y_n) = \phi(x_n) - \sup_{m\in\mathbf{N}}(\phi(x_n) \wedge m\textstyle\sum_{i<n} \phi(x_i)) = u_n$$

for each $n \in \mathbf{N}$. As M is sequentially inextensible, $y = \sup_{n\in\mathbf{N}} y_n$ exists in M, and $\phi(y) = \sup_{n\in\mathbf{N}} u_n$. Thus N is sequentially inextensible.

1·15. PROPOSITION. *Let M be a sequentially inextensible Riesz space. Then any increasing linear functional $f: M \to \mathbf{R}$ is expressible as the sum of a finite number of Riesz homomorphisms.*

Proof. Let F be the solid linear subspace

$$\{x: x \in M, f(|x|) = 0\}$$

of M. Let N be the Riesz space quotient M/E, and define $g: N \to \mathbf{R}$ by writing $g(x^\cdot) = f(x)$ for every $x \in M$. Then g is a strictly positive linear functional on N. It follows at once that N is Archimedean, and therefore (by 1·14) sequentially inextensible.

Now suppose that (u_n) is any disjoint sequence in N^+. For any sequence (α_n) in $\mathbf{R}^+$, $\sup_{n\in\mathbf{N}} \alpha_n u_n$ exists, and

$$\sup_{n\in\mathbf{N}} \alpha_n g(u_n) \leqslant g(\sup_{n\in\mathbf{N}} \alpha_n u_n) < \infty.$$

Thus $g(u_n) = 0$ except for finitely many n, and, as g is strictly positive, $u_n = 0$ except for finitely many n; i.e. there is no proper infinite disjoint sequence in N^+. It follows that $N \cong \mathbf{R}^n$ for some n ((6), Theorems 26·10 and 26·11). Consequently g is a sum of Riesz homomorphisms, $g_0, \ldots, g_r$ say. Now set

$$f_i(x) = g_i(x^\cdot) \; \forall x \in M, \; i \leqslant r;$$

as the map $x \mapsto x^\cdot: M \to N$ is a Riesz homomorphism, each f_i is a Riesz homomorphism, and $f = \sum_{i\leqslant r} f_i$.

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2. FATOU TOPOLOGIES ON INEXTENSIBLE SPACES

If X is a probability space, $L^0(X)$ has a familiar topology, that of 'convergence in measure' (see 2·8(*b*) below). Similarly, for any set X, $\mathbf{R}^X$ is naturally endowed with its product topology. In both cases, the topology is Fatou. In this section I shall discuss the general properties of Fatou topologies on inextensible spaces. We find that, on a given inextensible Riesz space $\boldsymbol{M}$, there may be no non-trivial Fatou topology (2·8(*c*)); but if $\boldsymbol{M}$ has any Hausdorff Fatou topology, it has only one (2·7), and this has several special properties (2·1–2·6). In the next section I will show that under normal circumstances there are no other locally solid topologies to trouble us.

I think it likely that there are parallel results concerning sequentially inextensible spaces, as in the first part of this paper. However, I have had little success in this direction.

2·1. LEMMA. *Let $\boldsymbol{E}$ be a Riesz space with a Hausdorff Fatou topology $\mathfrak{T}$. Let $A \subseteq \boldsymbol{E}^+$ be directed upwards and bounded for $\mathfrak{T}$. Then A is dominable.*

Proof. Let $x > 0$ in $\boldsymbol{E}$. Then there is a continuous Fatou pseudo-norm ρ on $\boldsymbol{E}$ such that $\epsilon = \rho(x) > 0$ (using the fact that $\mathfrak{T}$ is defined by the continuous Fatou pseudo-norms; (2), Theorem 23B). Now A is bounded, so there is an integer $k \geqslant 1$ such that $\rho(k^{-1}z) \leqslant \frac{1}{2}\epsilon$ for every $z \in A$. Set $B = \{x \wedge k^{-1}z : z \in A\}$. Then $B \uparrow$, and

$$\rho(x) > \tfrac{1}{2}\epsilon \geqslant \sup_{u \in B} \rho(u).$$

As ρ is a Fatou pseudo-norm, x cannot be the least upper bound of B; suppose that $y > 0$ is such that $x - y \geqslant u$ for every $u \in B$. Then

$$(kx - z)^+ = k(x - x \wedge k^{-1}z) \geqslant ky > 0 \; \forall \, z \in A.$$

As x is arbitrary, this shows that A is dominable.

2·2. THEOREM. *Let $\boldsymbol{M}$ be an inextensible Riesz space and $\mathfrak{T}$ a Fatou topology on $\boldsymbol{M}$. Then $\mathfrak{T}$ is Lebesgue, and is complete in the sense that every Cauchy filter has at least one limit. If $\mathfrak{T}$ is Hausdorff, it is Levi.*

Proof (*a*). From (2), Corollary 24J, we see that a Fatou topology on a Riesz space $\boldsymbol{E}$ is Lebesgue iff every disjoint order-bounded sequence in $\boldsymbol{E}^+$ is convergent to 0. But if (x_n) is any disjoint sequence in $\boldsymbol{M}^+$, (nx_n) is still disjoint, so $x = \sup_{n \in \mathbb{N}} nx_n$ exists in $\boldsymbol{M}$; because $\mathfrak{T}$ is locally solid, $[0, x]$ is bounded, and $x_n \to 0$ for $\mathfrak{T}$. Thus $\mathfrak{T}$ satisfies the condition and is Lebesgue.

(*b*) Suppose next that $\mathfrak{T}$ is Hausdorff. Let $A \subseteq \boldsymbol{M}^+$ be directed upwards and bounded for $\mathfrak{T}$. Then A is dominable (2·1). Because $\boldsymbol{M}$ is inextensible, A is bounded above (1·3). As A is arbitrary, $\mathfrak{T}$ is Levi.

(*c*) So if $\mathfrak{T}$ is Hausdorff, it is a Levi Hausdorff Fatou topology on a Dedekind complete Riesz space $\boldsymbol{M}$, and is therefore complete by Nakano's theorem ((2), Theorem 23K, or (7), chap. 4, Proposition 1·5 (p. 142)).

If $\mathfrak{T}$ is not Hausdorff, let N be the closure of $\{0\}$ for $\mathfrak{T}$, i.e.

$$N = \{x : x \in \boldsymbol{M}, \rho(x) = 0 \; \forall \text{ continuous Fatou pseudo-norms } \rho\}.$$

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Then N is a band in M, so $M = N \oplus N^d$ (as M is Dedekind complete). N^d is still an inextensible Riesz space and, with the topology induced by $\mathfrak{T}$, is isomorphic, as topological Riesz space, to the quotient M/N (this is because $\mathfrak{T}$ is locally solid). Now $\mathfrak{T}$ is a Hausdorff Fatou topology on N^d, therefore complete, so the quotient topology on M/N is complete; which is the same as saying that M is complete under $\mathfrak{T}$.

2·3. LEMMA. *Let M be an inextensible Riesz space, and ρ a Riesz pseudo-norm on M.*
(*a*) *If $x_n \downarrow 0$ in M, $\rho(x_n) \to 0$.*
(*b*) *If $F \subseteq M$ is a solid linear subspace such that ρ is strictly positive on F, then every disjoint set in F^+ is countable, and F has the countable sup property.*

Proof (*a*). Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\rho(\delta x_0) \leqslant \epsilon$, because part of the definition of a Riesz pseudo-norm is that $\lim_{\alpha \to 0} \rho(\alpha x) = 0$ for every x. Now

$$x = \sup_{n \in \mathbb{N}} n(x_n - \delta x_0)^+$$

exists in M, by 1·11. Again, there must be an $n \geqslant 1$ such that $\rho(n^{-1}x) \leqslant \epsilon$. So, for any $m \geqslant n$,

$$\begin{aligned} \rho(x_m) &\leqslant \rho(x_m - \delta x_0)^+ + \rho(x_m \wedge \delta x_0) \\ &\leqslant \rho(m^{-1}x) + \rho(\delta x_0) \\ &\leqslant \rho(n^{-1}x) + \epsilon \leqslant 2\epsilon. \end{aligned}$$

As ϵ is arbitrary, $\rho(x_n) \to 0$.

(*b*) If (x_n) is any disjoint sequence in F^+, $x = \sup_{n \in \mathbb{N}} n x_n$ exists in M, so

$$\rho(x_n) \leqslant \rho(n^{-1}x) \to 0 \quad \text{as} \quad n \to \infty.$$

Now if $A \subseteq F^+$ is any disjoint set,

$$\{x : x \in A, \quad \rho(x) \geqslant \epsilon\}$$

must be finite for every $\epsilon > 0$, so (because ρ is strictly positive on F) A is countable. Thus F has the countable sup property by the criterion in (2), Theorem 18D, or (6), Theorem 29·3.

2·4. The essential result we are seeking is 2·7 below; but the following lemma seems a convenient formulation of the underlying idea.

LEMMA. *Let E be a Riesz space with a Hausdorff Fatou topology $\mathfrak{T}$. Let M be the inextensible extension of E* (1·10), *and let θ be any Fatou pseudo-norm on M. Then θ is continuous on E for $\mathfrak{T}$.*

Proof (*a*). For any Fatou pseudo-norm ρ on E, set

$$F_\rho = \{z : z \in E, \rho(z) = 0\},$$

$$G_\rho = F_\rho^d = \{y : y \in E, |y| \wedge |z| = 0 \; \forall z \in F_\rho\}.$$

Then F_ρ and G_ρ are bands in E, and ρ is strictly positive on G_ρ. Moreover, if $x \in E^+$, then

$$A = \{y + z : y \in G_\rho, z \in F_\rho, 0 \leqslant y, z \leqslant x\} \uparrow x$$

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because $G_\rho = F_\rho^d$ and E is Archimedean. (Only Archimedean spaces can carry Hausdorff locally solid topologies.) As ρ is a Fatou pseudo-norm,

$$\rho(x) = \sup_{u \in A} \rho(u) = \sup\{\rho(y) : y \in G_\rho, 0 \leqslant y \leqslant x\}.$$

(*b*) Note that, because E is regularly embedded in M, the restriction of θ to E is a Fatou pseudo-norm on E. Consider the support G_θ of θ in E ((*a*) above). By 2·2 (*b*), every disjoint set in G_θ^+ is countable.

Let $\mathscr{A}$ be the set of all pairs (x, ρ) such that ρ is a continuous Fatou pseudo-norm on E, $x \in G_\rho \cap G_\theta$, and $x > 0$. Let $\mathscr{B} \subseteq \mathscr{A}$ be maximal subject to the requirement that $x_1 \wedge x_2 = 0$ whenever (x_1, ρ_1) and (x_2, ρ_2) are distinct members of $\mathscr{B}$. Then $\mathscr{B}$ must be countable. Enumerate it as (x_i, ρ_i). (I leave it to the reader to adjust details as necessary for the case in which $\mathscr{B}$ is finite.)

(*c*) Let $\mathfrak{T}_0$ be the topology on G_θ induced by the pseudo-norms ρ_i. Then $\mathfrak{T}_0$ is Hausdorff. For suppose that $x \in G_\theta$ and $x \neq 0$. As $\mathfrak{T}$ is Hausdorff, there is a continuous Fatou pseudo-norm ρ on E such that $\rho(x) \neq 0$. By the remarks in (*a*) above, there is a $y \in G_\rho$ such that $0 \leqslant y \leqslant |x|$ and $\rho(y) > 0$. Now $(y, \rho) \in \mathscr{A}$, so, by the maximality of $\mathscr{B}$, there must be some i such that $y \wedge x_i > 0$. Because x_i belongs to the support of ρ_i,

$$0 < \rho_i(y \wedge x_i) \leqslant \rho_i(|x|) = \rho_i(x).$$

As x is arbitrary, $\mathfrak{T}_0$ is Hausdorff.

(*d*) Suppose, if possible, that θ is not continuous for $\mathfrak{T}$. As θ is a pseudo-norm, it must be discontinuous at 0, and there is some $\epsilon > 0$ such that $\{x : x \in E, \theta(x) \leqslant \epsilon\}$ is not a neighbourhood of 0 for $\mathfrak{T}$. In particular, we must be able to choose, for each $n \in \mathbf{N}$, an $x_n \in E$ such that

$$\sum_{i \leqslant n} \rho_i(x_n) \leqslant 4^{-n}, \quad \theta(x_n) > \epsilon.$$

Again using (*a*) above, we can choose $y_n \in G_\theta$ such that $0 \leqslant y_n \leqslant |x_n|$ and $\theta(y_n) \geqslant \epsilon$; we still have $\sum_{i \leqslant n} \rho_i(y_n) \leqslant 4^{-n}$.

Let A be the set

$$\{\sum_{i \leqslant n} 2^i y_i : n \in \mathbf{N}\} \subseteq G_\theta.$$

Then A is directed upwards and bounded for $\mathfrak{T}_0$, as

$$\rho_i(2^n y_n) \leqslant 2^n \rho_i(y_n) \leqslant 2^{-n} \; \forall \, i \leqslant n.$$

Thus, by 2·1, A is dominable in G_θ. But it follows that A is dominable in M. For given $u > 0$ in M, either $u \wedge z = 0$ for every $z \in G_\theta$ and $(u - z)^+ = u > 0$ for every $z \in A$, or there is a $u_1 \in G_\theta$ such that $u \wedge u_1 > 0$. As E is order-dense in M, there is a $u_2 \in E$ such that $0 < u_2 \leqslant u \wedge u_1 \leqslant u$, and now $u_2 \in G_\theta$. So there is a $k \in \mathbf{N}$ and a $y \in G_\theta$ such that

$$0 < y \leqslant (ku_2 - z)^+ \leqslant (ku - z)^+ \; \forall \, z \in A.$$

As u is arbitrary, A is dominable in M.

So $x = \sup A$ exists in M. But in this case

$$\epsilon \leqslant \theta(y_n) \leqslant \theta(2^{-n} x) \; \forall \, n \in \mathbf{N},$$

which is impossible. This contradiction shows that θ is continuous for $\mathfrak{T}$.

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2·5. THEOREM. *Let E be a Riesz space with a Hausdorff Fatou topology, and let M be an inextensible Riesz space with a Fatou topology. Then any order-continuous Riesz homomorphism $T: E \to M$ is continuous.*

Proof. For let N be the inextensible extension of E (1·10). Then T has an extension to an order-continuous Riesz homomorphism $\tilde{T}: N \to M$ (1·6A). Let ρ be any continuous Fatou pseudo-norm on M, and let $\theta = \rho\tilde{T}: N \to \mathbf{R}^+$. Then θ is a Fatou pseudo-norm on N, so θ is continuous on E by 2·4. As the topology on M is defined by the continuous Fatou pseudo-norms, T is continuous.

2·6. COROLLARY. *Let M be an inextensible Riesz space with a Hausdorff Fatou topology $\mathfrak{T}$, and let E be any Riesz space with a Fatou topology. Then any order-continuous increasing linear map $T: M \to E$ must be continuous.*

Proof. For let $\mathfrak{T}_0$ be the topology on M with basic neighbourhoods of 0

$$\{x: T(|x|) \in U\},$$

where U runs through the neighbourhoods of 0 in E. Then $\mathfrak{T}_0$ is Fatou, so, applying 2·5 to the identity map $(M, \mathfrak{T}) \to (M, \mathfrak{T}_0)$, we see that $\mathfrak{T}_0 \subseteq \mathfrak{T}$, and that T is continuous.

2·7. COROLLARY. *Let M be an inextensible Riesz space. Then there is at most one Hausdorff Fatou topology on M.*

Proof. Immediate from 2·5 or 2·6.

2·8. EXAMPLES. (*a*) *If X is any set, the product topology is the Hausdorff Fatou topology on $\mathbf{R}^X$.*

(*b*) *If X is any measure space, we may endow $L^0(X)$ with the topology $\mathfrak{T}$ defined by the pseudo-norms*

$$\rho_u(x) = \int |x| \wedge u,$$

where u runs through $L^1(X)^+$; this is Fatou and Lebesgue ((2), 63K). *When X is semi-finite, $\mathfrak{T}$ is Hausdorff* ((2), *Lemma* 63L); *so when X is Maharam (i.e. is semi-finite and has a Dedekind complete measure algebra), $\mathfrak{T}$ is the Hausdorff Fatou topology on the inextensible space $L^0(X)$.*

(*c*) *Let $E = C([0, 1])$, and let M be the inextensible extension of E. Then the only Fatou topology on M is the indiscrete topology. For if ρ is any Fatou pseudo-norm on M, it induces a Lebesgue topology on M, and is therefore an order-continuous increasing functional on M^+; as E is regularly embedded in M, $\rho: E^+ \to \mathbf{R}$ is order-continuous. The argument of* (5), *Example* 24·5(ii), *although there applied to linear order-continuous increasing functionals, applies equally well in the present case to show that $\rho(\mathbf{1}) = 0$, where $\mathbf{1}$ is the unit of E; it follows that ρ is identically zero on E and therefore on M. Thus M has no non-trivial Fatou pseudo-norms and therefore no non-trivial Fatou topology.*

2·9. PROBLEM. Is there any inextensible Riesz space which does have a Hausdorff Fatou topology, but is not isomorphic to $L^0(X)$ for any Maharam measure space X?

2·10. METRIZABLE SPACES. As an example of the way in which Riesz space properties on an inextensible space are linked with topological properties, I give the following result.

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PROPOSITION. Let M be an inextensible Riesz space with a Hausdorff Fatou topology $\mathfrak{T}$. Then $\mathfrak{T}$ is metrizable iff M has the countable sup property.

Proof. (*a*) If M has the countable sup property, it has the stronger property that every disjoint set in M^+ is countable. For M has a weak order unit e (1·3); now, if $A \subseteq M^+$ is disjoint, then $\{e \wedge x : x \in A\}$ is an order-bounded disjoint set, so must be countable. But the map $x \mapsto e \wedge x : A \to M^+$ is one-to-one, so A is countable.

Now we may use the argument of the proof of 2·4, parts (*b*) and (*c*), but with M in the place of G_θ, to see that there is a countable family of Fatou pseudo-norms on M defining a Hausdorff topology $\mathfrak{T}_0$ on M, which is of course Fatou and metrizable. But by 2·7 we have $\mathfrak{T} = \mathfrak{T}_0$.

(*b*) On the other hand, if $\mathfrak{T}$ is metrizable, let (ρ_i) be a sequence of Fatou pseudo-norms on M defining $\mathfrak{T}$. Define $\rho : M \to \mathbf{R}^+$ by

$$\rho(x) = \sum_{i \in \mathbf{N}} \min(2^{-i}, \rho_i(x)) \; \forall \, x \in M.$$

Then it is easy to see that ρ is a strictly positive Fatou pseudo-norm on M. So by 2·3 (*b*) M has the countable sup property.

3. THE MEASURABLE CARDINAL PROBLEM

In this section I discuss various problems concerning inextensible Riesz spaces which are related to the famous measurable cardinal problem. They are of the same type as those in (4).

3·1. DEFINITION. Let X be a set. An *over-measure* on X is a function $\mu : \mathscr{P}X \to \mathbf{R}^+$ such that

(i) if $A \subseteq B \subseteq X$, then $\mu A \leqslant \mu B$;

(ii) if $A, B \subseteq X$, then $\mu(A \cup B) \leqslant \mu A + \mu B$;

(iii) if (A_n) is a decreasing sequence in $\mathscr{P}X$ and $\bigcap_n A_n = \varnothing$, then $\mu A_n \to 0$.

3·2. LEMMA. *If μ is an over-measure on X, then*

(*a*) *if (A_n) is a disjoint sequence in $\mathscr{P}X$, $\mu A_n \to 0$;*

(*b*) *$\mu(\bigcup_n A_n) \leqslant \sum_n \mu A_n$ for any sequence (A_n) in $\mathscr{P}X$.*

Proof. (*a*) Apply 3·1 (iii) to $B_n = \bigcup_{i \geqslant n} A_i$ and 3·1 (i) to $A_n \subseteq B_n$.

(*b*) By 3·1 (ii), $\mu(\bigcup_{i \leqslant n} A_i) \leqslant \sum_{i \leqslant n} \mu A_i$ for each $n \in \mathbf{N}$. Now

$$\mu(\bigcup_{i \in \mathbf{N}} A_i \setminus \bigcup_{i \leqslant n} A_i) \to 0 \quad \text{as} \quad n \to \infty$$

by 3·1 (iii), so the result follows on applying 3·1 (ii) again.

3·3. THEOREM. *If one of the following statements is true, so are the others.*

(i) *There is a set X and an over-measure μ on X such that $\mu X > 0$ but $\mu\{t\} = 0$ for every $t \in X$.*

(ii) EITHER (*mc*) *there is a set X and a non-principal ultrafilter $\mathscr{F}$ on X, closed under countable intersections,* OR (*qmc*) *there is a set X, of cardinal $\mathfrak{C} = 2^{\aleph_0}$, and an over-measure μ on X such that $\mu X > 0$ but $\mu\{t\} = 0$ for every $t \in X$.*

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(iii) *There is an inextensible Riesz space* $\mathbf{M}$ *and a Riesz pseudo-norm on* $\mathbf{M}$ *which is not Fatou.*

(iv) *There is an inextensible Riesz space* $\mathbf{M}$ *with a locally solid linear space topology which is not Fatou.*

Proof. (*a*) (*i*) $\Rightarrow$ (*ii*). Let X and μ be as in (i). If there is an $A \subseteq X$ such that $\mu A > 0$ but, for every $B \subseteq A$, either $\mu B = 0$ or $\mu(A \backslash B) = 0$, set $\mathscr{F} = \{B: B \subseteq A, \mu(A \backslash B) = 0\}$. Then $\mathscr{F}$ is a non-principal ultrafilter on A, closed under countable intersections, so (*mc*) is true.

Otherwise, given $A \subseteq X$ such that $\mu A > 0$, and any $\delta > 0$, there is a $B \subseteq A$ such that $0 < \mu B \leqslant \delta$. For choose (C_n) inductively so that $C_0 = A$, and $C_{n+1} \subseteq C_n$, $\mu C_{n+1} > 0$, and $\mu(C_n \backslash C_{n+1}) > 0$ for each $n \in \mathbf{N}$. Then $(C_{n+1} \backslash C_n)$ is a disjoint sequence in $\mathscr{P}X$, so $\mu(C_{n+1} \backslash C_n) \to 0$ (3·2(*a*)); let $B = C_{n+1} \backslash C_n$ for some n so large that $\mu B \leqslant \delta$.

Now, for each $n \in \mathbf{N}$, choose a set $\mathscr{A}_n \subseteq \mathscr{P}X$ which is maximal subject to being disjoint and having $0 < \mu A \leqslant 2^{-n}$ for every $A \in \mathscr{A}_n$. By 3·2(*a*) again, $\mathscr{A}_n$ is countable. Set $D_n = X \backslash \bigcup \mathscr{A}_n$; by the last paragraph and the maximality of $\mathscr{A}_n$, $\mu D_n = 0$. Let $\mathscr{B}_n = \mathscr{A}_n \cup \{D_n\}$, so that $\mathscr{B}_n$ is a countable partition of X into sets of measure $\leqslant 2^{-n}$.

Set $Y = \prod_n \mathscr{B}_n$, so that the cardinal of Y is equal to $\mathfrak{C}$. Define $f: X \to Y$ by saying that $f(t) = (B_n)$, where for each $n \in \mathbf{N}$ B_n is that member of $\mathscr{B}_n$ containing t. Then f induces an over-measure $\nu = \mu f^{-1}$ on Y, writing

$$\nu A = \mu f^{-1}[A] \forall A \subseteq Y.$$

We see that $\nu Y = \mu X > 0$, while if $u = (B_n) \in Y$,

$$\nu\{u\} = \mu(\textstyle\bigcap_n B_n) \leqslant \inf_n \mu B_n \leqslant \inf_n 2^{-n} = 0.$$

Thus Y and ν satisfy (*qmc*).

(*b*) (*ii*) $\Rightarrow$ (*i*). (*qmc*) $\Rightarrow$ (*i*) is obvious. On the other hand, if $\mathscr{F}$ is a non-principal ultrafilter on X, closed under countable intersections, set $\mu A = 1$ for $A \in \mathscr{F}$, $\mu A = 0$ if $A \subseteq X$ and $X \backslash A \in \mathscr{F}$; now μ is an over-measure of the required sort. So (*mc*) $\Rightarrow$ (*i*).

(*c*) (*i*) $\Rightarrow$ (*iv*). Given X and μ as in (i), define $U_n \subseteq \mathbf{R}^X$ to be

$$\{x: x \in \mathbf{R}^X, \mu\{t: |x(t)| \geqslant 2^{-n}\} \leqslant 2^{-n}\}$$

for each $n \in \mathbf{N}$. Then U_n is solid, by 3·1 (i), and $U_{n+1} + U_{n+1} \subseteq U_n$ by 3·1 (ii). Also, by 3·1 (iii), U_n is absorbent. So $\{U_n: n \in \mathbf{N}\}$ is a base of neighbourhoods of 0 for a locally solid linear space topology $\mathfrak{T}$ on $\mathbf{R}^X$, which is an inextensible Riesz space.

However, $\mathfrak{T}$ is not Fatou. For every U_n includes

$$s_0 = \{y: y \in \mathbf{R}^X, \{t: y(t) \neq 0\} \text{ is finite}\},$$

and for any $x \in (\mathbf{R}^X)^+$,

$$\{y: y \in s_0, 0 \leqslant y \leqslant x\} \uparrow x.$$

Thus any order-closed neighbourhood of 0 for $\mathfrak{T}$ must include the whole of $(\mathbf{R}^X)^+$. But $\mathfrak{T}$ is not the indiscrete topology, because $\mu X > 0$. So $\mathfrak{T}$ does not have a base of order-closed neighbourhoods of 0.

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(*d*) (*iv*) $\Rightarrow$ (*iii*). If $\mathfrak{T}$ is a locally solid linear space topology on $\boldsymbol{M}$, it is defined by a family of Riesz pseudo-norms. If $\mathfrak{T}$ is not Fatou, one of these pseudo-norms must fail to be Fatou.

(*e*) (*iii*) $\Rightarrow$ (*i*). Let ρ be a Riesz pseudo-norm on $\boldsymbol{M}$ which is not Fatou. Then there is a non-empty set $A \subseteq \boldsymbol{M}^+$ such that $A \uparrow x_0$ in $\boldsymbol{M}^+$ but $\rho(x_0) > \sup_{x \in A} \rho(x)$. Let $B \subseteq \boldsymbol{M}^+$ be maximal subject to the requirements:

$$\rho(x) = 0 \,\forall\, x \in B;$$

$$B \text{ is disjoint.}$$

Let $y_0 = \sup B$. I shall prove that $\rho(y_0) > 0$.

Let $z_0 = \sup_{n \in \mathbf{N}}(x_0 \wedge ny_0)$, so that $z_0 \wedge (x_0 - z_0) = 0$, and $x \wedge (x_0 - z_0) = 0$ for every $x \in B$. Let $\boldsymbol{F}$ be the band in $\boldsymbol{M}$ generated by $x_0 - z_0$. By the maximality of B, ρ is strictly positive on $\boldsymbol{F}$. Consequently $\boldsymbol{F}$ has the countable sup property (2·3(*b*)). Let

$$A_1 = \{x \wedge (x_0 - z_0) : x \in A\}.$$

Then $A_1 \uparrow x_0 - z_0$ in $\boldsymbol{F}$, so there is a sequence (u_n) in A_1 such that $u_n \uparrow x_0 - z_0$. Now

$$\begin{aligned} \rho(x_0 - z_0) &= \lim_{n \to \infty} \rho(u_n) \quad \text{(by 2·3(}a\text{))} \\ &\leqslant \sup_{x \in A} \rho(x) < \rho(x_0) \end{aligned}$$

by hypothesis. So $\rho(z_0) > 0$. But $x_0 \wedge ny_0 \uparrow z_0$, so there is some $n \in \mathbf{N}$ such that

$$0 < \rho(x_0 \wedge ny_0) \leqslant n\rho(y_0),$$

using 2·3(*a*) again, and $\rho(y_0) > 0$.

Now define $\mu : \mathscr{P}B \to \mathbf{R}$ by

$$\mu\varnothing = 0, \quad \mu C = \rho(\sup C) \quad \text{when} \quad \varnothing \subset C \subseteq B.$$

Because B is disjoint and ρ is sequentially order-continuous and subadditive on $\boldsymbol{M}^+$, μ is an over-measure; $\mu\{x\} = \rho(x) = 0$ for every $x \in B$, and $\mu B = \rho(y_0) > 0$.

3·4. REMARK. A negative answer to the problem in 2·9 would enable us to discard the concept of 'over-measure' and replace (i) above by the more familiar

(*rmc*) there is a set X and a measure μ on X, with domain $\mathscr{P}X$, such that $0 < \mu X < \infty$ and $\mu\{t\} = 0$ for every $t \in X$.

3·5. PROPOSITION. If one of the following statements is true, so are the others.

(*mc*) There is a non-empty set X and a non-principal ultrafilter $\mathscr{F}$ on X, closed under countable intersections.

(i) There is an inextensible Riesz space $\boldsymbol{M}$ and an increasing linear functional $f : \boldsymbol{M} \to \mathbf{R}$ which is not order-continuous.

(ii) There is an inextensible Riesz space $\boldsymbol{M} \neq \{0\}$, without discrete elements, and a Hausdorff locally convex locally solid linear space topology on $\boldsymbol{M}$.

Proof. (*a*) (*mc*) $\Rightarrow$ (*ii*). Let Y be the set of all (non-empty) finite sequences in X. For $\mathbf{t} = (t_0, \ldots, t_n) \in Y$ and $u \in X$, write $\mathbf{t}u = (t_0, \ldots, t_n, u) \in Y$.

On Y define a topology $\mathfrak{T}$ by saying that $G \in \mathfrak{T}$ iff, for every $\mathbf{t} \in G$, $\{u : \mathbf{t}u \in G\} \in \mathscr{F}$. Because $\mathscr{F}$ is a filter, $\mathfrak{T}$ is a topology.

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$\mathfrak{T}$ is extremally disconnected. For suppose that $G \subseteq Y$ is open and that $\mathbf{t} \in \bar{G}$. If $\mathbf{t} \in G$, then certainly

$$\{u\colon \mathbf{t}u \in \bar{G}\} \supseteq \{u\colon \mathbf{t}u \in G\} \in \mathscr{F}.$$

If $\mathbf{t} \in \bar{G} \backslash G$, then $\bar{G} \backslash \{\mathbf{t}\}$ cannot be closed, ie $H = \{\mathbf{t}\} \cup (Y \backslash \bar{G})$ is not open. So there is an $\mathbf{s}_0 \in H$ such that

$$\{u\colon \mathbf{s}_0 u \in Y \backslash \bar{G}\} \notin \mathscr{F}.$$

However, $\{u\colon \mathbf{s}u \in Y \backslash \bar{G}\} \in \mathscr{F}$ for every $\mathbf{s} \in Y \backslash \bar{G}$. So $\mathbf{s}_0 = \mathbf{t}$, and

$$\{u\colon \mathbf{t}u \in Y \backslash \bar{G}\} \notin \mathscr{F}.$$

As $\mathscr{F}$ is an ultrafilter,

$$\{u\colon \mathbf{t}u \in \bar{G}\} \in \mathscr{F}.$$

As $\mathbf{t}$ is arbitrary, $\bar{G}$ is open, as required.

Because $\mathscr{F}$ is non-principal, $\mathfrak{T}$ is Hausdorff; for

$$\{\mathbf{s}\colon \mathbf{s} \in Y, \quad \mathbf{s} \text{ is an extension of } \mathbf{t}\}$$

is an open-and-closed neighbourhood of $\mathbf{t}$ for every $\mathbf{t} \in Y$. Finally, the intersection of countably many open sets in Y is always open, because $\mathscr{F}$ is closed under countable intersections.

It follows that $C^\infty(Y)$, which is an inextensible Riesz space, is actually equal to $C(Y)$. For if $x \in C^\infty(Y)$, then

$$\{\mathbf{t}\colon |x(\mathbf{t})| = \infty\} = \cap_{n \in \mathbf{N}} \{t\colon |x(\mathbf{t})| > n\}$$

is open; but it is also nowhere dense, so it must be empty. As $\mathfrak{T}$ is Hausdorff and extremally disconnected, $C(Y)$ separates the points of Y; as Y has no isolated points, $C(Y)$ has no discrete elements. Thus $C(Y)$ is an inextensible Riesz space without discrete elements.

Now the product topology of $\mathbf{R}^Y$ induces a Hausdorff locally solid locally convex linear space topology on $C(Y)$.

(*b*) (*ii*) $\Rightarrow$ (*i*). There is certainly a non-zero continuous linear functional on M. As the topology on M is locally solid, this may be decomposed into positive and negative parts, so there is a non-zero increasing linear functional on M. Now by 1·14 there must be a non-zero Riesz homomorphism $f\colon M \to \mathbf{R}$.

But consider $N = \{x\colon fx = 0\}$. This is a solid linear subspace of M, and $M/N \cong \mathbf{R}$. If $N^d \neq \{0\}$, then $f\colon N^d \to \mathbf{R}$ would be a Riesz space isomorphism, and any non-zero element of N^d would be discrete. So $N^d = \{0\}$, and $N \subset M = N^{dd}$. Thus N is not a band and f is not order-continuous.

(*c*) (*i*) $\Rightarrow$ (*mc*). Again, if $f\colon M \to \mathbf{R}$ is an increasing linear functional, it is expressible as a sum of Riesz homomorphisms; so if f is not order-continuous, there is a Riesz homomorphism $g\colon M \to \mathbf{R}$ which is not order-continuous. Define $\rho\colon M \to \mathbf{R}^+$ by writing $\rho(x) = g(|x|)$ for each $x \in M$. Then ρ is a Riesz pseudo-norm on M which is not Fatou.

Follow the proof of 3·3, part (*e*), word for word up to the definition of $\mu\colon \mathscr{P}B \to \mathbf{R}$. Observe at this point that, because $\rho = g$ is a lattice homomorphism on M^+,

$$\mathscr{F} = \{C\colon C \subseteq B, \mu(B \backslash C) = g(\sup(B \backslash C)) = 0\}$$

is a non-principal ultrafilter on B, closed under countable intersections.

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Remark. In part (a) of the proof above, the algebra $\mathfrak{A}$ of open-and-closed sets in Y (which is isomorphic to the algebra of bands of $C^\infty(Y) = C(Y)$) is a Dedekind complete Boolean algebra, without atoms, with a non-trivial σ-complete maximal ideal, viz.

$$\{G: G \in \mathfrak{A}, \{u: (u) \in G\} \notin \mathscr{F}\},$$

where for each $u \in X$ I write (u) for the corresponding sequence with one term belonging to Y. The similar algebra $\mathfrak{B}$ of (10) can be represented as a subalgebra of $\mathfrak{A}$; conceivably they are isomorphic.

3·6. *Equiconsistency.* The statements of 3·3–3·5 are known to be unprovable in ordinary mathematics (by which I mean the customary rules of proof applied to the usual axioms, including the axiom of choice but excluding the continuum hypothesis). It remains open whether they are disprovable. However, they are equiconsistent, in the sense that if one is disprovable so are all the others. This follows directly from the work of Solovay ((8)). He is interested in (*rmc*) of 3·4, and shows that it is equiconsistent with (*mc*) (3·3 (ii) or 3·5). His method is to show that both are equiconsistent with

(*) there is a set X and a proper σ-ideal $\mathscr{I}$ of $\mathscr{P}X$ which contains all finite sets and is '$\aleph_1$-saturated', i.e. every uncountable disjoint family in $\mathscr{P}X$ meets $\mathscr{I}$.

[In the work of (8), §§ 2–3, Solovay uses a κ-complete ideal of $\mathscr{P}(\kappa)$, where κ is an uncountable cardinal. But such an ideal can easily be constructed from an ideal $\mathscr{I}$ as in (*) above by the method of (8), 1·1; κ is the least cardinal such that the union of κ members of $\mathscr{I}$ need not belong to $\mathscr{I}$.]

It is obvious that 3·3 (i) implies (*), setting

$$\mathscr{I} = \{A: \mu A = 0\},$$

while (*rmc*) implies 3·3 (i). So these three are equiconsistent with each other and with (*mc*).

3·7. From a practical point of view, supposing that we do not disbelieve entirely in measurable cardinals, we want to know when the phenomena of 3·3 or 3·5 are liable to arise. Examining the arguments above, in particular part (e) of the proof of 3·3, we see that this can happen only if M^+ includes suitable disjoint sets. For 3·5, M^+ must include a disjoint set large enough to carry a non-principal ultrafilter closed under countable intersections. Since such sets have to be unthinkably large ((3)), we can rule this out in all ordinary cases by cardinality arguments.

3·3 is harder to handle in this way, as the set X of (*qmc*) is always of cardinal $\leqslant \mathfrak{C}$; so that, for instance, we might have a non-Fatou locally solid topology on $\mathbf{R}^{[0,1]}$. However, it is perhaps worth remarking that in this case there must be unthinkably many cardinals between $\aleph_0$ and the cardinal of X ((8), Theorem 1), so that (to take a trivial example) if there is no disjoint set in M^+ of cardinal greater than $\aleph_1$, we are safe.

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REFERENCES

(1) FREMLIN, D. H. On the completion of locally solid vector lattices. *Pacific J. Math.* **43** (1972), 341–347.
(2) FREMLIN, D. H. *Topological Riesz spaces and measure theory* (Cambridge University Press, 1973).
(3) KEISLER, H. K. & TARSKI, A. From accessible to inaccessible cardinals, etc. *Fund Math.* **53** (1963–4), 225–308.
(4) LUXEMBURG, W. A. J. Is every integral normal? *Bull. American Math. Soc.* **73** (1967), 685–688.
(5) LUXEMBURG, W. A. J. & ZAANEN, A. C. Notes on Banach function spaces VII. *Nederl. Akad. Wetensch. Proc. Ser. A* **66** (1963), 669–681.
(6) LUXEMBURG, W. A. J. & ZAANEN, A. C. *Riesz spaces I* (North-Holland, 1971).
(7) PERESSINI, A. L. *Ordered topological vector spaces* (Harper & Row, 1967).
(8) SOLOVAY, R. M. Real-valued measurable cardinals. *Proc. Symposia in Pure Mathematics* **13** (Axiomatic Set Theory), Part 1 (*American Math. Soc.* 1971), pp. 397–428.
(9) VULIKH, B. Z. Introduction to the theory of partially ordered Spaces (Wolters-Noordhoff, 1967).
(10) PRIKRY, K. On σ-complete prime ideals in Boolean algebras. *Colloq. Math.* **22** (1971), 209–214.

Note added 9 *December* 1974. I have belatedly recalled seeing 'Measures on Metacompact Spaces' by W. Moran (Proc. London Math. Soc. (3) 20 (1970) 507-524) in which the 'over-measures' of §3 above are studied under the name of 'semi-measures'; the result 3.3 (i)⇔(ii) above is essentially the same as Moran's Proposition 3.5.

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