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Wave–wave interactions in two-fluid cosmic-ray hydrodynamics

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Multiple–scales perturbation methods are used to derive equations describing wave–wave interactions in two-fluid cosmic-ray hydrodynamics in one Cartesian space dimension. Two problems are considered: (*a*) the interaction between short-wavelength waves in a non-uniform large-scale background flow, and (*b*) wave interactions between long-wavelength waves propagating through a uniform background medium. The short-wavelength wave equations describe the interaction between the backward and forward thermal gas sound waves and the contact discontinuity eigenmode via 'wave mixing', in which the waves are reflected by gradients in the large scale background flow. The equations also contain quadratic wave interaction terms describing (i) the Burgers self-wave interaction term, (ii) mean–wave-field interaction terms in which a specific wave interacts with the mean field of the other waves, and (iii) three-wave resonant interactions that describe how a sound wave resonantly interacts with the contact discontinuity to generate a reverse sound wave. In the limit of no interaction between the cosmic rays and thermal gas, and for a uniform background state, the equations reduce to two coupled, integro-differential Burgers equations derived by Majda and Rosales for resonantly interacting waves in adiabatic gasdynamics. The short-wavelength equations also contain squeezing instability terms associated with the large-scale cosmic-ray pressure gradient, which were first investigated by Drury and Dorfi. A generalized wave-action equation, or canonical wave-energy equation, variational principles, and WKB analyses of the linearized equations are used to investigate the modification of the cosmic-ray squeezing instability by wave mixing. Coupled Burgers equations are also derived in the long-wavelength regime that describe resonant wave interactions for weak, diffusively smoothed cosmic-ray-modified shocks. In the latter equations, the cosmic-ray-modified sound waves can resonantly interact with either the contact discontinuity or the cosmic-ray pressure-balance mode to generate a reverse sound wave.

1. Introduction

The diffusive shock acceleration of energetic charged particles by the first-order Fermi mechanism in astrophysical shocks in general leads to a modification of the shock as the energetic particles extract energy from the background flow (Axford *et al.* 1977, 1982; Drury and Völk 1981). For steady state planar shocks in which

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particles do not significantly modify the flow, a systematic increase of the particle momentum on each acceleration cycle coupled with a finite escape probability downstream of the shock leads to the formation of a power law momentum spectrum of energetic particles from an originally monoenergetic distribution (Krymsky 1977; Axford *et al.* 1977; Bell 1978*a*, *b*; Blandford and Ostriker 1978). Reviews of diffusive shock acceleration have been given by Drury (1983), Forman and Webb (1985), Blandford and Eichler (1987), Berezhko and Krymsky (1988) and Jones and Ellison (1991).

The nonlinear two-fluid models of diffusive shock acceleration and cosmic-ray-modified shocks were originally developed by Drury and Völk (1981) and Axford *et al.* (1982) in order to account for the back-reaction of the accelerated energetic particles on the flow. An alternative, nonlinear one-fluid model has been developed by Ellison (1981) and Ellison *et al.* (1981) by means of Monte Carlo simulations (see the review by Jones and Ellison 1991). Other work on the two-fluid model using numerical codes has been carried out by Dorfi and Drury (1985), Falle and Giddings (1987), Bell (1987), Duffy (1992), Jones and Kang (1990), Kang *et al.* (1992), Donohue and Zank (1993) and Donohue *et al.* (1994).

The role of waves in two-fluid cosmic-ray hydrodynamics has been investigated by means of linear perturbation theory and weakly nonlinear asymptotics. Weakly nonlinear, long wavelength cosmic-ray-modified shocks in the simplest models are governed by Burgers equation (Webb and McKenzie 1984) or modified Burgers equations (Zank *et al.* 1988; Chalov 1990), or by Burgers equations modified by wave diffraction for shocks that are weakly curved (Zank and Webb 1990; Webb and Zank 1993). Hall-current effects in the small-cosmic-ray-pressure limit lead to models involving the Korteweg–de Vries–Burgers equation (Zank 1988*a*; Webb and Zank 1993). Alfvénic models of cosmic-ray-modified shocks in which the Alfvén waves that scatter the cosmic rays may be generated in part by the cosmic-ray streaming instability (Lerche 1967; Skilling 1975) were developed by McKenzie and Völk (1982) and Völk *et al.* (1984). The cosmic-ray-generated Alfvén waves in these models were shown by McKenzie and Webb (1984), Zank (1989), Begelman and Zweibel (1994) and Ko and Jeng (1994) to drive one of the modified slow magnetoacoustic waves unstable.

Numerical time-dependent hydrodynamical codes were initially developed to study aspects of the two-fluid model applicable to supernova remnant shocks by Dorfi (1984, 1985) and Dorfi and Drury (1985). Dorfi and Drury (1985) and Drury and Falle (1986) investigated in detail a short-wavelength squeezing instability for sound waves in the two-fluid model with **k** vector directed towards the shock in the energetic particle upstream precursor that were amplified by the cosmic-ray pressure gradient (see also Zank and McKenzie 1987; Zank *et al.* 1993; Berezhko 1986; Chalov 1988; Webb 1989; Kang *et al.* 1992; Begelman and Zweibel 1994). Zank *et al.* (1990) provided a more general analysis of instabilities of obliquely propagating magnetosonic modes in the upstream precursor of a cosmic-ray-modified shock, and found that the wave modes could be destabilized by squeezing and stratification effects, and by particle drifts. The above analyses of short-wavelength instabilities upstream of cosmic-ray-modified shocks were carried out by means of perturbation expansions that assume the wavelength of the wave $\lambda \ll L$, where L is the deceleration length of the background flow. Most of the short wavelength instability analyses have been restricted to the linear wave regime. However, Zank (1988*b*) showed that the unstable sound wave satisfied a generalized Burgers equation in

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which the disturbance would eventually steepen into a shocklet (see also Zank *et al.* 1993; Begelman and Zweibel 1994).

The two-fluid model of Axford *et al.* (1977) and Drury and Völk (1981) in one Cartesian space dimension admits four wave modes. The character of the wave modes depends on the wavelength. At short wavelengths, the wave modes consist of the backward- and forward-propagating thermal gas sound waves, the contact discontinuity and a diffusively damped mode. On the other hand, at long wavelengths, the wave modes consist of the backward and forward sound waves modified by a cosmic-ray-pressure contribution to the sound speed; the contact discontinuity, and a pressure-balance mode in which the sum of the cosmic-ray and thermal gas-pressure perturbations is zero. An analysis of the interaction between the contact discontinuity and the pressure-balance mode in the long-wavelength limit, and the evolution of the short-wavelength contact discontinuity in a large-scale background flow, was carried out in Webb *et al.* (1995) in order to elucidate the physics of the cosmic-ray-modified contact discontinuity and pressure-balance structure observed by Donohue and Zank (1993) in their numerical two fluid simulation of the interaction of an interplanetary travelling shock with the solar-wind termination shock.

The main purpose of the present paper is to develop a system of equations describing weakly nonlinear waves for the two-fluid model in one space dimension similar to equations obtained by Majda and Rosales (1984) for resonantly interacting, weakly nonlinear waves in gasdynamics in one Cartesian space dimension. For the case of compressible fluid flow, Majda and Rosales derived a pair of inviscid Burgers equations (one for the forward sound wave and one for the backward sound wave) coupled through a linear integral operator with known kernel dependent on the initial data for the entropy wave. For the forward sound-wave Burgers equation, the linear integral operator describes the resonant reflection of the backward sound wave off an entropy-wave disturbance to generate a forward sound wave. Similarly, the linear integral operator in the backward sound-wave Burgers equation describes the resonant reflection of the forward sound wave off an entropy wave to produce a backward sound wave. The equations are prototypical in describing three-wave resonant interactions for weakly nonlinear hyperbolic waves in one space variable. Resonant wave interactions in this theory are more liable to be significant for extended periodic or near-periodic wave trains than for isolated pulses, since the resonant interactions are strengthened the longer the period of time over which the waves interact.

The Majda and Rosales theory is based on weakly nonlinear geometric optics, and applies both for perturbations about a uniform background state and also for perturbations about a non-uniform background state. For perturbations about a non-uniform state, further linear wave-coupling terms appear between the different wave modes analogous to the wave-mixing terms obtained by Heinemann and Olbert (1980) in their bidirectional model for Alfvén waves in the solar wind. The linear coupling terms in the Heinemann and Olbert theory describe the reflection of short-wavelength Alfvén waves off the large-scale gradients in the background flow (see also Zhou and Matthaeus 1990*a*, *b*, and references therein). Further developments in the Majda and Rosales theory are described by Anile *et al.* (1993). The non-trivial extension of the theory to several space dimensions has been developed by Hunter *et al.* (1986). The character of resonant wave interactions in non-uniform background flows in one space dimension is complicated

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by the fact that resonance is in general limited to a localized region of (x, t) space, since the ray equations for the waves depend on the variable background flow.

Section 2 introduces the equations of the two-fluid model. Section 3 considers the propagation of short-wavelength waves in a large-scale non-uniform flow. This includes a discussion of the conditions for three-wave resonance for hyperbolic waves applicable to the cosmic-ray hydrodynamical equations in the short- and long-wavelength limits. At lowest order, the perturbation analysis yields the eigenvalues and eigenvectors for short-wavelength waves, as well as the equations for the background flow. At the next order in the perturbation analysis, one obtains equations describing the interaction of the different wave modes via resonant and non-resonant processes, as well as wave interactions with the large-scale background flow (this includes both reflection of the sound waves off the large-scale gradients, as well as cosmic-ray squeezing instability terms, and the Ptuskin (1981) cosmic-ray diffusive damping of the sound waves). Averaging over the fast time scale yields integro-differential equations of the type obtained by Majda and Rosales (1984). Section 4 shows the connection between the wave equations developed in Sec. 3 and the equations of resonant nonlinear acoustics for periodic waves in gasdynamics developed by Majda and Rosales (1984) for a uniform background state. Section 5 considers the linearized version of the short-wavelength equations for the two-fluid model derived in Sec. 3. It is shown how the linearized equations may be combined to obtain a wave-action-type equation, or canonical wave-energy equation describing the interaction of the waves with the large-scale background flow. The canonical wave-energy equation generalizes the corresponding wave-action equation for unidirectional sound waves derived by Drury and Falle (1986) and Zank and McKenzie (1987) to the case where all three short-wavelength waves (the backward and forward sound waves and the contact discontinuity eigenmode) are simultaneously present in the flow. A WKB analysis of the linear wave equations is then carried out in order to illustrate the phenomenon of wave mixing. The analysis shows how the cosmic-ray squeezing instability is modified when all three short-wavelength waves interact via the large-scale flow. Lagrangian and Hamiltonian variational principles for the linear wave-mixing equations for short-wavelength waves in a steady, large-scale background flow that take into account the spatial instability growth rates of the waves due to the cosmic rays are derived in Sec. 6. Equations describing the resonant and non-resonant interaction of the four long-wavelength waves (the backward and forward cosmic-ray-modified sound waves, the contact discontinuity and the cosmic-ray pressure-balance mode) for a uniform background state are derived in Sec. 7. Section 8 concludes with a summary and discussion.

2. Model and equations

The two-fluid model for cosmic-ray-modified flows that we use is the same as that introduced by Axford *et al.* (1977) and Drury and Völk (1981) in their models of cosmic-ray-modified shocks. The cosmic rays are scattered by waves or turbulence travelling in the background flow, and the phase velocity of the waves is assumed to be negliglible compared with the fluid speed.

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The dynamics of the system is constrained by the overall mass, momentum and energy conservation equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot [\rho \mathbf{u}\mathbf{u} + (p_g + p_c)\mathbf{I}] = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho u^2 + E_g + E_c) + \nabla \cdot [(\tfrac{1}{2}\rho u^2 + E_g + p_g)\mathbf{u} + \mathbf{F}_c] = 0. \tag{2.3}$$

where

$$E_g = \frac{p_g}{\gamma_g - 1}, \qquad E_c = \frac{p_c}{\gamma_c - 1}. \tag{2.4}$$

In the above equations $\mathbf{I}$ denotes the unit 3×3 dyadic; ρ, $\mathbf{u}$, p_g, E_g and γ_g denote the thermal gas density, velocity, pressure, energy density and adiabatic index respectively; p_c, E_c, γ_c and $\mathbf{F}_c$ denote the cosmic-ray pressure, energy density, adiabatic index and energy flux. In the above model the cosmic rays are assumed to be a hot low-density gas with negligible mass density, mass flux and momentum density compared with those of the thermal gas.

The cosmic-ray energy equation:

$$\frac{\partial E_c}{\partial t} + \nabla \cdot \mathbf{F}_c = \mathbf{u} \cdot \nabla p_c \tag{2.5}$$

governs the transfer of energy between the cosmic rays and the background flow where the cosmic-ray energy flux is given by

$$\mathbf{F}_c = \mathbf{u}(E_c + p_c) - \mathbf{K} \cdot \nabla E_c, \tag{2.6}$$

and $\mathbf{K}$ is the mean hydrodynamical diffusion tensor (see e.g. Drury and Völk 1981). The diffusion tensor in general contains terms corresponding to diffusion parallel and perpendicular to the mean magnetic field, as well as terms describing particle drifts (see e.g. Forman *et al.* 1974).

For the case of purely one dimensional flow along the x axis of a rectangular Cartesian coordinate system (x, y, z), the above equations may be rearranged to yield the system of equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{2.7}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\left(\frac{\partial p_c}{\partial x} + \frac{\partial p_g}{\partial x}\right), \tag{2.8}$$

$$\frac{\partial p_g}{\partial t} + u\frac{\partial p_g}{\partial x} + \gamma_g p_g \frac{\partial u}{\partial x} = 0, \tag{2.9}$$

$$\frac{\partial p_c}{\partial t} + u\frac{\partial p_c}{\partial x} + \gamma_c p_c \frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\kappa \frac{\partial p_c}{\partial x}\right) = 0, \tag{2.10}$$

governing the two fluids, where κ denotes the effective diffusion coefficient in the x direction. We note for later reference that the fluid-frame energy equation (2.9) may also be expressed in the form

$$\frac{dp_g}{dt} - \frac{\gamma_g p_g}{\rho}\frac{d\rho}{dt} \equiv \rho^{\gamma_g}\frac{d}{dt}\left(\frac{p_g}{\rho^{\gamma_g}}\right) = 0, \tag{2.11}$$

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where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{2.12}$$

denotes the Lagrangian time derivative. Equation (2.11) shows that the thermal gas is compressed adiabatically, moving with the flow, except possibly at shocks, where p_g/ρ^{γ_g} may jump discontinuously.

3. Short-wavelength waves in non-uniform flows

In this section we apply the method of multiple scales to derive equations describing the resonant and non-resonant interactions of short-wavelength waves in the two-fluid cosmic-ray hydrodynamical model in a large-scale non-uniform background flow. We first discuss in Sec. 3.1 the conditions for three-wave resonance for hyperbolic waves with scale-invariant dispersion relations in one space dimension that are applicable to the two fluid model in the short- and long-wavelength limits. Further discussion of three-wave resonance conditions for hyperbolic waves is given in Majda and Rosales (1984), Hunter *et al.* (1986) and Anile *et al.* (1993). In Sec. 3.2 we write (2.7)–(2.12) in dimensionless form, and introduce the perturbation expansion. We then proceed with the multiple scales perturbation analysis, in which the perturbation parameter $\epsilon = L_2/L_1$ ($0 < \epsilon \ll 1$) is the ratio of the wavelength of the wave $\lambda \equiv L_2$ to the scale length L_1 of the background flow. At lowest order ($O(1)$), one obtains equations determining the eigenvalues and eigenvectors for short-wavelength waves. Imposing the constraint that at $O(\epsilon)$ the perturbation expansion is a uniform expansion for times $\tau = O(1/\epsilon)$, where τ is the fast time scale leads to a coupled set of integro-differential evolution equations governing the different wave modes (Secs 3.3 and 3.4). The equations obtained differ from the corresponding equations for non-dissipative compressible gas dynamics obtained by Majda and Rosales (1984) in that the equations contain cosmic-ray diffusive-damping terms as well as squeezing instability terms associated with the large-scale cosmic-ray pressure gradient. Cehelsky and Rosales (1986) have discussed the formation of shocks, weak solutions and shock fitting for the pure gasdynamical case.

3.1. Conditions for three-wave resonance

The waves in the two-fluid model in the asymptotic long-wavelength limit are dispersionless. Similarly, in the short-wavelength limit the two sound waves and the entropy wave are dispersionless. Hence in these limits the waves have scale-invariant dispersion relations

$$D(\omega, k) = 0, \qquad \text{or} \quad \omega = W(k). \tag{3.1}$$

For scale invariant dispersion relations, $W(\alpha k) = \alpha W(k)$, where α is an arbitrary scalar. Consider the conditions for resonant interaction of two waves with frequencies and wavenumbers (ω_p, k_p) and (ω_q, k_q) to produce a third wave with $(\omega, k) = (\omega_j, k_j)$, where each wave mode satisfies the dispersion relation (3.1), so that

$$D(\omega_p, k_p) = D(\omega_q, k_q) = D(\omega_j, k_j) = 0. \tag{3.2}$$

The condition for three-wave resonance is that there exist three scalars ν_{pq}, ν_{qj} and ν_{jp} such that

$$\nu_{pq}\theta_j + \nu_{qj}\theta_p + \nu_{jp}\theta_q = 0, \tag{3.3}$$

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where

$$\theta_j = k_j\xi - \omega_j\tau, \qquad \theta_p = k_p\xi - \omega_p\tau, \qquad \theta_q = k_q\xi - \omega_q\tau, \tag{3.4}$$

denote the wave phases. The corresponding wave phase speeds are

$$\lambda_j = \frac{\omega_j}{k_j}, \quad \lambda_p = \frac{\omega_p}{k_p}, \quad \lambda_q = \frac{\omega_q}{k_q}. \tag{3.5}$$

Equations (3.3) and (3.4) imply

$$\nu_{pq}k_j + \nu_{qj}k_p + \nu_{jp}k_q = \boldsymbol{\nu}\cdot\mathbf{k} = 0, \tag{3.6}$$

$$\nu_{pq}\omega_j + \nu_{qj}\omega_p + \nu_{jp}\omega_q = \boldsymbol{\nu}\cdot\boldsymbol{\omega} = 0, \tag{3.7}$$

where the vectors $\boldsymbol{\nu}$, $\mathbf{k}$ and $\boldsymbol{\omega}$ are defined by

$$\boldsymbol{\nu} = (\nu_{pq}, \nu_{qj}, \nu_{jp})^t, \quad \mathbf{k} = (k_j, k_p, k_q)^t, \quad \boldsymbol{\omega} = (\omega_j, \omega_p, \omega_q)^t \tag{3.8}$$

(the superscript t indicates the transpose). A non-trivial solution of (3.6) and (3.7) for $\boldsymbol{\nu}$ is given by

$$\boldsymbol{\nu} = \boldsymbol{\omega}\times\mathbf{k}. \tag{3.9}$$

In component form, the solution (3.9) for $\boldsymbol{\nu}$ is

$$\nu_{pq} = (\lambda_p - \lambda_q)k_pk_q, \qquad \nu_{qj} = (\lambda_q - \lambda_j)k_qk_j, \qquad \nu_{jp} = (\lambda_j - \lambda_p)k_jk_p, \tag{3.10}$$

for the scalars ν_{pq}, ν_{qj} and ν_{jp}. An alternative procedure to arrive at the results (3.10) is to assume that the three wave modes coexist at the same point (ξ, τ). Elimination of ξ and τ from (3.4) then leads to (3.3) where ν_{pq}, ν_{qj} and ν_{jp} are given by (3.10). From (3.7),

$$\omega_j = -\frac{\nu_{qj}}{\nu_{pq}}\omega_p - \frac{\nu_{jp}}{\nu_{pq}}\omega_q. \tag{3.11}$$

The condition for periodic waves with frequencies and wavenumbers (ω_p, k_p) and (ω_q, k_q) to resonantly interact is that the numbers ν_{qj}/ν_{pq} and ν_{jp}/ν_{pq} should be rational.

3.2. The perturbation expansion

To carry out the analysis, we first write the equations (2.7)–(2.10) governing the system in terms of the dimensionless variables

$$\left.\begin{aligned} \bar{x} &= \frac{x}{L_1}, \qquad \bar{t} = \frac{t}{T_1}, \qquad \bar{p}_c = \frac{p_c}{p_{g0}}, \\ \bar{p}_g &= \frac{p_g}{p_{g0}}, \qquad \bar{\rho} = \frac{\rho}{\rho_0}, \qquad \bar{u} = \frac{u}{V_{p0}}, \end{aligned}\right\} \tag{3.12}$$

and choose the length (L_1) and time (T_1) scales such that

$$\frac{V_{p0}T_1}{L_1} = 1. \tag{3.13}$$

Here p_{g0} and ρ_0 are constant characteristic values of the gas pressure and density, and V_{p0} is a characteristic wave phase speed of the system. From (2.7)–(2.10), the normalized state vector of the system

$$\boldsymbol{\Psi} = (\bar{\rho}, \bar{u}, \bar{p}_g, \bar{p}_c)^t \tag{3.14}$$

then satisfies the matrix equation

$$\frac{\partial\boldsymbol{\Psi}}{\partial\bar{t}} + \mathbf{A}(\boldsymbol{\Psi})\cdot\frac{\partial\boldsymbol{\Psi}}{\partial\bar{x}} + \mathbf{B}(\boldsymbol{\Psi})\cdot\frac{\partial^2\boldsymbol{\Psi}}{\partial\bar{x}^2} = 0, \tag{3.15}$$

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where

$$\mathbf{A}(\mathbf{\Psi}) = \begin{pmatrix} \bar{u} & \bar{\rho} & 0 & 0 \\ 0 & \bar{u} & \bar{a}_{g0}^2/\gamma_g\bar{\rho} & \bar{a}_{g0}^2/\gamma_g\bar{\rho} \\ 0 & \gamma_g\bar{p}_g & \bar{u} & 0 \\ 0 & \gamma_c\bar{p}_c & 0 & \bar{u} \end{pmatrix}, \tag{3.16}$$

and

$$\mathbf{B}(\mathbf{\Psi})^{ij} = -\bar{\kappa}\delta^{i4}\delta^{j4}, \tag{3.17}$$

where δ^{ij} denotes the Kronecker delta symbol. In the above equations

$$\left.\begin{aligned} \bar{a}_{g0}^2 &= \frac{a_{g0}^2}{V_{p0}^2}, \qquad \bar{a}_{c0}^2 = \frac{a_{c0}^2}{V_{p0}^2}, \qquad \bar{\kappa} = \frac{\kappa}{V_{p0}L_1}, \\ a_{g0}^2 &= \frac{\gamma_g p_{g0}}{\rho_0}, \qquad a_{c0}^2 = \frac{\gamma_c p_{c0}}{\rho_0} \end{aligned}\right\} \tag{3.18}$$

define dimensionless forms of the thermal gas (a_{g0}) and cosmic-ray (a_{c0}) sound speeds, and diffusion coefficient (κ).

Consider the evolution of weakly nonlinear short-wavelength waves of length scale L_2 and time scale T_2 in a large-scale background flow with length and time scales L_1 and T_1 where $L_1 \gg L_2$ and $T_1 \gg T_2$. We develop a weakly nonlinear geometrical-optics expansion for the waves in the perturbation parameter

$$\epsilon = \frac{L_2}{L_1} = \frac{T_2}{T_1} \qquad (0 < \epsilon \ll 1), \tag{3.19}$$

of the form

$$\mathbf{\Psi} = \mathbf{\Psi}^{(0)} + \epsilon\mathbf{\Psi}^{(1)} + \epsilon^2\mathbf{\Psi}^{(2)} + \ldots, \tag{3.20}$$

where the the background state $\mathbf{\Psi}^{(0)} = \mathbf{\Psi}^{(0)}(\bar{x}, \bar{t})$ depends only on the slow variables $(\bar{x}, \bar{t})$, whereas the wave perturbations $\{\mathbf{\Psi}^{(i)}(\bar{x}, \bar{t}; \xi, \tau) : i \geqslant 1\}$ depend both on the slow variables $\bar{x}$, $\bar{t}$ and on the fast variables:

$$\xi = \frac{\bar{x}}{\epsilon}, \qquad \tau = \frac{\bar{t}}{\epsilon}. \tag{3.21}$$

The normalized cosmic-ray diffusion coefficient $\bar{\kappa}$ in (3.17) and (3.18) is taken to be a constant, with $\bar{\kappa} = O(1)$. Using the derivative transformations

$$\frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial \bar{t}} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial \bar{x}} + \frac{1}{\epsilon}\frac{\partial}{\partial \xi} \tag{3.22}$$

and the perturbation expansion (3.20) in (3.15) yields the equations

$$\begin{aligned} &\left[\frac{\partial}{\partial \bar{t}} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau} + [\mathbf{A}_0 + (\epsilon\mathbf{\Psi}^{(1)} + \epsilon^2\mathbf{\Psi}^{(2)} + \ldots)\cdot\nabla_{\mathbf{\Psi}}\mathbf{A}\right. \\ &+\tfrac{1}{2}(\epsilon\mathbf{\Psi}^{(1)} + \epsilon^2\mathbf{\Psi}^{(2)} + \ldots)(\epsilon\mathbf{\Psi}^{(1)} + \epsilon^2\mathbf{\Psi}^{(2)} + \ldots) : \nabla_{\mathbf{\Psi}}\nabla_{\mathbf{\Psi}}\mathbf{A} + \ldots] \cdot \left(\frac{\partial}{\partial \bar{x}} + \frac{1}{\epsilon}\frac{\partial}{\partial \xi}\right) \\ &\left.+\mathbf{B}_0 \cdot \left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial \xi\,\partial \bar{x}} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \xi^2}\right)\right](\mathbf{\Psi}^{(0)} + \epsilon\mathbf{\Psi}^{(1)} + \epsilon^2\mathbf{\Psi}^{(2)} + \ldots) = 0. \end{aligned} \tag{3.23}$$

Equating terms at $O(1/\epsilon^2)$ in (3.23) yields the balance equation

$$\mathbf{B}_0 \cdot \frac{\partial^2\mathbf{\Psi}^{(0)}}{\partial \xi^2} = -\bar{\kappa}\frac{\partial^2 p_c^0}{\partial \xi^2}(0, 0, 0, 1)^t = 0. \tag{3.24}$$

Equation (3.24) is automatically satisfied, since $\mathbf{\Psi}^{(0)}$ and p_c^0 depend only on the

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slow variables $\bar{x}$ and $\bar{t}$. The $O(1/\epsilon)$ balance equation is

$$\mathbf{B}_0 \cdot \frac{\partial^2 \mathbf{\Psi}^{(1)}}{\partial \xi^2} = -\bar{\kappa} \frac{\partial^2 p_c^1}{\partial \xi^2} (0, 0, 0, 1)^t = 0. \tag{3.25}$$

For a uniform expansion, we require p_c^1 to be sublinear in ξ, so that

$$p_c^1 = 0 \tag{3.26}$$

is the appropriate solution of (3.25) for p_c^1. Balance of terms at $O(1)$ in (3.23) yields the equation

$$\begin{aligned} &\frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{t}} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{x}} + \mathbf{B}_0 \cdot \frac{\partial^2 \mathbf{\Psi}^{(0)}}{\partial \bar{x}^2} \\ &+ \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \tau} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \xi} + \mathbf{B}_0 \cdot \left(\frac{\partial^2 \mathbf{\Psi}^{(2)}}{\partial \xi^2} + 2 \frac{\partial^2 \mathbf{\Psi}^{(1)}}{\partial \xi \, \partial \bar{x}} \right) = 0. \end{aligned} \tag{3.27}$$

Equation (3.27) in turn splits into two equations, namely the equations for the non-uniform background state,

$$\frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{t}} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{x}} + \mathbf{B}_0 \cdot \frac{\partial^2 \mathbf{\Psi}^{(0)}}{\partial \bar{x}^2} = 0, \tag{3.28}$$

and the $O(1)$ balance for the wave perturbations,

$$\frac{\partial \mathbf{\Psi}^{(1)}}{\partial \tau} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \xi} + \mathbf{B}_0 \cdot \left(\frac{\partial^2 \mathbf{\Psi}^{(2)}}{\partial \xi^2} + 2 \frac{\partial^2 \mathbf{\Psi}^{(1)}}{\partial \xi \, \partial \bar{x}} \right) = 0. \tag{3.29}$$

Taking into account the form of the diffusion term in (3.15) and the choice $p_c^1 = 0$ for the first-order cosmic-ray pressure perturbation, and introducing the thermal gas state vector $\tilde{\mathbf{\Psi}} = (\bar{\rho}, \bar{u}, \bar{p}_g)$, the first three equations in (3.29) reduce to

$$\frac{\partial \tilde{\mathbf{\Psi}}^{(1)}}{\partial \tau} + \mathscr{A}_0 \cdot \frac{\partial \tilde{\mathbf{\Psi}}^{(1)}}{\partial \xi} = 0, \tag{3.30}$$

where the matrix $\mathscr{A}_0 = \mathscr{A}(\mathbf{\Psi}^{(0)})$ is defined by

$$\mathscr{A} = \begin{pmatrix} \bar{u} & \bar{\rho} & 0 \\ 0 & \bar{u} & 1/\gamma_g \bar{\rho} \\ 0 & \gamma_g \bar{p}_g & \bar{u} \end{pmatrix}, \tag{3.31}$$

and consists of the first three rows and columns of the matrix $\mathbf{A}$ in (3.16), where we have normalized the velocities by setting $\bar{a}_{g0}^2 = 1$. The fourth equation in (3.29),

$$\gamma_c p_c^0 \frac{\partial u^1}{\partial \xi} - \bar{\kappa}_0 \frac{\partial^2 p_c^2}{\partial \xi^2} = 0, \tag{3.32}$$

couples the second-order cosmic-ray perturbation p_c^2 to u^1.

Following the approach of Majda and Rosales (1984), the solutions for $\tilde{\mathbf{\Psi}}^{(1)}$ and $\tilde{\mathbf{\Psi}}^{(2)}$ are expanded in terms of the eigenvector solutions associated with the matrix $\mathscr{A}$ of the form

$$\tilde{\mathbf{\Psi}}^{(1)} = \sum_{s=1}^{3} a_s(\bar{x}, \bar{t}, \theta_s) \mathbf{R}_s, \tag{3.33}$$

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$$\tilde{\boldsymbol{\Psi}}^{(2)} = \sum_{s=1}^{3} b_s(\bar{x}, \bar{t}, \theta_s, \tau)\mathbf{R}_s, \tag{3.34}$$

where $\{a_s\}$ and $\{b_s\}$ determine the wave amplitudes; and

$$\theta_s = k_s\xi - \omega_s\tau, \qquad \lambda_s = \omega_s/k_s, \tag{3.35}$$

are the phase and phase speed of the sth wave mode. The eigenvectors $\mathbf{R}_s$ and eigenvalues λ_s are determined from the eigenequation

$$(\mathscr{A}_0 - \lambda_s\mathbf{I})\cdot\mathbf{R}_s = 0, \tag{3.36}$$

where the λ_s satisfy the eigenvalue equation:

$$\det(\mathscr{A}_0 - \lambda\mathbf{I}) = (\bar{u}-\lambda)[(\bar{u}-\lambda)^2 - \bar{p}_g/\bar{\rho}] = 0. \tag{3.37}$$

It is straightforward to verify that the eigenvector solution (3.33) satisfies (3.30).

The eigenvalues and eigenvectors are those of one dimensional compressible fluid dynamics, namely the forward and backward sound waves and contact discontinuity solutions. We also need the corresponding left eigenvectors $\{\mathbf{L}_s\}$ of the matrix $\mathscr{A}$, satisfying the equation

$$\mathbf{L}_s\cdot(\mathscr{A}_0 - \lambda_s\mathbf{I}) = 0. \tag{3.38}$$

The left eigenvectors $\{\mathbf{L}_s\}$ are normalized so that

$$\mathbf{L}_s\cdot\mathbf{R}_k = \delta_{sk}, \tag{3.39}$$

where δ_{sk} is the Kronecker delta symbol. The eigenvalue and eigenvector solutions correspond to the backward sound wave, with

$$\lambda_1 = \bar{u} - \bar{a}_g, \qquad \mathbf{R}_1 = \left(1, -\frac{\bar{a}_g}{\bar{\rho}}, \frac{\gamma_g\bar{p}_g}{\bar{\rho}}\right)^t, \qquad \mathbf{L}_1 = \frac{1}{2}\left(0, -\frac{\bar{\rho}}{\bar{a}_g}, \frac{1}{\gamma_g\bar{a}_g^2}\right), \tag{3.40}$$

the contact discontinuity, with

$$\lambda_2 = \bar{u}, \qquad \mathbf{R}_2 = (1,0,0)^t, \qquad \mathbf{L}_2 = \left(1, 0, -\frac{\bar{\rho}}{\gamma_g\bar{p}_g}\right), \tag{3.41}$$

and the forward sound wave, with

$$\lambda_3 = \bar{u} + \bar{a}_g, \qquad \mathbf{R}_3 = \left(1, \frac{\bar{a}_g}{\bar{\rho}}, \frac{\gamma_g\bar{p}_g}{\bar{\rho}}\right)^t, \qquad \mathbf{L}_3 = \frac{1}{2}\left(0, \frac{\bar{\rho}}{\bar{a}_g}, \frac{1}{\gamma_g\bar{a}_g^2}\right). \tag{3.42}$$

The wave amplitudes $\{a_s\}$ in the expansion (3.33) correspond to the wave density perturbations.

The $O(\epsilon)$ balance equations split naturally into the matrix equation

$$\frac{\partial\tilde{\boldsymbol{\Psi}}^{(2)}}{\partial\tau} + \mathscr{A}_0\cdot\frac{\partial\tilde{\boldsymbol{\Psi}}^{(2)}}{\partial\xi} + \mathbf{F} = 0, \tag{3.43}$$

where

$$\mathbf{F} = \frac{\partial\tilde{\boldsymbol{\Psi}}^{(1)}}{\partial\bar{t}} + \mathscr{A}_0\cdot\frac{\partial\tilde{\boldsymbol{\Psi}}^{(1)}}{\partial\bar{x}} + \mathbf{C}\frac{\partial p_c^2}{\partial\xi} + \tilde{\boldsymbol{\Psi}}^{(1)}\cdot\nabla_{\tilde{\boldsymbol{\Psi}}}\mathbf{C}\frac{\partial p_c^0}{\partial\bar{x}}$$
$$+\tilde{\boldsymbol{\Psi}}^{(1)}\cdot\nabla_{\tilde{\boldsymbol{\Psi}}}\mathscr{A}\cdot\left(\frac{\partial\tilde{\boldsymbol{\Psi}}^{(0)}}{\partial\bar{x}} + \frac{\partial\tilde{\boldsymbol{\Psi}}^{(1)}}{\partial\xi}\right), \tag{3.44}$$

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and the scalar equation

$$\frac{\partial p_c^2}{\partial \tau} + \mathbf{D}\cdot\left(\frac{\partial \boldsymbol{\Psi}^{(2)}}{\partial \xi} + \frac{\partial \boldsymbol{\Psi}^{(1)}}{\partial \bar{x}}\right) + \boldsymbol{\Psi}^{(1)}\cdot\nabla_{\boldsymbol{\Psi}}\mathbf{D}\cdot\left(\frac{\partial \boldsymbol{\Psi}^{(0)}}{\partial \bar{x}} + \frac{\partial \boldsymbol{\Psi}^{(1)}}{\partial \xi}\right) - \bar{\kappa}\left(2\frac{\partial^2 p_c^2}{\partial \xi\,\partial \bar{x}} + \frac{\partial^2 p_c^3}{\partial \xi^2}\right) = 0. \tag{3.45}$$

In (3.44) and (3.45) the vectors

$$\mathbf{C} = \left(0, \frac{1}{\gamma_g \bar{\rho}}, 0\right)^t, \qquad \mathbf{D} = (0, \gamma_c p_c^0, 0, u^0)^t \tag{3.46}$$

correspond to part of the third column and the fourth row of the matrix $\mathbf{A}$ in (3.16).

Substituting the solution ansatz (3.33) and (3.34) for $\tilde{\boldsymbol{\Psi}}^{(1)}$ and $\tilde{\boldsymbol{\Psi}}^{(2)}$ into (3.43) and multiplying (3.43) on the left by the left eigenvector $\mathbf{L}_j$ yields the system of equations

$$\frac{\partial b_j(\theta_j, \tau)}{\partial \tau_j} + \mathbf{L}_j \cdot \mathbf{F} = 0, \qquad j = 1, 2, 3, \tag{3.47}$$

where $\partial/\partial\tau_j \equiv (\partial/\partial\tau)|_{\theta_j}$. For the perturbation expansion (3.20) to remain uniform for times $\tau = O(1/\epsilon)$ requires that $\Psi^{(2)}$ be sublinear in τ, and this in turn requires that the expansion coefficients $b_j(\theta_j, \tau)$ for $\tilde{\Psi}^{(2)}$ be sublinear in τ. Thus we require

$$\lim_{\tau\to\infty} \frac{b_j(\theta_j, \tau)}{\tau} = 0 \tag{3.48}$$

for a uniform expansion. Integration of (3.47) with respect to τ, with θ_j held fixed then leads to a system of three evolution equations

$$\langle \mathbf{L}_j \cdot \mathbf{F} \rangle_j = 0, \quad j = 1, 2, 3, \tag{3.49}$$

governing the short wavelength wave modes on the long time scale. The averaging operator in (3.49) is defined by

$$\langle f \rangle_j = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f|_{\theta_j}\, d\tau' \tag{3.50}$$

(i.e. the average defined in (3.50) is to be carried out with θ_j held fixed). The averaged wave equations (3.49) are integro-differential evolution equations of the type obtained by Majda and Rosales (1984) describing three wave resonant interactions for weakly nonlinear hyperbolic waves.

3.3. Wave evolution equations prior to averaging

The evolution equations (3.47) prior to averaging may be written in the form

$$\frac{da_j}{dt_j} + \sum_{s=1}^{3} \Lambda_{js} a_s + T_j + \frac{\partial b_j(\theta_j, \tau)}{\partial \tau} = 0, \qquad j = 1, \ldots, 3, \tag{3.51}$$

where

$$\frac{d}{dt_j} = \frac{\partial}{\partial \bar{t}} + \lambda_j \frac{\partial}{\partial \bar{x}}, \tag{3.52}$$

is the total time derivative along the jth characteristic. The nonlinear interaction term in (3.51) is of the form:

$$T_j = \sum_{p=1}^{3}\sum_{q=1}^{3} \Gamma_{jpq} a_q \frac{\partial a_p}{\partial \xi}. \tag{3.53}$$

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The non-symmetric interaction coefficients Γ_{jpq} in (3.53) are given by

$$\Gamma_{jpq} = \mathbf{L}_j \cdot (\mathbf{R}_q \cdot \boldsymbol{\nabla}_{\boldsymbol{\Psi}} \mathscr{A} \cdot \mathbf{R}_p). \tag{3.54}$$

The interaction coefficients Λ_{js} in (3.51) are the linear wave-coupling or wave-mixing terms describing the interaction of the waves with the large-scale flow and density gradients. The Λ_{js} are analogous to the linear wave-mixing coefficients obtained by Heinemann and Olbert (1980) in their bidirectional model for Alfvén waves in the solar wind.

By differentiating the eigenequation (3.36) in the directions of the right eigenvectors, the asymmetric interaction coefficients (3.54) may be written in the form

$$\Gamma_{jpq} = \delta_{jp} \mathbf{R}_q \cdot \boldsymbol{\nabla}_{\boldsymbol{\Psi}} \lambda_p + (\lambda_p - \lambda_j) \mathbf{L}_j \cdot (\mathbf{R}_q \cdot \boldsymbol{\nabla}_{\boldsymbol{\Psi}} \mathbf{R}_p). \tag{3.55}$$

The asymmetric interaction coefficients describe collisions between the pth and qth wave modes to produce the jth wave mode. In particular, the coefficients

$$\Gamma_{jjj} = \mathbf{R}_j \cdot \boldsymbol{\nabla}_{\boldsymbol{\Psi}} \lambda_j, \qquad \Gamma_{jjq} = \mathbf{R}_q \cdot \boldsymbol{\nabla}_{\boldsymbol{\Psi}} \lambda_j, \qquad q \neq j, \tag{3.56}$$

describe the nonlinear interaction of the jth wave with itself, and with the qth mode ($q \neq j$). From the perspective of differential geometry, the vector space $V = \{\mathbf{R}_p : p = 1, 2, 3\}$, spanned by the right eigenvectors associated with the manifold $M = \{(\bar{\rho}, \bar{u}, \bar{p}_g)\}$ has metric $g_{pq} = \mathbf{R}_p \cdot \mathbf{R}_q$, and affine connection coefficients G^j_{pq} defined by

$$\mathbf{R}_q \cdot \boldsymbol{\nabla}_{\boldsymbol{\Psi}} \mathbf{R}_p = G^j_{pq} \mathbf{R}_j, \qquad G^j_{pq} = \mathbf{L}_j \cdot (\mathbf{R}_q \cdot \boldsymbol{\nabla}_{\boldsymbol{\Psi}} \mathbf{R}_p), \tag{3.57}$$

where the left eigenvectors $\{\mathbf{L}_j : j = 1, 2, 3\}$ span the dual vector space. Hence the asymmetric interaction coefficients Γ_{jpq} for $p \neq j$ are directly related to the affine connection coefficents G^j_{pq}. Since the right eigenvectors are an anholonomic base, the affine connection coefficients are non symmetric in their lower indices p and q. We later show ((3.73) *et seq.*) that the averages of the nonlinear interaction terms $\langle T_j \rangle_j$ over the fast time scale τ depend on the torsion tensor of the connection.

Computing the various terms in (3.51), we find that the evolution of the backward sound wave is governed by the equation

$$\frac{da_1}{dt_1} + \Lambda_{11} a_1 + \Lambda_{12} a_2 + \Lambda_{13} a_3 + T_1 + \frac{\partial b_1(\theta_1, \tau)}{\partial \tau} = 0, \tag{3.58}$$

where the linear interaction coefficients $\{\Lambda_{1s}\}$ are given by

$$\Lambda_{11} = \frac{1}{2}\left[\frac{d}{dt_1} \ln\left(\frac{\bar{a}_g^3}{\bar{\rho}}\right) + (\gamma_g + 1)\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\bar{a}_g}{\gamma_g \bar{p}_g}\frac{\partial \bar{p}_c}{\partial \bar{x}} + \frac{\bar{a}_c^2}{\bar{\kappa}}\right], \tag{3.59a}$$

$$\Lambda_{12} = \frac{\bar{a}_g}{2\gamma_g \bar{p}_g}\frac{\partial}{\partial \bar{x}}(\bar{p}_c + \bar{p}_g), \tag{3.59b}$$

$$\Lambda_{13} = \frac{1}{2}\left[\frac{d}{dt_1} \ln(\bar{a}_g \bar{\rho}) + (\gamma_g - 1)\frac{\partial \bar{u}}{\partial \bar{x}} + 2\frac{\bar{a}_g}{\gamma_g}\frac{\partial}{\partial \bar{x}} \ln \bar{p}_g + \frac{\bar{a}_g}{\gamma_g \bar{p}_g}\frac{\partial \bar{p}_c}{\partial \bar{x}} - \frac{\bar{a}_c^2}{\bar{\kappa}}\right], \tag{3.59c}$$

and the nonlinear interaction term is given by

$$T_1 = \frac{\bar{a}_g}{2\bar{\rho}}\left[a_2 \frac{\partial}{\partial \xi}(a_1 + a_3) + (\gamma_g + 1)\left(a_3 \frac{\partial a_3}{\partial \xi} - a_1 \frac{\partial a_1}{\partial \xi}\right) + (\gamma_g + 1) a_1 \frac{\partial a_3}{\partial \xi} - (\gamma_g - 3) a_3 \frac{\partial a_1}{\partial \xi}\right]. \tag{3.60}$$

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Similarly, the entropy wave is governed by the equation

$$\frac{da_2}{dt_2} + \Lambda_{21}a_1 + \Lambda_{22}a_2 + \Lambda_{23}a_3 + T_2 + \frac{\partial b_2(\theta_2,\tau)}{\partial\tau} = 0, \tag{3.61}$$

where

$$\Lambda_{21} = -\Lambda_{23} = \frac{\bar{a}_g}{\gamma_g}\frac{\partial}{\partial\bar{x}}\ln\left(\frac{\bar{p}_g}{\bar{\rho}^{\gamma_g}}\right), \tag{3.62a}$$

$$\Lambda_{22} = \frac{\partial\bar{u}}{\partial\bar{x}}, \tag{3.62b}$$

$$T_2 = \frac{\bar{a}_g}{\bar{\rho}}\left\{\frac{\partial}{\partial\xi}\left[a_2(a_3-a_1) + \frac{\gamma_g-1}{2}(a_1^2-a_3^2)\right] + (\gamma_g-1)\left(a_3\frac{\partial a_1}{\partial\xi} - a_1\frac{\partial a_3}{\partial\xi}\right)\right\}. \tag{3.62c}$$

It is of interest to note in (3.61) that the linear coupling coefficients Λ_{21} and Λ_{23} of the entropy wave (a_2) to the sound waves (a_1) and (a_3) depends on the large-scale gradient of the entropy of the background flow. The corresponding equation for the forward sound wave is

$$\frac{da_3}{dt_3} + \Lambda_{31}a_1 + \Lambda_{32}a_2 + \Lambda_{33}a_3 + T_3 + \frac{\partial b_3(\theta_3,\tau)}{\partial\tau} = 0, \tag{3.63}$$

where

$$\Lambda_{31} = \frac{1}{2}\left[\frac{d}{dt_3}\ln(\bar{a}_g\bar{\rho}) + (\gamma_g-1)\frac{\partial\bar{u}}{\partial\bar{x}} - 2\frac{\bar{a}_g}{\gamma_g}\frac{\partial}{\partial\bar{x}}\ln\bar{p}_g - \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial\bar{x}} - \frac{\bar{a}_c^2}{\bar{\kappa}}\right], \tag{3.64a}$$

$$\Lambda_{32} = -\frac{\bar{a}_g}{2\gamma_g\bar{p}_g}\frac{\partial}{\partial\bar{x}}(\bar{p}_c+\bar{p}_g), \tag{3.64b}$$

$$\Lambda_{33} = \frac{1}{2}\left[\frac{d}{dt_3}\ln\left(\frac{\bar{a}_g^3}{\bar{\rho}}\right) + (\gamma_g+1)\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial\bar{x}}\right], \tag{3.64c}$$

$$T_3 = \frac{\bar{a}_g}{2\bar{\rho}}\left[-a_2\frac{\partial}{\partial\xi}(a_1+a_3) + (\gamma_g+1)\left(a_3\frac{\partial a_3}{\partial\xi} - a_1\frac{\partial a_1}{\partial\xi}\right) - (\gamma_g+1)a_3\frac{\partial a_1}{\partial\xi} + (\gamma_g-3)a_1\frac{\partial a_3}{\partial\xi}\right]. \tag{3.64d}$$

Equations (3.58), (3.59) and (3.63), (3.64) show that there is a linear coupling of the sound waves to the entropy wave via the coefficients Λ_{12} and Λ_{32}, which are proportional to the large-scale pressure gradients of the cosmic-ray and thermal gases $-\nabla(\bar{p}_g+\bar{p}_c)$, which in turn is proportional to the acceleration vector $d\bar{u}/d\bar{t}$ of the background flow.

3.4. Averaged wave evolution equations

The averaged wave evolution equations (3.49) may be written in the form

$$\frac{da_j}{dt_j} + \Lambda_{jj}a_j + \sum_s{}' \Lambda_{js}\langle a_s\rangle + \langle T_j\rangle_j = 0, \qquad j = 1,2,3, \tag{3.65}$$

where the summation symbol $\sum'_s$ denotes summation over the index s for all modes except $s = j$. The mean wave amplitudes $\langle a_s\rangle$ depend only on the slow variables $\bar{x}$ and $\bar{t}$ (Appendix A). The mean nonlinear interaction term $\langle T_j\rangle_j$ can be expressed in both a non-conservative form

$$\langle T_j\rangle_j = \tilde{\Gamma}_{jjj}a_j\frac{\partial a_j}{\partial\theta_j} + \sum_q{}'\tilde{\Gamma}_{jjq}\langle a_q\rangle\frac{\partial a_j}{\partial\theta_j} + \sum_{p<q}{}'\sum_q{}'\bar{\Gamma}_{jpq}\langle a_q a_p'\rangle_j, \tag{3.66}$$

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and a conservative form

$$\langle T_j\rangle_j = \frac{\partial}{\partial\theta_j}\left(\tilde\Gamma_{jjj}\frac{a_j^2}{2} + \tilde\Gamma_{jjq}\langle a_q\rangle a_j + \sum_{p<q}{}'\sum_q{}'\hat\Gamma_{jpq}\langle a_q a_p\rangle_j\right), \tag{3.67}$$

where the interaction coefficients $\tilde\Gamma_{jpq}$, $\bar\Gamma_{jpq}$ and $\hat\Gamma_{jpq}$ are given by

$$\tilde\Gamma_{jpq} = k_p\Gamma_{jpq}, \tag{3.68a}$$

$$\bar\Gamma_{jpq} = \tilde\Gamma_{jpq} - \frac{\nu_{jp}}{\nu_{jq}}\tilde\Gamma_{jqp} \equiv k_p(\lambda_p-\lambda_j)\mathbf{L}_j\cdot[\mathbf{R}_q,\mathbf{R}_p], \tag{3.68b}$$

$$\hat\Gamma_{jpq} = \frac{\nu_{qj}\tilde\Gamma_{jpq}-\nu_{pj}\tilde\Gamma_{jqp}}{\nu_{qp}} \equiv \frac{k_j(\lambda_q-\lambda_j)(\lambda_p-\lambda_j)}{\lambda_q-\lambda_p}\mathbf{L}_j\cdot[\mathbf{R}_q,\mathbf{R}_p], \tag{3.68c}$$

and

$$[\mathbf{R}_q,\mathbf{R}_p] = \mathbf{R}_q\cdot\boldsymbol{\nabla}_{\boldsymbol{\Psi}}\mathbf{R}_p - \mathbf{R}_p\cdot\boldsymbol{\nabla}_{\boldsymbol{\Psi}}\mathbf{R}_q. \tag{3.69}$$

In (3.66) and (3.67) we use the notation $a_p' = \partial a_p(\theta_p)/\partial\theta_p$, and assume that $\lambda_p \neq \lambda_q$.

The means $\langle a_q a_p'\rangle_j$ in (3.66) are defined by

$$\begin{aligned}\langle a_q a_p'\rangle_j &= \lim_{T\to\infty}\frac{1}{T}\int_0^T a_q\left(\frac{k_q\theta_j+\nu_{jq}\tau}{k_j}\right)a_p'\left(\frac{k_p\theta_j+\nu_{jp}\tau}{k_j}\right)d\tau\\ &\equiv \lim_{|T_{jp}|\to\infty}\frac{1}{T_{jp}}\int_{\alpha_{jp}}^{\alpha_{jp}+T_{jp}} a_q\left(\frac{\nu_{qp}\theta_j+\nu_{jq}\theta_p}{\nu_{jp}}\right)a_p'(\theta_p)\,d\theta_p,\end{aligned} \tag{3.70}$$

where

$$T_{jp} = \frac{T\nu_{jp}}{k_j}, \qquad \alpha_{jp} = \frac{k_p\theta_j}{k_j}. \tag{3.71}$$

The means $\langle a_q a_p\rangle$ are defined similarly, but with a_p' replaced by a_p in (3.70). The double summations over p and q in (3.66) and (3.67) are over all wave modes p and q with $p<q$ and $p\neq q\neq j$.

In (3.70) the phase θ_q of a_q is given by

$$\theta_q = \frac{\nu_{qp}\theta_j+\nu_{jq}\theta_p}{\nu_{jp}}, \tag{3.72}$$

which is precisely the condition for three-wave resonance discussed in Sec. 3.1. The results (3.66) and (3.67) for $\langle T_j\rangle_j$ show that the nonlinear wave–wave interactions split into three types, namely the interaction of the jth-wave with itself, the interaction of the jth wave with the mean wave field of the other waves (the $\tilde\Gamma_{jjq}$ term), and the resonant interaction between wave modes p and q to produce the jth-mode, where $p\neq q\neq j$. The conservative wave–wave interaction coefficient $\hat\Gamma_{jpq}$ is symmetric in the indices p and q, and depends on the torsion tensor

$$T^j_{pq} = \mathbf{L}_j\cdot[\mathbf{R}_q,\mathbf{R}_p] = G^j_{pq} - G^j_{qp}, \tag{3.73}$$

associated with the affine connection G^j_{pq}. The averaged wave equations (3.65) are integro-differential equations governing the evolution of the jth-wave mode of the type obtained by Majda and Rosales (1984).

For the two-fluid cosmic-ray model, the nonlinear wave–wave interaction term for the backward sound wave $\langle T_1\rangle_1$ may be expressed in both the non-conservative

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and conservative forms

$$\langle T_1\rangle_1 = \frac{\bar{a}_g}{2\bar{\rho}}k_1\frac{\partial}{\partial\theta_1}\left(-\frac{\gamma_g+1}{2}a_1^2 + a_1[\langle a_2\rangle - (\gamma_g-3)\langle a_3\rangle]\right) - \frac{\bar{a}_g}{4\bar{\rho}}k_2\langle a_3 a_2'\rangle_1$$
$$\equiv \frac{\bar{a}_g}{2\bar{\rho}}k_1\frac{\partial}{\partial\theta_1}\left(-\frac{\gamma_g+1}{2}a_1^2 + a_1[\langle a_2\rangle - (\gamma_g-3)\langle a_3\rangle] - \langle a_2 a_3\rangle_1\right). \qquad (3.74)$$

Similarly, for the entropy mode,

$$\langle T_2\rangle_2 = \frac{\bar{a}_g}{\bar{\rho}}k_2\frac{\partial}{\partial\theta_2}[a_2(\langle a_3\rangle - \langle a_1\rangle)], \qquad (3.75)$$

and the nonlinear wave–wave interaction term for the forward sound wave is

$$\langle T_3\rangle_3 = \frac{\bar{a}_g}{2\bar{\rho}}k_3\frac{\partial}{\partial\theta_3}\left(\frac{(\gamma_g+1)}{2}a_3^2 + a_3[(\gamma_g-3)\langle a_1\rangle - \langle a_2\rangle]\right) + \frac{\bar{a}_g}{4\bar{\rho}}k_2\langle a_1 a'_2\rangle_3$$
$$\equiv \frac{\bar{a}_g}{2\bar{\rho}}k_3\frac{\partial}{\partial\theta_3}\left(\frac{(\gamma_g+1)}{2}a_3^2 + a_3[(\gamma_g-3)\langle a_1\rangle - \langle a_2\rangle] + \langle a_2 a_1\rangle_3\right). \qquad (3.76)$$

Equations (3.74)–(3.76) show that the forward sound wave may be generated by resonant reflection of the backward sound wave off the entropy wave. Similarly, resonant reflection of the forward sound wave off the entropy wave produces a backward sound wave. The entropy wave on the other hand can only be generated by non-resonant interaction of the entropy wave with the mean wave field of the sound waves. The nonlinear wave–wave interaction terms (3.74)–(3.76) are in fact just those of ordinary compressible gasdynamics in the absence of cosmic rays. The mean wave equations (3.65) differ from those of ordinary gasdynamics in the form of the linear wave-mixing coefficients Λ_{js}.

The above formal derivation of (3.65) for wave–wave interactions does not address the important issue of the conditions under which three-wave resonant interactions will occur in a non-uniform background flow. This issue has been addressed by Almgren (1990) and Joly *et al.* (1993) for hyperbolic waves for semilinear and quasilinear equations. For the sake of argument, suppose that we have three waves in resonance with frequencies ω_1, ω_2 and ω_3 with corresponding wave numbers $k_1(\bar{x},\bar{t})$, $k_2(\bar{x},\bar{t})$ and $k_3(\bar{x},\bar{t})$ satisfying the resonance conditions

$$\omega_3 = \omega_2 - \omega_1, \qquad k_3 = k_2 - k_1, \qquad (3.77)$$

where $\omega_1 = k_1(\bar{u} - \bar{a}_g)$, $\omega_2 = k_2\bar{u}$ and $\omega_3 = k_3(\bar{u} + \bar{a}_g)$ correspond to the backward sound wave, the contact discontinuity and the forward sound wave. Using the dispersion equations for the waves, (3.77) imply

$$k_2 = 2k_1, \qquad k_3 = k_1 \qquad (3.78)$$

at resonance. Equations (3.77) and (3.78) can in general only be satisfied in a localized region of $(\bar{x},\bar{t})$ space, since the k vectors of the waves evolve along the rays (Appendix B). This implies that waves in resonance near $(\bar{x},\bar{t}) = (\bar{x}_0,\bar{t}_0)$ will in general move out of resonance as the waves progress. If the gradients in the background flow are sufficiently weak then the waves can remain in resonance over a larger region of $(\bar{x},\bar{t})$ space, whereas if the gradients are steeper then resonance will occur over a relatively localized region. The $(\bar{x},\bar{t})$ dependence of the background flow may lead to non-uniformity of the perturbation expansion (3.20) in which higher-order terms in the perturbation expansion become comparable to the lower-order terms. Similar problems concerning resonant wave interactions of dispersive

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internal gravity waves in a stratified shear flow have been considered by Grimshaw (1988), who considered the the role of resonant wave interactions for gravity waves near a critical level where the waves are reflected. We expect that in the general application of (3.65) to the problem of the propagation of short-wavelength waves in cosmic-ray modified shocks the resonant wave interaction terms in (3.65) and (3.74)–(3.76) can safely be neglected in most instances. In this case the waves are still subject to the Burgers self-wave interaction process; the waves also interact with the mean wave field of the other waves, and the waves undergo wave mixing or wave reflection with the non-uniform background flow.

The three-wave resonant interaction is expected to play an important role if the background state in (3.65) is uniform. The analysis in Sec. 7 of long-wavelength waves in weak cosmic-ray-modified shocks indicates that the three-wave resonant interaction is important in weak, long-wavelength, diffusively smoothed cosmic-ray shocks.

4. Resonant nonlinear acoustics

In this section we discuss the relation of the wave evolution equations (3.65) to equations obtained by Majda and Rosales (1984) (see also Anile *et al.* 1993) for resonant nonlinear acoustics in ideal gasdynamics.

Consider the form of (3.65) for perturbations about a uniform background state

$$\rho = \rho_0, \quad u = 0, \quad p_g = p_{g0}, \tag{4.1}$$

for the case of ideal gasdynamics in which all cosmic-ray effects are ignored. The wave amplitudes $\{a_j \ : \ j = 1, 2, 3\}$, which correspond to the density perturbations of the backward sound wave, the contact discontinuity and the forward sound wave, are assumed to have zero means ($\langle a_j \rangle = 0$, $j = 1, 2, 3$). Since the background state is uniform, the linear wave mixing coefficients Λ_{js} in (3.65) are zero, and

$$\bar{\rho} = 1, \qquad \bar{u} = 0, \qquad \bar{p}_g = 1, \qquad \bar{a}_g = 1, \tag{4.2}$$

are the normalized density, fluid velocity, pressure and sound speed of the gas.

Substitution of the nonlinear wave interaction terms (3.74)–(3.76) into (3.65) yields wave evolution equations for the sound waves and contact discontinuity in the form

$$\frac{\partial a_1}{\partial \bar{t}} - \frac{\partial a_1}{\partial \bar{x}} - k_1 \frac{\gamma_g + 1}{2} a_1 \frac{\partial a_1}{\partial \theta_1} - \frac{k_2}{4} \left\langle a_3 \frac{\partial a_2}{\partial \theta_2} \right\rangle_1 = 0, \tag{4.3}$$

$$\frac{\partial a_2}{\partial \bar{t}} = 0, \tag{4.4}$$

$$\frac{\partial a_3}{\partial \bar{t}} + \frac{\partial a_3}{\partial \bar{x}} + k_3 \frac{\gamma_g + 1}{2} a_3 \frac{\partial a_3}{\partial \theta_3} + \frac{k_2}{4} \left\langle a_1 \frac{\partial a_2}{\partial \theta_2} \right\rangle_3 = 0. \tag{4.5}$$

Equations (4.3)–(4.5) can also be written in conservative form by using the results

$$-\frac{k_2}{4} \left\langle a_3 \frac{\partial a_2}{\partial \theta_2} \right\rangle_1 = -\frac{k_1}{2} \frac{\partial}{\partial \theta_1} (\langle a_3 a_2 \rangle_1), \qquad \frac{k_2}{4} \left\langle a_1 \frac{\partial a_2}{\partial \theta_2} \right\rangle_3 = \frac{k_3}{2} \frac{\partial}{\partial \theta_3} (\langle a_1 a_2 \rangle_3), \tag{4.6}$$

for the three-wave resonant interaction terms.

Since $\bar{a}_g = 1$, the conditions (3.3) for three wave resonant interactions and the formulae (3.10) for the coefficients ν_{pq} yield

$$\nu_{12} = -k_1 k_2, \quad \nu_{23} = -k_2 k_3, \qquad \nu_{31} = 2 k_1 k_3, \tag{4.7}$$

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and the relationship between the wave phases reduces to

$$\theta_2 = \frac{k_2}{2k_1}\theta_1 + \frac{k_2}{2k_3}\theta_3. \tag{4.8}$$

Using the velocity perturbation amplitudes for the sound waves, $v_1 \equiv -a_1$ and $v_3 \equiv a_3$, (4.3) and (4.5) for the sound waves reduce to

$$\begin{aligned}
&\frac{\partial v_1}{\partial \bar{t}} - \frac{\partial v_1}{\partial \bar{x}} + k_1 \frac{\gamma_g + 1}{2} v_1 \frac{\partial v_1}{\partial \theta_1} \\
&+ \frac{k_2}{4} \lim_{|T_{13}|\to\infty} \frac{1}{T_{13}} \int_{\alpha_{13}}^{\alpha_{13}+T_{13}} v_3(\theta_3, \bar{x}, \bar{t}) a_2' \left(\frac{k_2}{2k_1}\theta_1 + \frac{k_2}{2k_3}\theta_3, \bar{x} \right) d\theta_3 = 0, \qquad (4.9) \\
&\frac{\partial v_3}{\partial \bar{t}} + \frac{\partial v_3}{\partial \bar{x}} + k_3 \frac{\gamma_g + 1}{2} v_3 \frac{\partial v_3}{\partial \theta_3} \\
&- \frac{k_2}{4} \lim_{|T_{31}|\to\infty} \frac{1}{T_{31}} \int_{\alpha_{31}}^{\alpha_{31}+T_{31}} v_1(\theta_1, \bar{x}, \bar{t}) a_2' \left(\frac{k_2}{2k_1}\theta_1 + \frac{k_2}{2k_3}\theta_3, \bar{x} \right) d\theta_1 = 0, \qquad (4.10)
\end{aligned}$$

where

$$\alpha_{13} = \frac{k_3\theta_1}{k_1}, \qquad \alpha_{31} = \frac{k_1\theta_3}{k_3}, \qquad T_{13} = -2k_3T, \qquad T_{31} = 2k_1T. \tag{4.11}$$

Note from (4.4) that the density perturbation $a_2(\theta_2, \bar{x})$ is independent of $\bar{t}$. For finite θ_3 and θ_1, α_{13} and α_{31} may be replaced by zero without affecting the limiting averages in (4.9) and (4.10). Below, we consider the form of the asymptotic equations for the case where v_1, a_2 and v_3 are 2π-periodic functions of θ_1, θ_2 and θ_3 respectively.

4.1. The case of periodic waves

If the wave amplitudes $\{a_j\}$ are periodic functions of the θ_j then the limiting integral in (4.9) vanishes unless $k_2/2k_3$ is a rational number. Similarly, the resonant integral in (4.10) vanishes unless $k_2/2k_1$ is rational. Thus the resonance conditions for periodic waves require

$$k_2 = nk_1 + mk_3, \quad \omega_2 = n\omega_1 + m\omega_3, \tag{4.12}$$

where

$$\omega_2 = 0, \quad \omega_1 = -k_1, \quad \omega_3 = k_3, \tag{4.13}$$

are the wave dispersion equations. Equations (4.12) and (4.13) imply

$$\frac{k_2}{2k_1} = n, \qquad \frac{k_2}{2k_3} = m, \qquad \theta_2 = n\theta_1 + m\theta_3 \tag{4.14}$$

for resonance of periodic waves.

Setting

$$w_1 = \tfrac{1}{2}(\gamma_g + 1)k_1 v_1, \qquad w_3 = \tfrac{1}{2}(\gamma_g + 1)k_3 v_3, \tag{4.15}$$

for the sound wave amplitudes, and

$$K(\theta_2, \bar{x}) = \frac{k_2}{4} \frac{\partial a_2(\theta_2, \bar{x})}{\partial \theta_2}, \tag{4.16}$$

and exploiting the 2π-periodicity of the wave amplitudes as functions of the θ_j,

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(4.9) and (4.10) reduce to the forms

$$\frac{\partial w_1}{\partial \bar{t}} - \frac{\partial w_1}{\partial \bar{x}} + w_1 \frac{\partial w_1}{\partial \theta} + \frac{m}{n}\frac{1}{2\pi}\int_0^{2\pi} K(m\theta_3 + n\theta, \bar{x}) w_3(\theta_3, \bar{x}, \bar{t})\, d\theta_3 = 0, \quad (4.17)$$

$$\frac{\partial w_3}{\partial \bar{t}} + \frac{\partial w_3}{\partial \bar{x}} + w_3 \frac{\partial w_3}{\partial \theta} - \frac{n}{m}\frac{1}{2\pi}\int_0^{2\pi} K(m\theta + n\theta_1, \bar{x}) w_1(\theta_1, \bar{x}, \bar{t})\, d\theta_1 = 0. \quad (4.18)$$

In (4.17) we write $w_1 = w_1(\theta, \bar{x}, \bar{t})$, where θ is understood to be the phase θ_1. Similarly, in (4.18) $w_3 \equiv w_3(\theta, \bar{x}, \bar{t})$, and θ stands for the phase θ_3. The forms of (4.17) and (4.18) studied in detail by Majda and Rosales (1984), Majda *et al.* (1988), and Pego (1988) correspond to the case of solutions that are independent of $\bar{x}$, and $m = 1$ and $n = -1$. In this case the wavelength of the sound waves is twice that of the entropy wave. Intuitively, one expects the resonant interactions to be strongest for the lower order harmonic interactions.

5. Linear waves in non-uniform large-scale flows

It is expected that the mean wave equations (3.65) under some circumstances will develop shocks. The procedure to fit shocks in the solutions of (3.65) has been investigated by Cehelsky and Rosales (1986) (see also Anile *et al.* 1993). In this section we consider linearized versions of the equations derived in Sec. 3 describing the evolution of linear waves in non-uniform flows. We assume that it is not necessary to fit shocks in the solutions. In the cases where it is necessary to fit shocks (in the weak sense), one expects on thermodynamical grounds an increase of entropy across the shock, and it is then not valid to use a linear wave theory.

In regions of the flow where no shocks develop, the jth wave equation (3.65) may be averaged over the fast phase variable θ_j to obtain the linear wave evolution equations:

$$\frac{d\langle a_j\rangle}{dt_j} + \Lambda_{jj}\langle a_j\rangle + \sum_s{}' \Lambda_{js}\langle a_s\rangle = 0, \quad (5.1)$$

Since the nonlinear interaction terms have the conservative form (3.67), they average to zero in (5.1) if the waves are quasiperiodic in θ_j.

Alternatively, one arrives at a linear wave perspective if the nonlinear terms T_j and higher-order terms $\partial b_j(\theta_j, \tau)/\partial\tau$ are dropped in (3.51). The non-averaged linear wave equations obtained in this fashion are of the form

$$\frac{da_j}{dt_j} + \Lambda_{jj} a_j + \sum_s{}' \Lambda_{js} a_s = Q_j, \quad (5.2)$$

where the nonlinear source terms Q_j are set equal to zero. Equations (5.2) are precisely of the same form as (5.1), except the mean wave amplitudes $\langle a_j\rangle$ are replaced by their un-averaged values a_j, which depend not only on the slow variables $\bar{x}$ and $\bar{t}$, but also on the fast phase variables θ_j.

It is of interest to speculate on the relation of equations of the form (5.1) and (5.2), and the phenomenon of wave mixing of Alfvén waves (Heinemann and Olbert (1980)), and models of Alfvénic turbulence in the solar wind (see e.g. Matthaeus and Goldstein, 1982; Zhou and Matthaeus, 1989, 1990*a*, *b*; Tu and Marsch 1990; Matthaeus *et al.* 1994). One of the problems that afflicts turbulence theory in general is the moment closure problem. Thus, in using a mean-field-electrodynamics

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approach to incompressible Alfvénic turbulence in the solar wind (see e.g. Zhou and Matthaeus 1989), equations for the turbulent fluctuations

$$\delta a_j = a_j - \langle a_j \rangle, \tag{5.3}$$

are derived that are of the same form as (5.2) with non-zero source terms Q_j on the right-hand side. The source term represents small-scale turbulent interactions involving quadratic correlations of the fluctuations. The manner in which the Q_j are modelled is related to the closure problem.

Our main concern in the present section is a study of the waves described by (5.1) and (5.2) with $Q_j = 0$ on the macroscopic scales characteristic of the slow variables $\bar{x}$ and $\bar{t}$. It turns out, for example, that the solution for the forward sound wave generates, via wave reflection with the large-scale flow, backward-propagating short-wavelength sound waves (however, on the long scale, these latter waves propagate in the forward direction). This phenomenon is known as wave mixing in the space physics community (see e.g. Heinemann and Olbert 1980; Matthaeus *et al.* 1994). Similar wave mixing phenomena are also associated with the entropy-wave solution. The effect of nonlinear interactions (e.g. the Burgers self-interaction term and the three-wave resonant-interaction terms) are important if the wave amplitudes become sufficiently large, leading to substantial modifications of (5.1). We ignore these complications in the present development.

For the non-averaged evolution equations (5.2), it is of interest to combine the equations to determine physical wave-energy equations and canonical wave-energy equations (or generalized wave-action equations; see e.g. Dewar 1970). The wave-action densities for the backward and forward sound waves are defined as

$$\mathscr{A}_1 = \frac{\bar{\rho}\langle v_1^2 \rangle}{\omega'_1}, \qquad \mathscr{A}_3 = \frac{\bar{\rho}\langle v_3^2 \rangle}{\omega'_3}, \tag{5.4}$$

where

$$\omega'_1 = -k_1 \bar{a}_g, \qquad \omega'_3 = k_3 \bar{a}_g \tag{5.5}$$

denote the wave frequencies in the fluid frame, and v_1 and v_3 denote the velocity perturbations of the sound waves. In determining equations for $\mathscr{A}_1$ and $\mathscr{A}_3$, one encounters the question of wave correlations since the wave mixing terms lead to moments of the form $\langle v_1 v_3 \rangle$, $\langle v_1 a_2 \rangle$ and $\langle v_3 a_2 \rangle$, which describe correlations between the different wave modes.

In Sec. 5.1 we provide a derivation of the canonical wave-energy equation (or generalized wave-action equation) for (5.2) with $Q_j = 0$. In Sec. 5.1 we also discuss the relation of (5.1) and (5.2) to the unidirectional JWKB equations used to describe the cosmic-ray squeezing instability investigated by Dorfi and Drury (1985), Drury and Falle (1986) and Zank and McKenzie (1987).

In Sec. 5.2 a WKB analysis of (5.1) and (5.2) is used to demonstrate the phenomenon of wave mixing. The WKB analysis describes the evolution of the waves at frequencies ω in the regime: $u/L \ll \omega \ll \Omega$, where Ω is a frequency characteristic of the fast phase variables θ_j, and L/u is a characteristic time scale of the background flow. Since

$$\frac{\partial a_j}{\partial \tau} + \lambda_j \frac{\partial a_j}{\partial \xi} = 0, \tag{5.6}$$

the analysis should also hold for wave with frequencies $\omega = O(\Omega)$ in (5.2). Two specific problems are addressed:

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(*a*) the interaction of backward and forward sound waves with the non-uniform background flow in the absence of entropy-wave disturbances;

(*b*) the generation of sound waves via wave mixing from entropy waves.

In a convenient abuse of notation, the variables x and t in the following analysis refer to $\bar{x}$ and $\bar{t}$ respectively.

5.1. The canonical wave-energy equation and wave-action equations

To obtain the wave-action equation, we first note from the form of the right eigenvectors $\mathbf{R}_1$ and $\mathbf{R}_3$ that

$$v_1 = -\frac{\bar{a}_g}{\bar{\rho}} a_1, \qquad v_3 = \frac{\bar{a}_g}{\bar{\rho}} a_3 \tag{5.7}$$

are the velocity perturbations for the sound waves. In terms of the sound-wave velocity perturbations v_1 and v_3, and the entropy-wave density perturbation a_2, (5.2) may be written as

$$\frac{dv_1}{dt_1} + \chi_{11} v_1 + \chi_{12} a_2 + \chi_{13} v_3 = 0, \tag{5.8}$$

$$\frac{da_2}{dt_2} + \frac{\partial \bar{u}}{\partial x} a_2 - \frac{\bar{\rho}}{\gamma_g} \frac{\partial}{\partial x} \ln\left(\frac{\bar{p}_g}{\bar{\rho}^{\gamma_g}}\right)(v_1 + v_3) = 0, \tag{5.9}$$

$$\frac{dv_3}{dt_3} + \chi_{31} v_1 + \chi_{32} a_2 + \chi_{33} v_3 = 0, \tag{5.10}$$

where

$$\chi_{11} = \frac{1}{2}\left(-\frac{d}{dt_3} \ln(\bar{a}_g \bar{\rho}) + \frac{\bar{a}_c^2}{\bar{\kappa}} + \frac{\bar{a}_g}{\gamma_g \bar{p}_g} \frac{\partial \bar{p}_c}{\partial x}\right), \tag{5.11a}$$

$$\chi_{12} = -\frac{1}{2\gamma_g \bar{\rho}^2} \frac{\partial}{\partial x}(\bar{p}_c + \bar{p}_g), \tag{5.11b}$$

$$\chi_{13} = \frac{1}{2}\left[\frac{d}{dt_3} \ln\left(\frac{\bar{a}_g}{\bar{\rho}}\right) - 2\frac{\bar{a}_g}{\gamma_g} \frac{\partial}{\partial x} \ln\left(\frac{\bar{p}_g}{\bar{\rho}^{\gamma_g}}\right) + \frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{a}_g}{\gamma_g \bar{p}_g} \frac{\partial \bar{p}_c}{\partial x}\right], \tag{5.11c}$$

and

$$\chi_{31} = \frac{1}{2}\left[\frac{d}{dt_1} \ln\left(\frac{\bar{a}_g}{\bar{\rho}}\right) + 2\frac{\bar{a}_g}{\gamma_g} \frac{\partial}{\partial x} \ln\left(\frac{\bar{p}_g}{\bar{\rho}^{\gamma_g}}\right) + \frac{\bar{a}_c^2}{\bar{\kappa}} + \frac{\bar{a}_g}{\gamma_g \bar{p}_g} \frac{\partial \bar{p}_c}{\partial x}\right], \tag{5.12a}$$

$$\chi_{32} = -\frac{1}{2\gamma_g \bar{\rho}^2} \frac{\partial}{\partial x}(\bar{p}_c + \bar{p}_g), \tag{5.12b}$$

$$\chi_{33} = \frac{1}{2}\left(-\frac{d}{dt_1} \ln(\bar{a}_g \bar{\rho}) + \frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{a}_g}{\gamma_g \bar{p}_g} \frac{\partial \bar{p}_c}{\partial x}\right). \tag{5.12c}$$

Following the approach of Heinemann and Olbert (1980), we introduce new variables

$$f = \frac{(\bar{u} - \bar{a}_g) v_1}{(\bar{a}_g/\bar{\rho})^{1/2}}, \qquad g = \frac{(\bar{u} + \bar{a}_g) v_3}{(\bar{a}_g/\bar{\rho})^{1/2}} \tag{5.13}$$

to replace v_1 and v_3. For steady background flows, the gas-entropy equation reduces to

$$\frac{\partial}{\partial x} \ln\left(\frac{\bar{p}_g}{\bar{\rho}^{\gamma_g}}\right) = 0, \tag{5.14}$$

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and the entropy-wave equation (5.9) decouples from the sound-wave equations to yield the continuity equation

$$\frac{\partial a_2}{\partial t} + \frac{\partial}{\partial x}(\bar{u}a_2) = 0, \tag{5.15}$$

expressing the fact that for steady flows a_2 is advected with the flow. In terms of the variables f and g, the backward sound-wave equation (5.8) for steady background flows may be written in the form

$$\begin{aligned}\frac{\partial f}{\partial t} + (\bar{u} - \bar{a}_g)\frac{\partial f}{\partial x} &= -\frac{(\bar{u} - \bar{a}_g)}{2(\bar{a}_g\bar{\rho})^{1/2}}\frac{d\bar{u}}{dt}a_2 - \frac{1}{2}\left(\frac{\bar{a}_c^2}{\bar{\kappa}} + \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right) f \\ &\quad -\frac{1}{2}\left[(\bar{u} - \bar{a}_g)\frac{\partial}{\partial x}\ln\left(\frac{\bar{a}_g}{\bar{\rho}}\right) + \frac{\bar{u} - \bar{a}_g}{\bar{u} + \bar{a}_g}\left(\frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right)\right] g,\end{aligned} \tag{5.16}$$

where

$$\frac{d\bar{u}}{dt} = \bar{u}\frac{\partial \bar{u}}{\partial x}, \tag{5.17}$$

is the acceleration vector of the fluid. Similarly, the forward sound-wave equation (5.10) may be written in the form

$$\begin{aligned}\frac{\partial g}{\partial t} + (\bar{u} + \bar{a}_g)\frac{\partial g}{\partial x} &= -\frac{(\bar{u} + \bar{a}_g)}{2(\bar{a}_g\bar{\rho})^{1/2}}\frac{d\bar{u}}{dt}a_2 - \frac{1}{2}\left(\frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right) g \\ &\quad -\frac{1}{2}\left[(\bar{u} + \bar{a}_g)\frac{\partial}{\partial x}\ln\left(\frac{\bar{a}_g}{\bar{\rho}}\right) + \frac{\bar{u} + \bar{a}_g}{\bar{u} - \bar{a}_g}\left(\frac{\bar{a}_c^2}{\bar{\kappa}} + \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right)\right] f.\end{aligned} \tag{5.18}$$

Equations (5.16) and (5.18) are analogous to equations (9) in Heinemann and Olbert's (1980) model for bidirectional Alfvén waves in the solar wind, if the entropy wave and cosmic-ray source terms on the right-hand sides of these equations are omitted.

Multiplying (5.18) by $2g/(\bar{u} + \bar{a}_g)$ and subtracting $2f/(\bar{u} - \bar{a}_g)$ times (5.16) we obtain the wave-action-type equation

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{g^2}{\bar{u} + \bar{a}_g} - \frac{f^2}{\bar{u} - \bar{a}_g}\right) + \frac{\partial}{\partial x}(g^2 - f^2) \\ &= -\frac{1}{(\bar{a}_g\bar{\rho})^{1/2}}\frac{d\bar{u}}{dt}a_2(g - f) - \left(\frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right)\frac{g^2 - fg}{\bar{u} + \bar{a}_g} \\ &\quad + \left(\frac{\bar{a}_c^2}{\bar{\kappa}} + \frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right)\frac{f^2 - fg}{\bar{u} - \bar{a}_g},\end{aligned} \tag{5.19}$$

which is similar to equation (26) of Heinemann and Olbert (1980). It is clear that equations analogous to (5.16)–(5.19) also hold for the averaged wave equations derived from (5.1).

From the definitions (5.13) of f and g and (5.4), we have the correspondences

$$\begin{aligned}\omega_1\mathscr{A}_1 &= -\frac{\langle f^2\rangle}{\bar{u} - \bar{a}_g}, & \omega_1 V_{g1}\mathscr{A}_1 &= -\langle f^2\rangle, \\ \omega_3\mathscr{A}_3 &= \frac{\langle g^2\rangle}{\bar{u} + \bar{a}_g}, & \omega_3 V_{g3}\mathscr{A}_3 &= \langle g^2\rangle,\end{aligned} \tag{5.20}$$

where

$$\left.\begin{aligned}\omega_1 &= k_1(\bar{u} - \bar{a}_g), & \omega_3 &= k_3(\bar{u} + \bar{a}_g), \\ V_{g1} &= \bar{u} - \bar{a}_g, & V_{g3} &= \bar{u} + \bar{a}_g\end{aligned}\right\} \tag{5.21}$$

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give the wave frequencies ω_1 and ω_3 and group velocities V_{g1} and V_{g3} of the sound waves in the fixed frame.

Equation (5.19) averaged over a suitable short time scale yields the canonical wave-energy equation or wave-action equation

$$\begin{aligned}\frac{\partial}{\partial t}\ (\omega_1\mathscr{A}_1+\omega_3\mathscr{A}_3)+\frac{\partial}{\partial x}(V_{g1}\omega_1\mathscr{A}_1+V_{g3}\omega_3\mathscr{A}_3)\\ =-\frac{1}{(\bar{a}_g\bar{\rho})^{1/2}}\frac{d\bar{u}}{dt}[\langle a_2 g\rangle-\langle a_2 f\rangle]\\ -\left(\frac{\bar{a}_c^2}{\bar{\kappa}}-\frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x}\right)\frac{V_{g3}\omega_3\mathscr{A}_3-\langle fg\rangle}{V_{g3}}\\ -\left(\frac{\bar{a}_c^2}{\bar{\kappa}}+\frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x}\right)\frac{V_{g1}\omega_1\mathscr{A}_1+\langle fg\rangle}{V_{g1}}.\end{aligned}\tag{5.22}$$

In (5.22) it is implicitly assumed that the periods of the sound waves and the contact discontinuity are commensurate: $\omega_1:\omega_2:\omega_3=p:q:r$, where p, q and r are integers so that a meaningful short-time average may be made. Note the presence of the wave correlations $\langle a_2 f\rangle$, $\langle a_2 g\rangle$ and $\langle fg\rangle$ in (5.22). If the three waves are uncorrelated, (5.22) simplifies to

$$\begin{aligned}\frac{\partial}{\partial t}(\omega_1\mathscr{A}_1+\omega_3\mathscr{A}_3)+\frac{\partial}{\partial x}(V_{g1}\omega_1\mathscr{A}_1+V_{g3}\omega_3\mathscr{A}_3)\\ =-\left(\frac{\bar{a}_c^2}{\bar{\kappa}}-\frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x}\right)\omega_3\mathscr{A}_3-\left(\frac{\bar{a}_c^2}{\bar{\kappa}}+\frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x}\right)\omega_1\mathscr{A}_1.\end{aligned}\tag{5.23}$$

The more complex form of the canonical wave-energy equation (5.22) applies if $\langle a_2 f\rangle\neq 0$, $\langle a_2 g\rangle\neq 0$ and $\langle fg\rangle\neq 0$. It is worth noting that in the plasma physics literature (Dewar 1970) $\omega_1\mathscr{A}_1$ and $\omega_3\mathscr{A}_3$ correspond to the canonical energy densities of the backward and forward sound waves, in distinction to the physical energy densities $\omega_1'\mathscr{A}_1$ and $\omega_3'\mathscr{A}_3$. In the absence of entropy disturbances, and cosmic-ray source terms in (5.21), and for supersonic flows, one can think of $\mathscr{A}_3$ and $\mathscr{A}_1$ as representing particle and antiparticle quantum number densities. This interpretation is similar to that of Heinemann and Olbert (1980) in their model of bidirectional Alfvén waves. However, the interpretation is more complicated for subsonic flows, where $\mathscr{A}_1<0$ and $\mathscr{A}_3>0$ but $\omega_1\mathscr{A}_1$ and $\omega_3\mathscr{A}_3$ are both positive.

5.1.1. Unidirectional waves and the wave-action equation. For steady background flows, Hamilton's equations for the rays imply that

$$\frac{d\omega_j}{dt_j}=\frac{\partial\omega_j}{\partial t}+V_{gj}\frac{\partial\omega_j}{\partial x}=0\tag{5.24}$$

(Appendix B), and the frequency ω_j is constant along the ray. Using the result (5.24), and assuming that the wave-entropy disturbance $a_2=0$, and that only one of the sound waves is present, (5.22) reduces to the wave-action equation

$$\frac{\partial\mathscr{A}_j}{\partial t}+\frac{\partial}{\partial x}(V_{gj}\mathscr{A}_j)=-\left(\frac{\bar{a}_c^2}{\bar{\kappa}}-\frac{\bar{V}_{pj}'}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x}\right)\mathscr{A}_j,\tag{5.25}$$

where $j=1$ or 3, and $V_{pj}'=\omega_j'/k_j=\pm\bar{a}_g$ is the phase speed of the wave mode of interest as measured in the fluid frame. Equation (5.25) is the wave action equation for the squeezing instability, obtained by Drury and Falle (1986) and Zank and

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McKenzie (1987) for unidirectional sound waves upstream of a cosmic-ray-modified shock. Equations (5.25) show that the backward sound wave in the supersonic flow upstream of a cosmic-ray-modified shock is damped, whereas a forward-propagating sound wave is driven unstable by the large-scale cosmic-ray pressure gradient if the pressure gradient is sufficiently large and positive to counteract the diffusive damping term proportional to $\bar{a}_c^2/\bar{\kappa}$. The wave-action equation (5.25) may also be written in the form of a wave-energy equation as

$$\frac{\partial E_{wj}}{\partial t} + \frac{\partial}{\partial x}(V_{gj}E_{wj} + \bar{u}p_{wj}) = \bar{u}\frac{\partial p_{wj}}{\partial x} - \left(\frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{V}'_{pj}}{\gamma_g \bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right)E_{wj}, \tag{5.26}$$

where

$$E_{wj} = \omega'_j \mathscr{A}_j, \qquad p_{wj} = \tfrac{1}{2}(\gamma_g + 1)E_{wj} \tag{5.27}$$

define the wave energy density and the xx component of the wave pressure tensor respectively (see e.g. Dewar 1970; Webb and McKenzie 1984; Webb 1989).

For the case where only one sound wave is present, we can consistently choose $b_j = b_j(\theta_j)$ in equation (3.51), and use the wave evolution equation for the sound wave before averaging, and without invoking the sublinearity constraint on b_j. Hence the density perturbation of the sound wave a_j in this case satisfies the evolution equation

$$\frac{da_j}{dt_j} + \frac{(\gamma_g + 1)\bar{V}'_{pj}}{2\bar{\rho}}a_j\frac{\partial a_j}{\partial \xi} = -\Lambda_{jj}a_j, \tag{5.28}$$

where $j = 1$ or $j = 3$, and

$$\Lambda_{jj} = \frac{1}{2}\left[\frac{d}{dt_j}\ln\left(\frac{\bar{a}_g^3}{\bar{\rho}}\right) + (\gamma_g + 1)\frac{\partial \bar{u}}{\partial x} - \frac{\bar{V}'_{pj}}{\gamma_g \bar{p}_g}\frac{\partial \bar{p}_c}{\partial x} + \frac{\bar{a}_c^2}{\bar{\kappa}}\right], \tag{5.29}$$

As shown in Webb *et al.* (1995), the modified Burgers equation (5.28) may be written in the form

$$\frac{\partial A_j}{\partial t} + \frac{\partial}{\partial x}(V_{gj}A_j) + \frac{\partial}{\partial \xi}\left(\frac{(\gamma_g + 1)A_j v_j}{3}\right) = -\left(\frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{V}'_{pj}}{\gamma \bar{p}_g}\frac{\partial \bar{p}_c}{\partial x}\right)A_j, \tag{5.30}$$

where

$$A_j = \frac{\bar{\rho}v_j^2}{\omega'_j}, \tag{5.31}$$

is the non-averaged form of the wave action (i.e. $\mathscr{A}_j = \langle A_j \rangle$). The results (5.28) and (5.30) also hold for a time-varying background flow. A modified Burgers equation similar to (5.28) governing the short-wavelength sound wave was obtained by Zank (1988*b*) for the case of wave perturbations about a uniform background state. He showed how the solutions of (5.28) for a uniform background state develop into shocks in a finite time.

5.2. WKB analysis of wave mixing in steady flows

Our main concern in this section is to demonstrate the phenomenon of wave mixing described by (5.15)–(5.18) for the case of steady background flows. We consider two simplified problems:

(*a*) the interaction between the backward and forward sound waves of frequency ω in the absence of the contact discontinuity eigenmode;

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(*b*) the generation of sound waves by wave mixing arising from contact discontinuities initially present in the flow.

In both cases the waves generated by wave mixing have amplitudes $O(1/\omega)$ that of the original wave, and have characteristic phase shifts relative to the original wave.

The equations for the contact discontinuity (a_2), backward sound wave (f) and forward sound wave (g) from (5.15)–(5.18) for a steady background flow may be written in the form

$$\frac{\partial a_2}{\partial t} + \frac{\partial}{\partial x}(\bar{u} a_2) = 0, \tag{5.32}$$

$$\frac{\partial f}{\partial t} + \lambda_1 \frac{\partial f}{\partial x} = \lambda_1(\nu_{11} f + \nu_{12} a_2 + \nu_{13} g), \tag{5.33}$$

$$\frac{\partial g}{\partial t} + \lambda_3 \frac{\partial g}{\partial x} = \lambda_3(\nu_{31} f + \nu_{32} a_2 + \nu_{33} g), \tag{5.34}$$

where

$$\nu_{11} = -\frac{1}{2(\bar{u} - \bar{a}_g)} \left(\frac{\bar{a}_c^2}{\bar{\kappa}} + \frac{\bar{a}_g}{\gamma_g \bar{p}_g} \frac{\partial \bar{p}_c}{\partial x} \right), \tag{5.35a}$$

$$\nu_{12} = \nu_{32} = -\frac{1}{2(\bar{a}_g \bar{\rho})^{1/2}} \bar{u} \frac{d\bar{u}}{dx}, \tag{5.35b}$$

$$\nu_{33} = -\frac{1}{2(\bar{u} + \bar{a}_g)} \left(\frac{\bar{a}_c^2}{\bar{\kappa}} - \frac{\bar{a}_g}{\gamma_g \bar{p}_g} \frac{\partial \bar{p}_c}{\partial x} \right), \tag{5.35c}$$

$$\nu_{13} = \frac{\partial \psi}{\partial x} + \nu_{33}, \qquad \nu_{31} = \frac{\partial \psi}{\partial x} + \nu_{11}, \tag{5.35d,e}$$

$$\psi = -\frac{1}{2} \ln \left(\frac{\bar{a}_g}{\bar{\rho}} \right) = \frac{3 - \gamma_g}{4} \ln \bar{\rho}, \tag{5.35f}$$

and $\lambda_1 = \bar{u} - \bar{a}_g$, $\lambda_2 = \bar{u}$ and $\lambda_3 = \bar{u} + \bar{a}_g$ denote the speeds of the backward sound wave, the contact discontinuity and the forward sound wave.

Introducing the Fourier transform

$$\bar{P}(x, \omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(i\omega t)\, P(x, t)\, dt, \tag{5.36}$$

(where $P(x, t)$ denotes an arbitrary function of x and t with a well-defined Fourier transform) (5.32)–(5.34) in Fourier space reduce to

$$\bar{u} \frac{d\bar{a}_2}{dx} + \left(\frac{d\bar{u}}{dx} - i\omega \right) \bar{a}_2 = 0, \tag{5.37}$$

$$\frac{d\bar{f}}{dx} - \left(\frac{i\omega}{\lambda_1} + \nu_{11} \right) \bar{f} = \nu_{12} \bar{a}_2 + \nu_{13} \bar{g}, \tag{5.38}$$

$$\frac{d\bar{g}}{dx} - \left(\frac{i\omega}{\lambda_3} + \nu_{33} \right) \bar{g} = \nu_{31} \bar{f} + \nu_{32} \bar{a}_2. \tag{5.39}$$

Equation (5.37) for the contact discontinuity may be integrated to yield the explicit solution

$$\bar{a}_2(x, \omega) = \bar{a}_{20}(\omega) \frac{\bar{u}_0}{\bar{u}(x)} \exp(i\omega \phi_2), \qquad \phi_2 = \int_{x_0}^{x} \frac{dx'}{\bar{u}(x')}. \tag{5.40}$$

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Thus the differential equation system (5.37)–(5.39) reduces to the differential equations (5.38) and (5.39) for the two sound waves, in which the entropy-wave solution appears as a source term.

To further analyse (5.38) and (5.39), it is useful to introduce auxiliary functions $\bar{F}(x,\omega)$ and $\bar{G}(x,\omega)$ for the backward and forward sound waves:

$$\bar{F}(x,\omega) = \bar{f}\,\exp\left(-i\omega\phi_1 - \gamma_1\right), \qquad \bar{G}(x,\omega) = \bar{g}\exp\left(-i\omega\phi_3 - \gamma_3\right), \tag{5.41}$$

where

$$\phi_j = \int_{x_0}^{x} \frac{dx'}{\lambda_j}, \qquad \gamma_1 = \int_{x_0}^{x} \nu_{11}\,dx', \qquad \gamma_3 = \int_{x_0}^{x} \nu_{33}\,dx'. \tag{5.42}$$

For the further development it is also useful to introduce the notation

$$\phi = \phi_3 - \phi_1 = \int_{x_0}^{x} dx' \left(\frac{1}{\bar{u}+\bar{a}_g} - \frac{1}{\bar{u}-\bar{a}_g}\right), \qquad \gamma = \gamma_3 - \gamma_1. \tag{5.43}$$

The quantities γ_1 and γ_3 in (5.42) correspond to spatial growth rates for the backward and forward sound waves due to diffusive damping of the waves by the cosmic rays, and to a squeezing instability associated with the large-scale cosmic-ray pressure gradient (see e.g. Drury and Falle 1986; Zank and McKenzie 1987). The quantities ϕ_j measure the time taken for the jth wave to travel from a fiducial point x_0 to position x in the flow.

In terms of $\bar{F}$ and $\bar{G}$, the backward and forward sound-wave equations (5.38) and (5.39) reduce to

$$\frac{d\bar{F}}{dx} = \nu_{13}\,\exp\left(i\omega\phi + \gamma\right)\bar{G} + \nu_{12}\,\exp\left(-i\omega\phi_1 - \gamma_1\right)\bar{a}_2, \tag{5.44}$$

$$\frac{d\bar{G}}{d\bar{x}} = \nu_{31}\,\exp\left(-i\omega\phi - \gamma\right)\bar{F} + \nu_{32}\exp\left(-i\omega\phi_3 - \gamma_3\right)\bar{a}_2. \tag{5.45}$$

Equations (5.44) and (5.45) may be combined to yield second-order differential equations for $\bar{F}$ and $\bar{G}$:

$$\frac{d^2\bar{F}}{dx^2} - (i\omega\phi_x + M_{13})\frac{d\bar{F}}{dx} - \nu_{13}\nu_{31}\bar{F} = \bar{S}_F, \tag{5.46}$$

$$\frac{d^2\bar{G}}{dx^2} + (i\omega\phi_x - M_{31})\frac{d\bar{G}}{dx} - \nu_{13}\nu_{31}\bar{G} = \bar{S}_G, \tag{5.47}$$

where

$$M_{13} = \frac{\partial}{\partial x}(\ln\,\nu_{13} + \gamma), \qquad M_{31} = \frac{\partial}{\partial x}(\ln\,\nu_{31} - \gamma). \tag{5.48}$$

The source terms $\bar{S}_F$ and $\bar{S}_G$ in (5.46) and (5.47) associated with the entropy wave are given by

$$\begin{aligned}\bar{S}_F = {} & \frac{d}{dx}[\nu_{12}\bar{a}_2\exp\left(-i\omega\phi_1 - \gamma_1\right)] \\ & + \nu_{12}\bar{a}_2\exp\left(-i\omega\phi_1 - \gamma_1\right)\left(\nu_{13} - \frac{d}{dx}\ln\left[\nu_{13}\exp\left(i\omega\phi + \gamma\right)\right]\right),\end{aligned} \tag{5.49}$$

$$\begin{aligned}\bar{S}_G = {} & \frac{d}{dx}[\nu_{12}\bar{a}_2\exp\left(-i\omega\phi_3 - \gamma_3\right)] \\ & + \nu_{12}\bar{a}_2\exp\left(-i\omega\phi_3 - \gamma_3\right)\left(\nu_{31} - \frac{d}{dx}[\nu_{31}\exp\left(-i\omega\phi - \gamma\right)]\right).\end{aligned} \tag{5.50}$$

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Equations (5.46) and (5.47) form the basis of our WKB analysis of wave mixing. It should be noted that the solutions for $\bar{F}(x,\omega)$ and $\bar{G}(x,\omega)$ are not independent, but must also satisfy the coupled first-order (5.44) and (5.45).

5.2.1. WKB analysis of wave mixing in the absence of entropy waves. In this subsection we use standard WKB analysis (see e.g. Bender and Orszag 1978, Chap. 10) to consider the coupling between the backward and forward sound-wave solutions in the absence of entropy waves ($\tilde{a}_2 = 0$). In this case (5.46) and (5.47) have zero source terms (i.e. $\bar{S}_F = 0$ and $\bar{S}_G = 0$). In the WKB analysis the frequency ω is taken as a large parameter, so that

$$\epsilon = 1/\omega \tag{5.51}$$

plays the role of a perturbation parameter.

To obtain WKB solutions of the homogeneous equation (5.46) for $\bar{F}$, we use the WKB solution ansatz

$$\bar{F}(x,\omega) = \exp\left(\frac{1}{\epsilon}\sum_{n=0}^{\infty}\epsilon^n S_n\right) \tag{5.52}$$

for $\bar{F}(x,\omega)$. Equating powers of $\epsilon \equiv 1/\omega$ in (5.46) (with $\bar{S}_F = 0$) yields the balance equations

$$O\left(\epsilon^{-1}\right): \qquad S_0'(S_0' - i\phi_x) = 0, \tag{5.53a}$$

$$O(1): \qquad S_0'' + 2S_0'S_1' - (i\phi_x S_1' + M_{13})S_0' = 0, \tag{5.53b}$$

$$O(\epsilon): \qquad S_1'' + (S_1')^2 + 2S_0'S_2' - (i\phi_x S_2' + M_{13}S_1') - \nu_{13}\nu_{31} = 0, \tag{5.53c}$$

$$O(\epsilon^2): \qquad S_2'' + 2S_1'S_2' - (i\phi_x S_3' + M_{13}S_2') = 0, \tag{5.53d}$$

where $S_j' \equiv dS_j/dx$. Higher-order balance equations can clearly be obtained.

Equation (5.53 *a*) admits two possible solutions for S_0, namely $S_0 = \text{const}$ and $S_0 = i\phi$. Thus there are two possible solutions for the $S_j(x)$ depending on whether $S_0 = \text{const}$ or $S_0 = i\phi$ is chosen as the solution for S_0 (without loss of generality, we choose $S_0 = 0$ for the constant solution in the following analysis).

The net result from solving the balance equations (5.53) for the $S_j(x)$ is that the homogeneous equation (5.46) possesses approximate WKB solutions of the form

$$\bar{F}(x,\omega) = c_1\bar{F}_1(x,\omega) + c_2\bar{F}_2(x,\omega), \tag{5.54}$$

where

$$\bar{F}_1(x,\omega) = \exp\left[\frac{i}{\omega}\int^x \frac{\nu_{13}\nu_{31}}{\phi_x}dx + \frac{1}{\omega^2}\int^x dx\frac{\nu_{13}\nu_{31}}{\phi_x}\left(\frac{M_{31}}{\phi_x} - \frac{\phi_{xx}}{\phi_x^2}\right) + \ldots\right], \tag{5.55a}$$

$$\bar{F}_2(x,\omega) = \frac{\nu_{13}}{\phi_x}\exp\left[i\omega\phi + \gamma + \frac{i}{\omega}\left(\frac{M_{13}}{\phi_x} - \frac{\phi_{xx}}{\phi_x^2} - \int^x \frac{\nu_{13}\nu_{31}}{\phi_x}dx\right) + \ldots\right], \tag{5.55b}$$

and c_1 and c_2 can be arbitrary functions of ω. Similarly the homogeneous equation (5.47) for the backward sound wave possesses approximate WKB solutions of the form

$$\bar{G}(x,\omega) = d_1\bar{G}_1(x,\omega) + d_2\bar{G}_2(x,\omega), \tag{5.56}$$

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where d_1 and d_2 are arbitrary functions of ω, and the solutions $\bar{G}_1(x,\omega)$ and $\bar{G}_2(x,\omega)$ are given by

$$\bar{G}_1(x,\omega) = \exp\left[-\frac{i}{\omega}\int^x \frac{\nu_{13}\nu_{31}}{\phi_x}dx + \frac{1}{\omega^2}\int^x dx\frac{\nu_{13}\nu_{31}}{\phi_x}\left(\frac{M_{13}}{\phi_x} - \frac{\phi_{xx}}{\phi_x^2}\right) + \ldots\right], \quad (5.57a)$$

$$\bar{G}_2(x,\omega) = \frac{\nu_{31}}{\phi_x}\exp\left[-i\omega\phi - \gamma - \frac{i}{\omega}\left(\frac{M_{31}}{\phi_x} - \frac{\phi_{xx}}{\phi_x^2} - \int^x \frac{\nu_{13}\nu_{31}}{\phi_x}dx\right) + \ldots\right]. \quad (5.57b)$$

Before proceeding with further analysis, it is useful to note some limitations of the WKB analysis. The usual criteria for WKB analysis to apply (in the case where $S_0 \neq 0$) are that the successive terms in the WKB expansion (5.52) should be ordered according to

$$|\omega S_0| \gg |S_1| \gg |S_2/\omega| \gg |S_3/\omega^2| \gg \ldots \quad . \quad (5.58)$$

The lowest-order criterion $|\omega S_0| \gg |S_1|$ for the solution $\bar{F}_2(x,\omega)$ reduces to

$$|\omega\phi_x| \gg \left|M_{13} - \frac{\phi_{xx}}{\phi_x}\right| \equiv L^{-1}, \quad (5.59)$$

where L is a characteristic scale of the background flow. Using the definition (5.43) ϕ, the criterion (5.59) for the WKB analysis to apply may be expressed as

$$\omega\left|\frac{L}{\lambda_3} - \frac{L}{\lambda_1}\right| \equiv \omega\left|\frac{2\bar{a}_g L}{\bar{u}^2 - \bar{a}_g^2}\right| \gg 1. \quad (5.60)$$

The criterion (5.60) is essentially the same as that derived by Heinemann and Olbert (1980) in their description of non-WKB toroidal Alfvén waves in the solar wind, namely that the wave period $T_\omega = 2\pi/\omega$ should be much less than the difference between the characteristic propagation time scales L/λ_3 and L/λ_1 for the forward and backward waves. More complex criteria result from the higher-order conditions (5.58). Our analysis assumes that the solutions for $\bar{F}$ and $\bar{G}$ do not involve turning points, in which case it would be necessary to match exponential-type solutions to oscillatory solutions through the turning-point region with Airy function approximations.

For the wave-mixing problem described by the solutions (5.54) and (5.56), the parameters c_1, c_2 and d_1 and d_2 are not independent, since the solutions for $\bar{F}(\bar{x},\omega)$ and $\bar{G}(\bar{x},\omega)$ must satisfy the coupled first-order differential equations (5.44) and (5.45). This requires that

$$c_2 = -\frac{id_1}{\omega}, \qquad d_2 = \frac{ic_1}{\omega} \quad (5.61)$$

in the solutions (5.54) and (5.56) for $\bar{F}$ and $\bar{G}$, so that

$$\bar{F}(x,\omega) = c_1\bar{F}_1 - i\frac{d_1}{\omega}\bar{F}_2, \qquad \bar{G}(x,\omega) = d_1\bar{G}_1 + i\frac{c_1}{\omega}\bar{G}_2 \quad (5.62)$$

give the appropriate solutions for $\bar{F}$ and $\bar{G}$.

Using the relations (5.41) and (5.13) relating $\bar{F}$, $\bar{G}$, $\bar{f}$, $\bar{g}$ and $\bar{v}_1$ and $\bar{v}_3$ leads to solutions for the sound velocity perturbations v_1 and v_3 in (x,t) space in the form

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$$v_1 = \frac{(\bar{a}_g/\bar{\rho})^{1/2}}{\bar{u}-\bar{a}_g}\left\{c_1 \exp\left[-i\omega(t-\phi_1)+\gamma_1+\frac{i}{\omega}\int_{x_0}^{x}\frac{\nu_{13}\nu_{31}}{\phi_x}\,dx\right.\right.$$
$$\left.+\frac{1}{\omega^2}\int_{x_0}^{x}dx\,\frac{\nu_{13}\nu_{31}}{\phi_x}\left(\frac{M_{31}}{\phi_x}-\frac{\phi_{xx}}{\phi_x^2}\right)+\ldots\right]$$
$$-i\frac{d_1}{\omega}\frac{\nu_{13}}{\phi_x}\exp\left[-i\omega(t-\phi_3)+\gamma_3+\right.$$
$$\left.\left.\frac{i}{\omega}\left(\frac{M_{13}}{\phi_x}-\frac{\phi_{xx}}{\phi_x^2}-\int_{x_0}^{x}\frac{\nu_{13}\nu_{31}}{\phi_x}\,dx\right)+\ldots\right]+\text{c.c.}\right\}, \quad (5.63a)$$

$$v_3 = \frac{(\bar{a}_g/\bar{\rho})^{1/2}}{\bar{u}+\bar{a}_g}\left\{d_1 \exp\left[-i\omega(t-\phi_3)+\gamma_3-\frac{i}{\omega}\int_{x_0}^{x}\frac{\nu_{13}\nu_{31}}{\phi_x}\,dx\right.\right.$$
$$\left.+\frac{1}{\omega^2}\int_{x_0}^{x}dx\,\frac{\nu_{13}\nu_{31}}{\phi_x}\left(\frac{M_{13}}{\phi_x}-\frac{\phi_{xx}}{\phi_x^2}\right)+\ldots\right]$$
$$+i\frac{c_1}{\omega}\frac{\nu_{31}}{\phi_x}\exp\left[-i\omega(t-\phi_1)+\gamma_1\right.$$
$$\left.\left.-\frac{i}{\omega}\left(\frac{M_{31}}{\phi_x}-\frac{\phi_{xx}}{\phi_x^2}-\int_{x_0}^{x}\frac{\nu_{13}\nu_{31}}{\phi_x}\,dx\right)+\ldots\right]+\text{c.c.}\right\}. \quad (5.63b)$$

In (5.63) c_1 and d_1 are taken as constants corresponding to solutions with one Fourier component, and the notation '+c.c.' means 'add the complex conjugate'. The solutions (5.63) describe the large-space-scale and large-time-scale dependence of v_1 and v_3 only. The results (5.63) show the basic character of wave mixing for the sound-wave solutions. If, for example, $d_1 = 0$ in (5.63) then the solution for v_1 may be regarded as the primary solution, and the solution for v_3 represents the reflected wave. Note that the reflected wave for v_3 in this case has its amplitude reduced by a factor of $1/\omega$ times that of the primary wave, and has a $\frac{1}{2}\pi$ phase shift to lowest order relative to the primary wave. Note also that the reflected wave is proportional to the wave coupling coefficient ν_{31}. The other main point to note about the solutions (5.63) is the presence of the spatial growth terms

$$\gamma_1 = \int^x \nu_{11}\,dx' = -\frac{1}{2}\int^x\left(\frac{\bar{a}_c^2}{\bar{\kappa}}+\frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x'}\right)\frac{dx'}{\bar{u}-\bar{a}_g}, \quad (5.64)$$

$$\gamma_3 = \int^x \nu_{33}\,dx' = -\frac{1}{2}\int^x\left(\frac{\bar{a}_c^2}{\bar{\kappa}}-\frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x'}\right)\frac{dx'}{\bar{u}+\bar{a}_g} \quad (5.65)$$

associated with the cosmic rays. In the smooth decelerating, supersonic flow upstream of a cosmic-ray-modified shock where $\partial\bar{p}_c/\partial x > 0$, the primary backward sound wave is damped ($\gamma_1 < 0$), whereas the primary forward sound wave is driven unstable if

$$\frac{\bar{a}_g}{\gamma_g\bar{p}_g}\frac{\partial\bar{p}_c}{\partial x} > \frac{\bar{a}_c^2}{\bar{\kappa}}. \quad (5.66)$$

The results (5.63)–(5.66) show the role of the cosmic-ray squeezing instability (see e.g. Drury and Falle 1986; Zank and McKenzie 1987) in situations where wave mixing occurs. The linear wave solutions (5.63) become problematic near the gas sonic point ($\bar{u}^2 = \bar{a}_g^2$), where at least one of the linear wave solutions diverges.

5.2.2. WKB analysis of the role of entropy waves. In the case where entropy waves are present in the flow, the second order differential equations (5.46) and (5.47) for

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$\bar{F}(x,\omega)$ and $\bar{G}(x,\omega)$ have non-zero source terms. We restrict our attention to the case of supersonic flows. In this case waves generated at a point x_0 at time t_0 are forced to travel downstream of x_0. A particular solution of (5.46) for $\bar{F}$ in this case, using the method of variation of parameters, is of the form (Morse and Feshbach 1953)

$$\bar{F}(x,\omega) = -\bar{F}_1(x,\omega)\int_{x_0}^{x}\frac{\bar{S}_F(x')\bar{F}_2(x',\omega)}{W(\bar{F}_1,\bar{F}_2)}\,dx' + \bar{F}_2(x,\omega)\int_{x_0}^{x}\frac{\bar{S}_F(x')\bar{F}_1(x',\omega)}{W(\bar{F}_1,\bar{F}_2)}\,dx', \tag{5.67}$$

where $\bar{F}_1(x,\omega)$ and $\bar{F}_2(x,\omega)$ are independent solutions of the homogeneous equation (5.46), and

$$W(\bar{F}_1,\bar{F}_2) = \bar{F}_1\bar{F}_2' - \bar{F}_2\bar{F}_1', \tag{5.68}$$

is their Wronskian. The general solution of (5.46) for $\bar{F}(x,\omega)$ consists of the particular solution (5.67) plus the complementary solution, which consists of an arbitrary linear combination of $\bar{F}_1(x,\omega)$ and $\bar{F}_2(x,\omega)$. Since the contact discontinuity is frozen into the background flow, the entropy-wave solution (5.40) in Fourier space, or its equivalent

$$a_2(x,t) = \frac{\bar{u}_0}{\bar{u}(x)}a_{20}[t-\phi_2(x)], \qquad \phi_2(x) = \int_{x_0}^{x}\frac{dx'}{\bar{u}(x')} \tag{5.69}$$

in (x,t) space, represents an entropy disturbance downstream of the point $x = x_0$. Hence the solution (5.67) corresponds to a backward sound wave that has been generated from the entropy wave by wave mixing. Similarly, the solution

$$\bar{G}(x,\omega) = -\bar{G}_1(x,\omega)\int_{x_0}^{x}\frac{\bar{S}_G(x',\omega)\bar{G}_2(x',\omega)}{W(\bar{G}_1,\bar{G}_2)}\,dx' + \bar{G}_2(x,\omega)\int_{x_0}^{x}\frac{\bar{S}_G(x',\omega)\bar{G}_1(x',\omega)}{W(\bar{G}_1,\bar{G}_2)}\,dx', \tag{5.70}$$

is the solution for the forward sound wave generated by the entropy wave. In (5.70) $\bar{G}_1(x,\omega)$ and $\bar{G}_2(x,\omega)$ are independent solutions of the homogeneous equation (5.47) for $\bar{G}(x,\omega)$.

Using the approximate WKB solutions (5.55) for $\bar{F}_1$ and $\bar{F}_2$ and the approximate solutions (5.57) for $\bar{G}_1$ and $\bar{G}_2$ leads to the formulae

$$W(\bar{F}_1,\bar{F}_2) = i\omega\nu_{13}\exp(i\omega\phi+\gamma)\,[1+O(1/\omega^2)], \tag{5.71a}$$

$$W(\bar{G}_1,\bar{G}_2) = -i\omega\nu_{31}\exp(-i\omega\phi-\gamma)\,[1+O(1/\omega^2)], \tag{5.71b}$$

for the Wronskians $W(\bar{F}_1,\bar{F}_2)$ and $W(\bar{G}_1,\bar{G}_2)$ in (5.67) and (5.70). Note that if $\bar{F}_1$ and $\bar{F}_2$ were exact solutions of (5.46), and similarly $\bar{G}_1$ and $\bar{G}_2$ were exact solutions of (5.47) then the Wronskians in (5.71) would be exact.

Substituting the Wronskians (5.71) into (5.67) and (5.70) shows that the solutions for $\bar{F}$ and $\bar{G}$ may be written in the form

$$\bar{F}(x,\omega) = \frac{i}{\omega}[\bar{F}_1(x,\omega)V_1(x,\omega) - \bar{F}_2(x,\omega)V_2(x,\omega)], \tag{5.72a}$$

$$\bar{G}(x,\omega) = -\frac{i}{\omega}[\bar{G}_1(x,\omega)U_1(x,\omega) - \bar{G}_2(x,\omega)U_2(x,\omega)], \tag{5.72b}$$

where the functions U_1, U_2, V_1 and V_2 are given by

$$V_1(x,\omega) = \int_{x_0}^{x}\exp\left[-i\omega(\phi_1-\phi_2)-\gamma_1+O\left(\frac{1}{\omega}\right)\right]\frac{\nu_{12}\bar{a}_{20}}{\phi_x}\frac{\bar{u}_0}{\bar{u}(x')}q_1(x',\omega)\,dx', \tag{5.73a}$$

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$$V_2(x,\omega) = \int_{x_0}^{x} \exp\left[-i\omega(\phi_3-\phi_2) - \gamma_3 + O\left(\frac{1}{\omega}\right)\right] \frac{\nu_{12}\bar{a}_{20}}{\nu_{13}} \frac{\bar{u}_0}{\bar{u}(x')} q_1(x',\omega)\, dx', \quad (5.73b)$$

$$U_1(x,\omega) = \int_{x_0}^{x} \exp\left[-i\omega(\phi_3-\phi_2) - \gamma_3 + O\left(\frac{1}{\omega}\right)\right] \frac{\nu_{12}\bar{a}_{20}}{\phi_x} \frac{\bar{u}_0}{\bar{u}(x')} q_2(x',\omega)\, dx', \quad (5.73c)$$

$$U_2(x,\omega) = \int_{x_0}^{x} \exp\left[-i\omega(\phi_1-\phi_2) - \gamma_1 + O\left(\frac{1}{\omega}\right)\right] \frac{\nu_{12}\bar{a}_{20}}{\nu_{31}} \frac{\bar{u}_0}{\bar{u}(x')} q_2(x',\omega)\, dx', \quad (5.73d)$$

where

$$q_1(x,\omega) = \frac{\partial}{\partial x} \ln\left(\frac{\nu_{12}\bar{a}_{20}\bar{u}_0/\bar{u}}{\nu_{13}(\bar{a}_g/\bar{\rho})^{1/2}} \exp\left[i\omega(\phi_2-\phi_3)\right]\right), \quad (5.74a)$$

$$q_2(x,\omega) = \frac{\partial}{\partial x} \ln\left(\frac{\nu_{12}\bar{a}_{20}\bar{u}_0/\bar{u}}{\nu_{31}(\bar{a}_g/\bar{\rho})^{1/2}} \exp\left[i\omega(\phi_2-\phi_1)\right]\right). \quad (5.74b)$$

Using (5.72)–(5.74) the solutions for the velocity perturbations for the backward and forward sound waves, assuming only one Fourier component is present, have the form

$$v_1 = \frac{i(\bar{a}_g/\bar{\rho})^{1/2}}{\omega(\bar{u}-\bar{a}_g)} \left\{ \exp\left[-i\omega(t-\phi_1) + \gamma_1 + O\left(\frac{1}{\omega}\right)\right] V_1(x,\omega) \right.$$
$$\left. - \frac{\nu_{13}}{\phi_x} \exp\left[-i\omega(t-\phi_3) + \gamma_3 + O\left(\frac{1}{\omega}\right)\right] V_2(x,\omega) + \text{c.c.} \right\}, \quad (5.75)$$

$$v_3 = \frac{i(\bar{a}_g/\bar{\rho})^{1/2}}{\omega(\bar{u}+\bar{a}_g)} \left\{ -\exp\left[-i\omega(t-\phi_3) + \gamma_3 + O\left(\frac{1}{\omega}\right)\right] U_1(x,\omega) \right.$$
$$\left. + \frac{\nu_{31}}{\phi_x} \exp\left[-i\omega(t-\phi_1) + \gamma_1 + O\left(\frac{1}{\omega}\right)\right] U_2(x,\omega) + \text{c.c.} \right\}. \quad (5.76)$$

The main point to note about the solutions (5.75) and (5.76) for the sound-wave perturbations is that their amplitude is $O(1/\omega)$ that of the primary entropy wave. The solutions (5.75) only apply downstream of x_0 (i.e. $x > x_0$), since the background flow is assumed to be supersonic and the entropy-wave source terms are non-zero only for $x > x_0$.

Inspection of the solution (5.75), for example, reveals that the solution term

$$\exp\left[-i\omega(t-\phi_1) + \gamma_1 + O\left(\frac{1}{\omega}\right)\right] V_1(x,\omega) \quad (5.77)$$

depends on the phase factor

$$\exp\left(-i\omega\{t - [\phi_1(x) - \phi_1(x')] - \phi_2(x')\} + \gamma_1(x) - \gamma_1(x')\right), \quad (5.78)$$

where x' denotes the source-point location in the integral for $V_1(x,\omega)$ in (5.73). This accords with the physical intuition that the phase of the sound wave generated by the entropy wave source at position x' should depend on the phase $\omega\phi_2(x')$ of the entropy wave, and on the travel time $\phi_1(x) - \phi_1(x')$ of the sound wave between the source and observation points. The spatial growth-rate factor $\gamma_1(x) - \gamma_1(x')$ in (5.78) represents the growth or damping of the sound waves between the source (x') and observation (x) points due to cosmic-ray diffusive damping and squeezing instability effects. The factors $\pm i/\omega$ in the solutions (5.75) and (5.76) for v_1 and v_3 correspond to phase shifts of $\pm\frac{1}{2}\pi$ of the sound waves relative to the entropy-wave source due to wave reflection.

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6. Variational principles for linear wave mixing

In this section we show that the linear wave mixing equations (5.15)–(5.18) for the sound waves and contact discontinuity for steady background flows may be written in terms of both Lagrangian and Hamiltonian variational principles. One of the more interesting results to emerge from the variational formulation in (x,t) space is that the wave coupling equations may be written in the Hamiltonian form

$$\frac{\partial p}{\partial x} = -\frac{\delta \mathscr{H}}{\delta q}, \qquad \frac{\partial q}{\partial x} = \frac{\delta \mathscr{H}}{\delta p}, \tag{6.1}$$

where $q(x,t)$ and $p(x,t)$ are suitable canonical variables and $\mathscr{H}(p,q)$ is the variational functional. The analysis is restricted to steady background flows. This has the advantage that one can develop variational principles first in (x,ω) Fourier space, and subsequently Fourier invert back to (x,t) space (in the present section, in a convenient abuse of notation, x and t refer to the large space and time variables).

In Sec. 6.1 we Fourier-invert the equations for $\bar{F}(x,\omega)$ and $\bar{G}(x,\omega)$ derived in Sec. 5.2 to establish wave equations for $F(x,t)$ and $G(x,t)$ in (x,t) space. In Sec. 6.2 Lagrangian and Hamiltonian variational principles for the Fourier-space equations are derived. In Sec. 6.3 variational principles for the equations in (x,t) space are derived by Fourier inversion. It is worth noting that the Hamiltonian formulation can be carried out in terms of many different canonical variables other than those used in the present analysis.

6.1. Equations for $F(x,t)$ and $G(x,t)$

Using the inverse Fourier transform

$$F(x,t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(-i\omega t)\, \bar{F}(x,\omega)\, d\omega, \tag{6.2}$$

in (5.41) yields the formula

$$F(x,t) = \exp(-\gamma_1)\, f(x, t+\phi_1), \tag{6.3}$$

relating $F(x,t)$ to the function $f(x,t)$, (5.13) describing the backward sound wave. Similarly, in (5.41) the inverse Fourier transform of $\bar{G}(x,\omega)$ is

$$G(x,t) = \exp(-\gamma_3)\, g(x, t+\phi_3) \tag{6.4}$$

for the forward sound wave.

The inverse Fourier transforms of (5.44) and (5.45) yield the first-order partial differential equation system

$$\frac{\partial F(x,t)}{\partial x} = \nu_{13} \exp(\gamma)\, G(x, t-\phi) + \nu_{12} \exp(-\gamma_1)\, a_2(x, t+\phi_1), \tag{6.5}$$

$$\frac{\partial G(x,t)}{\partial x} = \nu_{31} \exp(-\gamma)\, F(x, t+\phi) + \nu_{32} \exp(-\gamma_3)\, a_2(x, t+\phi_3), \tag{6.6}$$

governing the coupling of the backward and forward sound waves. The inverse Fourier transforms of (5.46) and (5.47) result in the wave equations

$$\frac{\partial^2 F}{\partial x^2} + \phi_x \frac{\partial^2 F}{\partial x \partial t} - \frac{\partial}{\partial x}(\ln \nu_{13} + \gamma)\frac{\partial F}{\partial x} - \nu_{13}\nu_{31} F(x,t) = S_F, \tag{6.7}$$

$$\frac{\partial^2 G}{\partial x^2} - \phi_x \frac{\partial^2 G}{\partial x \partial t} - \frac{\partial}{\partial x}(\ln \nu_{31} - \gamma)\frac{\partial G}{\partial x} - \nu_{13}\nu_{31} G(x,t) = S_G, \tag{6.8}$$

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for $F(x,t)$ and $G(x,t)$, where the source terms

$$S_F = \nu_{12}\exp(-\gamma_1)\Big\{D_x[a_2(x,t+\phi_1)] + \phi_x\frac{\partial}{\partial t}a_2(x,t+\phi_1) + a_2(x,t+\phi_1)\frac{\partial}{\partial x}\ln\left(\frac{\nu_{12}}{\nu_{13}(\bar{a}_g/\bar{\rho})^{1/2}}\right)\Big\}, \tag{6.9}$$

$$S_G = \nu_{12}\exp(-\gamma_3)\Big\{D_x[a_2(x,t+\phi_3)] - \phi_x\frac{\partial}{\partial t}a_2(x,t+\phi_3) + a_2(x,t+\phi_3)\frac{\partial}{\partial x}\left[\ln\left(\frac{\nu_{12}}{\nu_{31}(\bar{a}_g/\bar{\rho})^{1/2}}\right)+\gamma\right]\Big\} \tag{6.10}$$

represent the effects of coupling of the sound waves to the entropy wave. In (6.9) and (6.10) D_x denotes the total derivative operator with respect to x.

Inversion of the solution (5.40) for $\bar{a}_2(x,\omega)$ yields the explicit solution

$$a_2(x,t) = \frac{u_0}{u(x)}a_{20}[t-\phi_2(x)], \qquad \phi_2 = \int_{x_0}^{x}\frac{dx'}{\bar{u}(x')}, \tag{6.11}$$

for the entropy wave, where the function $a_{20}(z)$ determines the initial entropy-wave distribution.

Equations (6.7) and (6.8) are hyperbolic equations for F and G provided that ϕ_x is not zero or infinite (note that $|\phi_x| \to \infty$ near the gas sonic point). The characteristics $\xi = \xi(x,t)$ for (6.7) satisfy the first-order partial differential equation

$$\xi_x^2 + \phi_x\xi_x\xi_t = 0, \tag{6.12}$$

with solutions

$$\xi_1 = t, \quad \xi_2 = t-\phi. \tag{6.13}$$

The characteristic $\xi_1 = t$ corrresponds to the backward sound-wave characteristic, whereas $\xi_2 = t-\phi$ corresponds to the forward sound-wave characteristic. Since one of the characteristics is $\xi_1 = t$, the initial-value problem for (6.7) (and also (6.8)) is ill posed.

6.2. Variational formulations in Fourier space

In this section we consider both Lagrangian and Hamiltonian variational principles for the wave-mixing equations (5.44) and (5.45) in (x,ω) space, namely

$$\frac{d\bar{F}(x,\omega)}{dx} = \nu_{13}\exp(i\omega\phi+\gamma)\,\bar{G}(x,\omega) + \nu_{12}\exp(-i\omega\phi_1-\gamma_1)\,\bar{a}_2(x,\omega), \tag{6.14}$$

$$\frac{d\bar{G}(x,\omega)}{dx} = \nu_{31}\exp(-i\omega\phi-\gamma)\,\bar{F}(x,\omega) + \nu_{32}\exp(-i\omega\phi_3-\gamma_3)\,\bar{a}_2(x,\omega), \tag{6.15}$$

where the entropy wave solution $\bar{a}_2(x,\omega)$ is given by (5.40). Variational formulations of the decoupled second-order differential equations (5.46) and (5.47) for $\bar{F}$ and $\bar{G}$ can also be developed, but the latter equations do not show how the waves are coupled.

Equations (6.14) and (6.15) may be obtained by extremizing the variational functional

$$\bar{\mathscr{L}}_1 = \int_{x_a}^{x_b}\bar{L}_1\,dx, \tag{6.16}$$

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where

$$\bar{L}_1 = \bar{G}\bar{F}_x + \tfrac{1}{2}\nu_{31}\exp(-i\omega\phi-\gamma)\,\bar{F}^2 - \tfrac{1}{2}\nu_{13}\exp(i\omega\phi+\gamma)\,\bar{G}^2 + \nu_{32}\exp(-i\omega\phi_3-\gamma_3)\,\bar{a}_2\bar{F} - \nu_{12}\exp(-i\omega\phi_1-\gamma_1)\,\bar{a}_2\bar{G}, \tag{6.17}$$

is the Lagrangian density. The variational equations $\delta\bar{\mathscr{L}}_1/\delta\bar{G}=0$ and $\delta\bar{\mathscr{L}}_1/\delta\bar{F}=0$ yield (6.14) and (6.15) respectively.

Equations (6.14) and (6.15) can also be written in the Hamiltonian form

$$\frac{d\bar{p}_1}{dx} = -\frac{\partial\bar{H}_1}{\partial\bar{q}_1}, \qquad \frac{d\bar{q}_1}{dx} = \frac{\partial\bar{H}_1}{\partial\bar{p}_1}, \tag{6.18}$$

where

$$\bar{q}_1 = \bar{F}(x,\omega), \qquad \bar{p}_1 = \bar{G}(x,\omega), \tag{6.19}$$

are the canonical variables and

$$\bar{H}_1 = \tfrac{1}{2}\nu_{13}\exp(i\omega\phi+\gamma)\,\bar{p}_1^2 - \tfrac{1}{2}\nu_{31}\exp(-i\omega\phi-\gamma)\,\bar{q}_1^2 + \bar{a}_2[\nu_{12}\exp(-i\omega\phi_1-\gamma_1)\,\bar{p}_1 - \nu_{32}\exp(-i\omega\phi_3-\gamma_3)\,\bar{q}_1] \tag{6.20}$$

is the Hamiltonian.

The above formulation is not unique. Thus, for example, the transformations

$$\left.\begin{aligned} \bar{\Phi} &= \bar{F}(x,\omega)I_1^{1/2}, & \bar{\Gamma} &= \bar{G}(x,\omega)I_2^{1/2}, \\ I_1 &= \frac{\exp(-i\omega\phi-\gamma)}{\nu_{13}}, & I_2 &= \frac{\exp(i\omega\phi+\gamma)}{\nu_{31}}, \end{aligned}\right\} \tag{6.21}$$

applied to the second-order differential equations (5.46) and (5.47) for $\bar{F}(x,\omega)$ and $\bar{G}(x,\omega)$ yield the second-order differential equations:

$$\bar{\Phi}_{xx} + q_\Phi\bar{\Phi} = \bar{S}_F I_1^{1/2}, \tag{6.22}$$

$$\bar{\Gamma}_{xx} + q_\Gamma\bar{\Gamma} = \bar{S}_G I_2^{1/2} \tag{6.23}$$

for $\bar{\Phi}$ and $\bar{\Gamma}$, where

$$q_\Phi = -\left[\nu_{13}\nu_{31} + \frac{1}{4}\left(\frac{d\ln I_1}{dx}\right)^2 + \frac{1}{2}\frac{d^2\ln I_1}{dx^2}\right], \tag{6.24}$$

$$q_\Gamma = -\left[\nu_{13}\nu_{31} + \frac{1}{4}\left(\frac{d\ln I_2}{dx}\right)^2 + \frac{1}{2}\frac{d^2\ln I_2}{dx^2}\right]. \tag{6.25}$$

Equations (6.14) and (6.15), when written in terms of $\bar{\Phi}$ and $\bar{\Gamma}$, lead to the equations

$$\frac{d\bar{\Phi}}{dx} = \frac{1}{2}\frac{d\ln I_1}{dx}\bar{\Phi} + (\nu_{13}\nu_{31})^{1/2}\bar{\Gamma} + \nu_{12}\bar{a}_2\exp(-i\omega\phi_1-\gamma_1)I_1^{1/2}, \tag{6.26}$$

$$\frac{d\bar{\Gamma}}{dx} = \frac{1}{2}\frac{d\ln I_2}{dx}\bar{\Gamma} + (\nu_{13}\nu_{31})^{1/2}\bar{\Phi} + \nu_{32}\bar{a}_2\exp(-i\omega\phi_3-\gamma_3)I_2^{1/2}, \tag{6.27}$$

governing the coupling of the backward and forward sound waves to the entropy wave.

Equations (6.26) and (6.27) may also be obtained by extremizing the variational functional

$$\bar{\mathscr{L}}_2 = \int_{x_a}^{x_b}\bar{L}_2\,dx, \tag{6.28}$$

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where

$$\bar{L}_2 = (\nu_{13}\nu_{31})^{1/2}\bar{\Gamma}\bar{\Phi}_x + \tfrac{1}{2}\nu_{13}\nu_{31}(\bar{\Phi}^2 - \bar{\Gamma}^2) - \frac{(\nu_{13}\nu_{31})^{1/2}}{2}\frac{d\ln I_1}{dx}\bar{\Gamma}\bar{\Phi}$$
$$+(\nu_{13}\nu_{31})^{1/2}\bar{a}_2[\nu_{32}I_2^{1/2}\exp(-i\omega\phi_3-\gamma_3)\bar{\Phi} - \nu_{12}I_1^{1/2}\exp(-i\omega\phi_1-\gamma_1)\bar{\Gamma}] \quad (6.29)$$

is the Lagrangian density. The variational equations $\delta\bar{\mathscr{L}}_2/\delta\bar{\Gamma} = 0$ and $\delta\bar{\mathscr{L}}_2/\delta\bar{\Phi} = 0$ yield (6.26) and (6.27) respectively. The Lagrangian density $\bar{L}_2$, when written in terms of $\bar{F}$ and $\bar{G}$, reduces to the expression (6.17) for $\bar{L}_1$. Using the Lagrangian density $\bar{L}_2$, it is also possible to write (6.26) and (6.27) in Hamiltonian form in which $\bar{q}_2 = \bar{\Phi}$ and $\bar{p}_2 = \bar{\Gamma}$ are the canonical variables.

6.3. Variational principles in (x,t) space

Consider the variational functional

$$\mathscr{L}_1 = \int_{x_a}^{x_b} dx \int_{-\infty}^{\infty} d\omega \exp(-i\omega\tau)\,\bar{L}_1, \quad (6.30)$$

where τ is a constant and $\bar{L}_1$ is the Lagrangian density (6.17). Using the convolution theorem for Fourier transforms, the functional (6.30) reduces to

$$\mathscr{L}_1 = \int_{x_a}^{x_b} dx \int_{-\infty}^{\infty} dt \left(G(x,t)\frac{\partial F(x,\tau-t)}{\partial x} + \tfrac{1}{2}\nu_{31}\exp(-\gamma)\,F(x,\tau+\phi-t)F(x,t)\right.$$
$$-\tfrac{1}{2}\nu_{13}\exp(\gamma)\,G(x,\tau-\phi-t)G(x,t)$$
$$+\nu_{32}\exp(-\gamma_3)\,a_2(x,\tau+\phi_3-t)F(x,t)$$
$$\left.-\nu_{12}\exp(-\gamma_1)\,a_2(x,\tau+\phi_1-t)G(x,t)\right) \quad (6.31)$$

for the Lagrangian variational functional in (x,t) space. From (6.31),

$$\frac{\delta\mathscr{L}_1}{\delta F(x,t)} = \nu_{31}\exp(-\gamma)\,F(x,\phi+t') + \nu_{32}\exp(-\gamma_3)\,a_2(x,\phi_3+t') - \frac{\partial G(x,t')}{\partial x}, \quad (6.32)$$

$$\frac{\delta\mathscr{L}_1}{\delta G(x,t)} = \frac{\partial F(x,t')}{\partial x} - [\nu_{13}\exp(\gamma)\,G(x,t'-\phi) + \nu_{12}\exp(-\gamma_1)\,a_2(x,\phi_1+t')], \quad (6.33)$$

where $t' = \tau - t$. Hence the equations $\delta\mathscr{L}_1/\delta G = 0$ and $\delta\mathscr{L}_1/\delta F = 0$ lead to the first-order partial differential equations (6.5) and (6.6) for $F(x,t)$ and $G(x,t)$ respectively.

Now consider the functional

$$\mathscr{H}_1 = \int_{-\infty}^{\infty} d\omega\,\bar{H}_1, \quad (6.34)$$

where $\bar{H}_1$ is the Hamiltonian (6.20) in (x,ω) space. Using the convolution theorem for Fourier transforms, the functional (6.34) can be expressed in the form

$$\mathscr{H}_1 = \int_{-\infty}^{\infty} dt\,[\,\tfrac{1}{2}\nu_{13}\exp(\gamma)\,G(x,-\phi-t)G(x,t) - \tfrac{1}{2}\nu_{31}\exp(-\gamma)\,F(x,\phi-t)F(x,t)$$
$$+\nu_{12}\exp(-\gamma_1)\,a_2(x,\phi_1-t)G(x,t)$$
$$-\nu_{32}\exp(-\gamma_3)\,a_2(x,\phi_3-t)F(x,t)\,]. \quad (6.35)$$

Straightforward computation of the variational derivatives $\delta\mathscr{H}_1/\delta F(x,t)$ and

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$\delta\mathscr{H}_1/\delta G(x,t)$ yields the formulae

$$\frac{\delta\mathscr{H}_1}{\delta F(x,t)} = -[\nu_{31}\exp(-\gamma)\,F(x,\phi-t) + \nu_{32}\exp(-\gamma_3)a_2\,(x,\phi_3-t)], \tag{6.36}$$

$$\frac{\delta\mathscr{H}_1}{\delta G(x,t)} = \nu_{13}\exp(\gamma)\,G(x,-\phi-t) + \nu_{12}\exp(-\gamma_1)\,a_2(x,\phi_1-t). \tag{6.37}$$

Using the results (6.36) and (6.37), it follows that the first-order partial differential equations (6.5) and (6.6) for $F(x,t)$ and $G(x,t)$ may be cast in the Hamiltonian form

$$\frac{\partial p}{\partial x} = -\frac{\delta\mathscr{H}_1}{\delta q}, \qquad \frac{\partial q}{\partial x} = \frac{\delta\mathscr{H}_1}{\delta p}, \tag{6.38}$$

where

$$q = F(x,t), \quad p = G(x,-t) \tag{6.39}$$

are the canonical variables. Alternatively, one can use $q = F(x,-t)$ and $p = G(x,t)$ as the canonical variables.

Using the standard Poisson bracket $\{U,V\}$ for functionals $U(p,q)$ and $V(p,q)$,

$$\{U,V\} = \int_{-\infty}^{\infty}\left(\frac{\delta U}{\delta q}\frac{\delta V}{\delta p} - \frac{\delta U}{\delta p}\frac{\delta V}{\delta q}\right)dt, \tag{6.40}$$

Hamilton's equations (6.38) may be written in the form

$$\frac{\partial p}{\partial x} = \{p,\mathscr{H}_1\}, \qquad \frac{\partial q}{\partial x} = \{q,\mathscr{H}_1\}. \tag{6.41}$$

This latter formulation is useful in the consideration of canonical transformations, since the Poisson bracket is invariant under canonical transformations.

In terms of the non-canonical variables f and g the Poisson bracket (6.40) reduces to

$$\{U,V\} = \exp(\gamma_1+\gamma_3)\int_{-\infty}^{\infty}dt\ \left(\frac{\delta U}{\delta f(x,\phi_1+t)}\frac{\delta V}{\delta g(x,\phi_3-t)} - \frac{\delta U}{\delta g(x,\phi_3-t)}\frac{\delta V}{\delta f(x,\phi_1-t)}\right). \tag{6.42}$$

Use of the Poisson bracket (6.42) leads to the results

$$\frac{\partial}{\partial x}[\exp(-\gamma_1)\,f(x,t)] = \{\exp(-\gamma_1)\,f(x,t),\mathscr{H}_1\} = \frac{\delta\mathscr{H}_1}{\delta[\exp(-\gamma_3)\,g(x,t')]}, \tag{6.43a}$$

$$\frac{\partial}{\partial x}[\exp(-\gamma_3)\,g(x,t')] = \{\exp(-\gamma_3)\,g(x,t'),\mathscr{H}_1\} = -\frac{\delta\mathscr{H}_1}{\delta[\exp(-\gamma_1)\,f(x,t)]}, \tag{6.43b}$$

where

$$t' = \phi_1+\phi_3-t. \tag{6.44}$$

Equations (6.43) and (6.44) show that

$$q_2 = \exp(-\gamma_1)\,f(x,t), \qquad p_2 = \exp(-\gamma_3)\,g(x,\phi_1+\phi_3-t) \tag{6.45}$$

may also be used as canonical variables in Hamilton's equations (6.38).

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7. Long-wavelength wave–wave interactions

In this section we consider small-amplitude, weakly nonlinear waves for the two-fluid model for cases where the wavelength of the wave λ is much greater than the characteristic diffusive length scale κ/V_{p0}, where κ is the diffusion coefficient and V_{p0} is a characteristic wave speed. We investigate the interaction between the four long-wavelength modes, namely the backward and forward sound waves, the contact discontinuity, and the pressure-balance mode. The pressure balance mode and contact discontinuity are both advected with the flow. The resonant wave–wave interactions in the long-wavelength limit differ from short-wavelength waves owing to the fact that the two fluids are more effectively coupled on the long scale length, and owing to the role of the pressure-balance mode at long wavelengths.

We consider long-wavelength waves about a uniform background state $\boldsymbol{\Psi}^{(0)} = (\rho^0, u^0, p_g^0, p_c^0)$ of the form:

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}^{(0)} + \epsilon\boldsymbol{\Psi}^{(1)} + \epsilon^2\boldsymbol{\Psi}^{(2)} + \dots, \tag{7.1}$$

where the wave perturbations $\{\boldsymbol{\Psi}^{(i)}(\bar{x}, \bar{t}, \tau) \,:\, i \geqslant 1\}$ depend on the normalized space and time variables $\bar{x}$ and $\bar{t}$ of (3.12), namely $\bar{x} = x/L_1$ and $\bar{t} = t/T_1$. The length scale L_1 is chosen to be of the order of the wavelength of the wave, which is much larger than the diffusive scale length κ/V_{p0}. Hence the appropriate scaling for the normalized diffusion coefficient is

$$\bar{\kappa} = \frac{\kappa}{V_{p0}L_1} = \kappa_0\epsilon, \tag{7.2}$$

where κ_0 is a constant of order unity. We are interested in the evolution of the wave on the long time scale

$$\tau = \epsilon\bar{t}. \tag{7.3}$$

When only one sound wave is present, the canonical wave equation obtained is the dissipative Burgers equation (see e.g. Webb and McKenzie 1984; Zank *et al.* 1988). On the other hand, if only the pressure-balance mode and contact discontinuity are present, one obtains at lowest-order coupled linear equations for the density and thermal gas pressure in which the gas pressure perturbation is diffusively smoothed by the cosmic rays (Webb *et al.* 1995). The presence of the pressure-balance mode ensures that the sum of the gas pressure and cosmic-ray pressure perturbations is zero across the structure, whereas the change of the gas pressure across the structure deviates from that expected from adiabatic gasdynamics owing to the presence of the entropy wave modes.

Substitution of the expansion (7.1) into (3.15) yields the set of perturbation equations

$$\begin{aligned}\Bigg[\frac{\partial}{\partial\bar{t}} + \epsilon\frac{\partial}{\partial\tau} + [\mathbf{A}_0 + (\epsilon\boldsymbol{\Psi}^{(1)} + \epsilon^2\boldsymbol{\Psi}^{(2)} + \dots)\cdot\nabla_{\boldsymbol{\Psi}}\mathbf{A} \\ + \tfrac{1}{2}(\epsilon\boldsymbol{\Psi}^{(1)} + \epsilon^2\boldsymbol{\Psi}^{(2)} + \dots)(\epsilon\boldsymbol{\Psi}^{(1)} + \epsilon^2\boldsymbol{\Psi}^{(2)} + \dots) : \nabla_{\boldsymbol{\Psi}}\nabla_{\boldsymbol{\Psi}}\mathbf{A}]\cdot\frac{\partial}{\partial\bar{x}} \\ + \epsilon\tilde{\mathbf{B}}_0\frac{\partial^2}{\partial\bar{x}^2}\Bigg](\boldsymbol{\Psi}^{(0)} + \epsilon\boldsymbol{\Psi}^{(1)} + \dots) = 0,\end{aligned} \tag{7.4}$$

where

$$B_{0ij} = \epsilon\tilde{B}_{0ij} = -\kappa_0\epsilon\delta_{i4}\delta_{j4} \tag{7.5}$$

is the cosmic-ray diffusion matrix.

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Balance of terms at $O(\epsilon)$ in (7.4) yields the matrix equation

$$\frac{\partial \mathbf{\Psi}^{(1)}}{\partial \bar{t}} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \bar{x}} = 0. \tag{7.6}$$

Similarly, balance of terms at $O(\epsilon^2)$ yields the equations

$$\frac{\partial \mathbf{\Psi}^{(2)}}{\partial \bar{t}} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(2)}}{\partial \bar{x}} + \mathbf{F} = 0, \tag{7.7}$$

where

$$\mathbf{F} = \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \tau} + \mathbf{\Psi}^{(1)} \cdot \nabla_{\mathbf{\Psi}} \mathbf{A} \cdot \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \bar{x}} + \tilde{\mathbf{B}}_0 \cdot \frac{\partial^2 \mathbf{\Psi}^{(1)}}{\partial \bar{x}^2}. \tag{7.8}$$

Following a development similar to the short-wavelength perturbation analysis of Sec. 3, we search for solutions for $\mathbf{\Psi}^{(1)}$ and $\mathbf{\Psi}^{(2)}$ of the form

$$\mathbf{\Psi}^{(1)} = \sum_{s=1}^{4} a_s(\theta_s, \tau)\mathbf{R}_s, \tag{7.9}$$

$$\mathbf{\Psi}^{(2)} = \sum_{s=1}^{4} b_s(\theta_s, \bar{t}, \tau)\mathbf{R}_s, \tag{7.10}$$

where a_s and b_s determine the wave amplitudes, and

$$\theta_s = k_s \bar{x} - \omega_s \bar{t}, \qquad \lambda_s = \frac{\omega_s}{k_s} \tag{7.11}$$

are the phase and wave phase speed of the sth mode. Substitution of the solution ansatz (7.9) into (7.6) yields the matrix eigenequation:

$$(\mathbf{A}_0 - \lambda_s \mathbf{I}) \cdot \mathbf{R}_s = 0, \tag{7.12}$$

where the eigenvalues satisfy the eigenvalue equation

$$\det(\mathbf{A}_0 - \lambda \mathbf{I}) = (\lambda - \bar{u})^2[(\lambda - \bar{u})^2 - \bar{a}_*^2] = 0. \tag{7.13}$$

The long-wavelength sound speed $\bar{a}_*$ in (7.13) is given by

$$\bar{a}_*^2 = \bar{a}_c^2 + \bar{a}_g^2, \qquad \bar{a}_g^2 = \frac{\bar{p}_g}{\bar{\rho}}, \qquad \bar{a}_c^2 = \frac{\gamma_c \bar{p}_c}{\gamma_g \bar{\rho}}, \tag{7.14}$$

where we have adopted the normalization $\bar{a}_{g0}^2 = 1$. Without loss of generality, we can choose $\bar{u} = u^0 = 0$ for the background state. Since the cosmic rays effectively couple to the thermal gas on the long scale length, the effective long-wavelength sound speed $\bar{a}_*$ has both cosmic-ray and thermal gas-pressure contributions in (7.14) (see Ptuskin 1981). For further analysis we also need suitably normalized left eigenvectors of the matrix $\mathbf{A}_0$ satisfying

$$\mathbf{L}_s \cdot (\mathbf{A}_0 - \lambda_s \mathbf{I}) = 0, \qquad \mathbf{L}_s \cdot \mathbf{R}_k = \delta_{sk}, \tag{7.15}$$

where δ_{sk} is the Kronecker delta symbol.

The eigenvalue and eigenvector solutions of (7.12)–(7.15) correspond to the backward sound wave, with

$$\lambda_1 = \bar{u} - \bar{a}_*, \qquad \mathbf{R}_1 = \left(1, -\frac{\bar{a}_*}{\bar{\rho}}, \frac{\gamma_g \bar{p}_g}{\bar{\rho}}, \frac{\gamma_c \bar{p}_c}{\bar{\rho}}\right)^t, \qquad \mathbf{L}_1 = \frac{1}{2}\left(0, -\frac{\bar{\rho}}{\bar{a}_*}, \frac{1}{\gamma_g \bar{a}_*^2}, \frac{1}{\gamma_g \bar{a}_*^2}\right), \tag{7.16}$$

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the contact discontinuity, with

$$\lambda_2 = \bar{u}, \qquad \mathbf{R}_2 = (1,0,0,0)^t, \qquad \mathbf{L}_2 = \left(1, 0, -\frac{1}{\gamma_g \bar{a}_*^2}, -\frac{1}{\gamma_g \bar{a}_*^2}\right), \tag{7.17}$$

the pressure-balance mode, with

$$\lambda_3 = \bar{u}, \qquad \mathbf{R}_3 = (0,0,1,-1)^t, \qquad \mathbf{L}_3 = \left(0, 0, \frac{\bar{a}_c^2}{\bar{a}_*^2}, -\frac{\bar{a}_g^2}{\bar{a}_*^2}\right), \tag{7.18}$$

and the forward sound wave, with

$$\lambda_4 = \bar{u} + \bar{a}_*, \qquad \mathbf{R}_4 = \left(1, \frac{\bar{a}_*}{\bar{\rho}}, \frac{\gamma_g \bar{p}_g}{\bar{\rho}}, \frac{\gamma_c \bar{p}_c}{\bar{\rho}}\right)^t, \qquad \mathbf{L}_4 = \frac{1}{2}\left(0, \frac{\bar{\rho}}{\bar{a}_*}, \frac{1}{\gamma_g \bar{a}_*^2}, \frac{1}{\gamma_g \bar{a}_*^2}\right). \tag{7.19}$$

The above eigenvector solution (7.9) for $\boldsymbol{\Psi}^{(1)}$ is readily verified to satisfy the $O(\epsilon)$ balance equation (7.6).

Taking the scalar product of (7.7) with the left eigenvectors $\{\mathbf{L}_j : j = 1, \ldots, 4\}$ leads to a system of four wave evolution equations

$$\frac{\partial b_j(\theta_j, \bar{t}, \tau)}{\partial \bar{t}} + \mathbf{L}_j \cdot \mathbf{F} = 0, \qquad j = 1, \ldots, 4, \tag{7.20}$$

analogous to (3.47) in the short-wavelength limit. For a uniform perturbation expansion, we require $b_j(\theta_j, \bar{t}, \tau)$ to be sublinear in $\bar{t}$. Thus we require

$$\lim_{\bar{t}\to\infty} \frac{b_j(\theta_j, \bar{t}, \tau)}{\bar{t}} = 0, \tag{7.21}$$

for a uniform expansion. As in the short-wavelength analysis, integration of (7.20) over the short time scale $\bar{t}$, with θ_j held fixed, leads to averaged wave evolution equations of the form

$$\langle \mathbf{L}_j \cdot \mathbf{F} \rangle_j = 0, \qquad j = 1, \ldots, 4, \tag{7.22}$$

where the averaging operator is as defined in (3.50), but with τ replaced by $\bar{t}$, i.e. the average of a function f dependent on θ_j, $\bar{t}$ and τ is defined as

$$\langle f \rangle_j = \lim_{\bar{t}\to\infty} \frac{1}{\bar{t}} \int_0^{\bar{t}} f|_{\theta_j} d\bar{t}'. \tag{7.23}$$

7.1. Wave evolution equations

The wave evolution equations (7.20) prior to averaging may be expressed in the form

$$\frac{\partial b_j(\theta_j, \bar{t}, \tau)}{\partial \bar{t}} + \frac{\partial a_j(\theta_j, \tau)}{\partial \tau} + T_j + D_j = 0, \tag{7.24}$$

where

$$T_j = \sum_{p=1}^{4} \sum_{q=1}^{4} \Gamma_{jpq} a_q \frac{\partial a_p}{\partial \bar{x}} \tag{7.25}$$

is the nonlinear interaction term, and

$$\mathbf{D} = -\frac{\kappa_0}{2\gamma_g \bar{a}_*^2} \frac{\partial^2 p_c^1}{\partial \bar{x}^2} (1, -2, -2\gamma_g \bar{a}_g^2, 1)^t, \tag{7.26}$$

$$p_c^1 = \frac{\gamma_c \bar{p}_c}{\bar{\rho}}(a_1 + a_4) - a_3, \tag{7.27}$$

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describes the dissipative effect of cosmic-ray diffusion. Note that the cosmic-ray pressure perturbation p_c^1 in (7.27) consists of contributions from the backward and forward cosmic-ray-modified sound waves (a_1 and a_4) as well as a contribution from the pressure-balance mode (the $-a_3$ term). The asymmetric interaction coefficients Γ_{jpq} are given by (3.54) and (3.55), except that the matrix $\mathscr{A}$ is replaced by $\mathbf{A}$, and the eigenvectors $\{\mathbf{R}_p\}$ and $\{\mathbf{L}_j\}$ now refer to the long-wavelength eigenvectors. The detailed forms of the nonlinear interaction terms (7.25) are given in Appendix C. Since the expansion is carried out about a uniform background state, there are no linear wave-mixing terms in (7.24).

By directly averaging the nonlinear interaction terms in Appendix C, or by using the formulae (3.54)–(3.55) and (3.68)–(3.69) of Sec. 3, we obtain expressions for $\langle T_j\rangle_j$ in the averaged wave evolution equations (7.22). The averaged equations are equivalent to the equations obtained by averaging (7.24) over $\bar{t}$, with θ_j held constant, in which the $\partial b_j(\theta_j,\bar{t},\tau)/\partial\bar{t}$ term averages to zero by the sublinearity constraint (7.21). The averaged equation for the backward sound wave is

$$\frac{\partial a_1(\theta_1,\tau)}{\partial\tau} + \langle T_1\rangle_1 - \frac{\kappa_0}{2\gamma_g\bar{a}_*^2}\frac{\partial^2\langle p_c^1\rangle_1}{\partial\bar{x}^2} = 0. \tag{7.28}$$

The nonlinear interaction term $\langle T_1\rangle_1$ may be expressed in both the non-conservative and conservative forms

$$\begin{aligned}\langle T_1\rangle_1 &= k_1\frac{\partial\hat{T}_1}{\partial\theta_1} - \frac{\bar{a}_*}{4\bar{\rho}}\left(k_2\langle a_4a_2'\rangle_1 + \frac{\gamma_g-\gamma_c}{\gamma_g\bar{a}_*^2}k_3\langle a_4a_3'\rangle_1\right)\\ &\equiv k_1\frac{\partial\hat{T}_1}{\partial\theta_1} - \frac{\bar{a}_*}{2\bar{\rho}}k_1\frac{\partial}{\partial\theta_1}\left(\langle a_4a_2\rangle_1 + \frac{\gamma_g-\gamma_c}{\gamma_g\bar{a}_*^2}\langle a_4a_3\rangle_1\right),\end{aligned} \tag{7.29}$$

where

$$\begin{aligned}\hat{T}_1 =& -\frac{[(\gamma_g+1)\bar{a}_g^2+(\gamma_c+1)\bar{a}_c^2]}{4\bar{\rho}\bar{a}_*}a_1^2\\ &+a_1\left(\frac{(3-\gamma_g)\bar{a}_g^2+(3-\gamma_c)\bar{a}_c^2}{2\bar{\rho}\bar{a}_*}\langle a_4\rangle + \frac{\bar{a}_*}{2\bar{\rho}}\langle a_2\rangle - \frac{\gamma_g-\gamma_c}{2\gamma_g\bar{\rho}\bar{a}_*}\langle a_3\rangle\right).\end{aligned} \tag{7.30}$$

The nonlinear interaction terms in (7.29) and (7.30) show that the backward sound wave (a_1) may be produced by resonantly scattering the forward sound wave (a_4) off either the entropy wave (a_2) or off the pressure-balance mode (a_3). Hence in the long wavelength limit, as opposed to the short-wavelength limit, resonant wave reflection can also occur between the pressure-balance mode and a sound wave to produce the reverse sound wave. The nonlinear terms in $\hat{T}_1$ in (7.29) and (7.30) represent the interaction of the sound wave with itself (the usual nonlinear Burgers term $\propto a_1^2$ for unidirectional waves), and other terms representing the interaction of the backward sound wave with the mean wave fields of the other modes. It is of interest to note that the terms in $\langle T_1\rangle_1$ involving the pressure-balance mode are small in the limit of a soft cosmic-ray momentum spectrum, for which $\gamma_c \approx \gamma_g$.

The averaged equation for the entropy mode is

$$\frac{\partial a_2(\theta_2,\tau)}{\partial\tau} + \langle T_2\rangle_2 + \frac{\kappa_0}{\gamma_g\bar{a}_*^2}\frac{\partial^2\langle p_c^1\rangle_2}{\partial\bar{x}^2} = 0, \tag{7.31}$$

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where the wave–wave interaction term is given by

$$\langle T_2\rangle_2 = \frac{\bar{a}_*}{\bar{\rho}} k_2 \frac{\partial}{\partial\theta_2}[a_2(\langle a_4\rangle - \langle a_1\rangle)]. \tag{7.32}$$

Similarly the pressure-balance-mode equation is

$$\frac{\partial a_3(\theta_3,\tau)}{\partial\tau} + \langle T_3\rangle_3 + \frac{\kappa_0\bar{a}_g^2}{\bar{a}_*^2}\frac{\partial^2\langle p_c^1\rangle_3}{\partial\bar{x}^2} = 0, \tag{7.33}$$

where

$$\langle T_3\rangle_3 = \frac{\bar{a}_*}{\bar{\rho}} k_3 \frac{\partial}{\partial\theta_3}[a_3(\langle a_4\rangle - \langle a_1\rangle)]. \tag{7.34}$$

Thus both the entropy mode and the pressure-balance mode can be generated by non-resonant interaction between the modes and the net mean velocity field

$$\langle v_4\rangle + \langle v_1\rangle = \frac{\bar{a}_*}{\bar{\rho}}(\langle a_4\rangle - \langle a_1\rangle), \tag{7.35}$$

of the sound waves, but cannot be generated by resonant wave–wave interactions. It is of interest to note that the cosmic-ray diffusive terms in (7.31) and (7.33) for the entropy-mode and pressure-balance-mode equations are of opposite sign to the corresponding diffusion term in the sound-wave equation (7.28).

The equation for the forward sound wave is similar to the backward sound-wave equation, namely

$$\frac{\partial a_4(\theta_4,\tau)}{\partial\tau} + \langle T_4\rangle_4 - \frac{\kappa_0}{2\gamma_g\bar{a}_*^2}\frac{\partial^2\langle p_c^1\rangle_4}{\partial\bar{x}^2} = 0. \tag{7.36}$$

The non-conservative and conservative forms of the nonlinear interaction term $\langle T_4\rangle_4$ are given by

$$\begin{aligned}\langle T_4\rangle_4 &= k_4\frac{\partial\hat{T}_4}{\partial\theta_4} + \frac{\bar{a}_*}{4\bar{\rho}}\left(k_2\langle a_1 a_2'\rangle_4 + \frac{\gamma_g-\gamma_c}{\gamma_g\bar{a}_*^2}k_3\langle a_1 a_3'\rangle_4\right)\\ &\equiv k_4\frac{\partial\hat{T}_4}{\partial\theta_4} + \frac{\bar{a}_*}{2\bar{\rho}}k_4\frac{\partial}{\partial\theta_4}\left(\langle a_1 a_2\rangle_4 + \frac{\gamma_g-\gamma_c}{\gamma_g\bar{a}_*^2}\langle a_1 a_3\rangle_4\right),\end{aligned} \tag{7.37}$$

where

$$\begin{aligned}\hat{T}_4 = {}& \frac{(\gamma_g+1)\bar{a}_g^2 + (\gamma_c+1)\bar{a}_c^2}{4\bar{\rho}\bar{a}_*}a_4^2\\ &+a_4\left(\frac{(\gamma_g-3)\bar{a}_g^2 + (\gamma_c-3)\bar{a}_c^2}{2\bar{\rho}\bar{a}_*}\langle a_1\rangle - \frac{\bar{a}_*}{2\bar{\rho}}\langle a_2\rangle + \frac{\gamma_g-\gamma_c}{2\gamma_g\bar{\rho}\bar{a}_*}\langle a_3\rangle\right).\end{aligned} \tag{7.38}$$

Hence the forward sound wave may be generated by resonant reflection of the backward sound wave off the entropy wave and pressure-balance mode.

8. Concluding remarks

In this paper we have applied a formalism developed by Majda and Rosales (1984) and Hunter *et al.* (1986) to describe both non-resonant and resonant three-wave interactions for hyperbolic waves to the equations of two-fluid cosmic-ray hydrodynamics in the short- and long-wavelength limits.

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Equations have been obtained that describe the evolution of short-wavelength waves in a non-uniform large-scale flow (Sec. 3). The wave evolution equations obtained consist of Burgers equations for the backward and forward sound waves coupled by linear integral operators that describe resonant reflection of a sound wave off an entropy-wave disturbance to produce a reverse sound wave. Resonant wave interactions are more liable to be significant for extended periodic or near-periodic wave trains than for isolated pulses. More precise conditions for significant resonant wave interactions to occur are discussed by Majda and Rosales (1984) and Hunter *et al.* (1986). On the other hand, the entropy wave in the model may be generated by non-resonant interaction between the entropy wave and the mean wave velocity field of the sound waves. The equations also contain terms describing the reflection and interaction of the sound waves with the large-scale background flow, which are analogous to the wave mixing terms obtained by Heinemann and Olbert (1980) in their model of bidirectional Alfvén waves in the solar wind (see also Zhou and Matthaeus 1990*a*,*b*). The study of three-wave resonant interactions in non-uniform flows is in general complicated by the fact that waves in resonance in a localized region of (x, t) space will in general pass out of resonance, since the ray equations for the waves depend on the large space and time variables characteristic of the flow.

In Sec. 4 the equations describing the resonant interaction of periodic sound waves and entropy waves in a uniform background medium derived by Majda and Rosales (1984) have been obtained as a limiting case of the wave evolution equations for the cosmic-ray model.

In Sec. 5 we have showed how the linearized versions of the equations are related to a wave-action equation (or canonical wave-energy equation) for both backward and forward sound waves, similar to the wave-action equation obtained by Heinemann and Olbert (1980) for bidirectional Alfvén waves in the solar wind. The wave-action equation generalizes the corresponding equation obtained by Drury and Falle (1986) and Zank and McKenzie (1987) to the case where all three short-wavelength modes are present and interact with the large-scale background flow. The formulation thus gives a generalization of the wave-action equation for the squeezing instability first investigated in detail by Drury and Falle (1986) and Zank and McKenzie (1987). Perhaps the major limitation of the wave-action-equation formulation is that it applies to regions of the flow that are free of shocks or shocklets, which in general are expected to form owing to wave steepening associated with the nonlinear Burgers term. A WKB analysis of the role of wave mixing in the linear wave evolution equations has also been carried out in Sec. 5. Two simplified problems have been considered: (*a*) the interaction between the backward and forward sound waves of frequency ω in the absence of the contact discontinuity, and (*b*) the generation of sound waves arising from contact discontinuities initially present in the flow. These problems illustrate how the cosmic-ray squeezing instability is modified by wave mixing. In both problems the waves generated by wave mixing have amplitudes $O(1/\omega)$ that of the original wave, and phase shifts to lowest order (in the perturbation parameter $1/\omega$) of $\pm\frac{1}{2}\pi$ relative to the original wave at the point of generation. For the case of steady flows the contact discontinuity appears as a source in the sound-wave equations.

Lagrangian and Hamiltonian variational principles for the linear wave-mixing equations (5.15)–(5.18) for the sound waves and the contact discontinuity propagating through a steady large-scale background flow have been derived in Sec. 6.

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The equations may be written in the Hamiltonian form (6.1), in which the canonical coordinate $q(x,t)$ corresponds to the backward sound-wave perturbation amplitude, and the canonical momentum $p(x,t)$ is related to the time-reversed, forward sound-wave perturbation amplitude. Further work on the variational form of the equations is under investigation.

In the long-wavelength limit there are four basic wave modes of interest, namely the backward and forward cosmic-ray-modified sound waves, the contact discontinuity, and the cosmic-ray pressure-balance mode (Sec. 7). Both the contact discontinuity and the cosmic-ray pressure-balance mode are advected with the background flow, and have non-zero entropy perturbations, whereas the sound waves are isentropic. In this limit the sound waves can be resonantly reflected off either the contact discontinuity or the pressure-balance mode. However, the entropy wave and pressure balance mode cannot be generated by resonant wave interactions, but may be generated by non-resonant interaction between these modes and the mean wave velocity field of the sound waves. In the limit that only one sound wave is present one obtains the usual Burgers-equation description of weak cosmic-ray-modified shocks (Webb and McKenzie 1984; Zank *et al.* 1988). For the case where only the entropy wave and the pressure-balance mode are present, one obtains coupled linear equations for the density and thermal gas pressure in which the gas pressure perturbation is diffusively smoothed by the cosmic rays, and a transfer of entropy occurs between the two modes during the evolution (Webb *et al.* 1995). The pressure-balance mode ensures that the sum of the cosmic-ray and thermal gas pressure perturbations is zero across the structure, whereas the change of the gas pressure across the structure deviates from that expected from isentropic gas-dynamics owing to the presence of the entropy disturbances initially present in the flow.

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Appendix A

In this appendix we show that the mean of the wave amplitude $\langle a_p \rangle_j$ over the short time scale τ, with θ_j held fixed, is independent of θ_j. By the definition (3.50),

$$\begin{aligned} \langle a_p \rangle_j &= \lim_{T\to\infty} \frac{1}{T} \int_0^T a_p \left(\frac{k_p \theta_j + \nu_{jp} \tau}{k_j} \right) d\tau, \\ &\equiv \lim_{|T_{jp}|\to\infty} \frac{1}{T_{jp}} \int_{\alpha_{jp}}^{\alpha_{jp}+T_{jp}} a_p(\theta_p) d\theta_p \qquad (p \neq j), \end{aligned} \tag{A 1}$$

where

$$\alpha_{jp} = \frac{k_p \theta_j}{k_j}, \qquad T_{jp} = \frac{T \nu_{jp}}{k_j}. \tag{A 2}$$

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For large but finite T, differentiation of (A1) yields

$$\frac{\partial\langle a_p\rangle_j}{\partial\theta_j} = \lim_{|T_{jp}|\to\infty}[a_p(\theta_p)]_{\alpha_{jp}}^{\alpha_{jp}+T_{jp}}\left(\frac{k_p}{k_jT_{jp}}\right) = 0. \tag{A 3}$$

Hence $\langle a_p\rangle_j$ is independent of θ_j for $p \neq j$. The proof that $\langle a_j\rangle_j$ is independent of θ_j is trivial. For the case of perturbations about a uniform background state, the $\langle a_p\rangle_j$ are determined by the initial data. Since $\langle a_p\rangle$ is independent of θ_j, it is appropriate to drop the subscript j on the average.

Appendix B

In this appendix we provide a derivation of (5.24) for steady background flows, namely

$$\frac{d\omega_j}{dt_j} = \frac{\partial\omega_j}{\partial\bar{t}} + V_{gj}\frac{\partial\omega_j}{\partial\bar{x}} = 0. \tag{B 1}$$

Equation (B1) follows from the dispersion equation, written in the form

$$\frac{\partial\theta_j}{\partial\bar{t}} + W(\nabla\theta_j, \bar{x}, \bar{t}) = 0, \tag{B 2}$$

with $\omega_j \equiv W(\nabla\theta_j, \bar{x}, \bar{t})$. Equation (B2) is the Hamilton–Jacobi equation for the phase θ_j. The Cauchy characteristics of the first-order partial differential equation (B2) are

$$\frac{d\bar{x}}{d\bar{t}} = \frac{\partial W}{\partial k} \equiv V_{gj}, \qquad \frac{dk}{dt} = -\frac{\partial W}{\partial\bar{x}}, \tag{B 3}$$

where $k \equiv k_j$. Equations (B3) are Hamilton's equations for the rays, with Hamiltonian W. It follows from (B3) that, on the ray paths,

$$\frac{d\omega_j}{dt_j} \equiv \frac{dW}{d\bar{t}} = \frac{\partial W}{\partial\bar{t}} + \frac{\partial W}{\partial k}\frac{dk}{d\bar{t}} + \frac{\partial W}{\partial\bar{x}}\frac{d\bar{x}}{d\bar{t}} = \frac{\partial W}{\partial\bar{t}}. \tag{B 4}$$

Similarly, on the characteristics,

$$\frac{d\theta_j}{dt} = \frac{\partial\theta_j}{\partial\bar{t}} + \frac{\partial\theta_j}{\partial\bar{x}}\frac{d\bar{x}}{d\bar{t}} = -W + k\frac{\partial W}{\partial k}. \tag{B 5}$$

For a time-independent background flow, $\partial W/\partial\bar{t} = 0$, and (B4) then reduces to (B1). One can also view the result (B1) as a consequence of Noether's theorem for a Hamiltonian system in which the Hamiltonian does not depend explicitly on time $\bar{t}$. A more direct proof of (B1) for time-independent background flows follows from noting that

$$\frac{d\omega_j}{dt_j} = \frac{\partial\omega_j}{\partial k_j}\frac{\partial k_j}{\partial\bar{t}} + V_{gj}\frac{\partial\omega_j}{\partial\bar{x}} = V_{gj}\left(\frac{\partial k_j}{\partial\bar{t}} + \frac{\partial\omega_j}{\partial\bar{x}}\right) = 0, \tag{B 6}$$

since

$$\frac{\partial k_j}{\partial\bar{t}} + \frac{\partial\omega_j}{\partial\bar{x}} = \frac{\partial^2\theta_j}{\partial\bar{t}\partial\bar{x}} - \frac{\partial^2\theta_j}{\partial\bar{x}\partial\bar{t}} = 0. \tag{B 7}$$

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Appendix C

In this appendix we provide expressions for the nonlinear wave–wave interaction terms in (7.25) in the long-wavelength limit. For the backward sound wave, we have

$$T_1 = \frac{(\gamma_g+1)\bar{a}_g^2+(\gamma_c+1)\bar{a}_c^2}{2\bar{\rho}\bar{a}_*}\left(-a_1\frac{\partial a_1}{\partial\bar{x}}+a_1\frac{\partial a_4}{\partial\bar{x}}+a_4\frac{\partial a_4}{\partial\bar{x}}\right)$$
$$-\frac{[(\gamma_g-3)\bar{a}_g^2+(\gamma_c-3)\bar{a}_c^2]}{2\bar{\rho}\bar{a}_*}a_4\frac{\partial a_1}{\partial\bar{x}}+\frac{\bar{a}_*}{2\bar{\rho}}a_2\left(\frac{\partial a_1}{\partial\bar{x}}+\frac{\partial a_4}{\partial\bar{x}}\right)$$
$$+\frac{\gamma_g-\gamma_c}{2\gamma_g\bar{\rho}\bar{a}_*}a_3\left(\frac{\partial a_4}{\partial\bar{x}}-\frac{\partial a_1}{\partial\bar{x}}\right). \tag{C 1}$$

The nonlinear term for the entropy wave is

$$T_2 = \frac{(\gamma_g+1)\bar{a}_g^2+(\gamma_c+1)\bar{a}_c^2}{\bar{\rho}\bar{a}_*}a_1\frac{\partial a_1}{\partial\bar{x}}+\frac{\bar{a}_*}{\bar{\rho}}\frac{\partial}{\partial\bar{x}}(a_2a_4-a_2a_1)$$
$$+\frac{(\gamma_g-1)\bar{a}_g^2+(\gamma_c-1)\bar{a}_c^2}{\bar{\rho}\bar{a}_*}\left(-a_1\frac{\partial a_4}{\partial\bar{x}}+a_4\frac{\partial a_1}{\partial\bar{x}}-a_4\frac{\partial a_4}{\partial\bar{x}}\right)$$
$$+\frac{\gamma_g-\gamma_c}{\gamma_g\bar{\rho}\bar{a}_*}a_3\frac{\partial}{\partial\bar{x}}(a_1-a_4). \tag{C 2}$$

The nonlinear term T_3 for the pressure-balance mode is

$$T_3 = \frac{\gamma_g(\gamma_g-\gamma_c)\bar{a}_c^2\bar{a}_g^2}{\bar{\rho}\bar{a}_*}(a_1+a_4)\frac{\partial}{\partial\bar{x}}(a_4-a_1)+\frac{\bar{a}_*}{\bar{\rho}}(a_4-a_1)\frac{\partial a_3}{\partial\bar{x}}$$
$$+\frac{\gamma_g\bar{a}_c^2+\gamma_c\bar{a}_g^2}{\bar{\rho}\bar{a}_*}a_3\frac{\partial}{\partial\bar{x}}(a_4-a_1). \tag{C 3}$$

The nonlinear term T_4 for the forward sound wave is

$$T_4 = \frac{(\gamma_g+1)\bar{a}_g^2+(\gamma_c+1)\bar{a}_c^2}{2\bar{\rho}\bar{a}_*}\left(-a_1\frac{\partial a_1}{\partial\bar{x}}-a_4\frac{\partial a_1}{\partial\bar{x}}+a_4\frac{\partial a_4}{\partial\bar{x}}\right)$$
$$+\frac{(\gamma_g-3)\bar{a}_g^2+(\gamma_c-3)\bar{a}_c^2}{2\bar{\rho}\bar{a}_*}a_1\frac{\partial a_4}{\partial\bar{x}}-\frac{\bar{a}_*}{2\bar{\rho}}a_2\frac{\partial}{\partial\bar{x}}(a_1+a_4)$$
$$+\frac{\gamma_g-\gamma_c}{2\gamma_g\bar{\rho}\bar{a}_*}a_3\frac{\partial}{\partial\bar{x}}(a_4-a_1). \tag{C 4}$$

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