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A note on composition operators on the disc and bidisc

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Abstract. In this note, we give a new necessary condition for the boundedness of the composition operator on the Dirichlet-type space on the disc, via a two dimensional change of variables formula. With the same formula, we characterize the bounded composition operators on the anisotropic Dirichlet-type spaces $\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$ induced by holomorphic self maps of the bidisc $\mathbb{D}^2$ of the form $\Phi(z_1, z_2) = (\phi_1(z_1), \phi_2(z_2))$. We also consider the problem of boundedness of composition operators $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2(\mathbb{D}^2)$ for general self maps of the bidisc, applying some recent results about Carleson measures on the Dirichlet space of the bidisc.

1 Introduction

In the present work, we will study the composition operators in the Dirichlet-type spaces on the setting of the unit disc and the bidisc. Moreover, the problem of boundedness of the composition operator between the Bergman and Dirichlet spaces on the bidisc $\mathbb{D}^2$ is studied. A standard reference for composition operators in the Hardy and Bergman space is the celebrated work of Shapiro [15]. Another very influential work is the article of Pau and Perez, in which the authors provide criteria for the boundedness, compactness and closed range composition operators on the Dirichlet-type space on the unit disc, mainly using the *generalized Nevanlinna counting function* and the pull-back measure. In this note we will provide a new sufficient condition for the boundedness of the composition operator acting on the Dirichlet space, which utilizes the *generalized Nevanlinna counting function*, along with some other novel techniques. More precisely, we will follow the approach of Pau and Perez on the bidisc setting, for symbols $\Phi = (\varphi(z_1), \varphi(z_2))$, and obtain a characterization for boundedness of the composition operator on the bidisc via the *generalized Nevanlinna counting function*. Also, we will provide some insight on the problem of boundedness of the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2(\mathbb{D}^2)$ using known results for Carleson measures on the Dirichlet space of the bidisc, as well as new results on the matter.

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The problem of boundedness of the composition operator on the bidisc and tridisc for the Hardy and Bergman spaces, has been treated by Kosinski and Bayart in their papers [6, 7, 12], respectively, for symbols Φ who are $\mathcal{C}^2$ smooth on the closed bidisc $\overline{\mathbb{D}^2}$. In our note, we establish a connection of the above mentioned problem with the Carleson measures for Dirichlet-type spaces on the bidisc. Let us now give some essential notation. For the whole length of the article, we denote by $\mathbb{D}$ the unit disc on the one-dimensional complex plane, and by $\varphi : \mathbb{D} \to \mathbb{D}$ a holomorphic self-map of the disc. For a holomorphic function f on the disc or the bidisc, we will write $f \in H(\mathbb{D})$ or $f \in \mathbb{D}^2$, respectively. By $\mathbb{D}^2 = \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$, we denote the bidisc in the two-dimensional complex plane, while by $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ we denote a holomorphic self-map of the unit bidisc. Consider now the composition operators

$$C_\varphi f = f \circ \varphi(z), z \in \mathbb{D}$$

and

$$C_\Phi f = f \circ \Phi(z_1, z_2) = f(\varphi_1(z_1, z_2), \varphi_2(z_1, z_2)),$$

where $z_1, z_2 \in \mathbb{D}$ and both φ_1, φ_2 map the bidisc to the unit disc. Of course, if f is a holomorphic function in one, or two variables, then the composition operator induces a holomorphic function on the disc or the bidisc, respectively.

1.1 Plan of the article

The presentation of the results of this note will go as follows: In Section 2, we prove a two-dimensional change of variables for *separated symbols*. We apply this useful tool to prove the two main results of this section, namely, a new necessary condition for boundedness of the composition operator acting on the Dirichlet space of the disc, and a characterization of the holomorphic self-maps of the bidisc of the form

$$\Phi = (\varphi_1(z_1), \varphi_2(z_2)).$$

Note that these symbols have one-dimensional functions as co-ordinate functions. By following the standard techniques of [14] we will prove a characterization of the boundedness of the composition operator acting on the anisotropic Dirichlet space $\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$ for these symbols.

In Section 3, we give a sufficient condition for the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2(\mathbb{D}^2)$ to be bounded, by proving a sufficient condition for a positive Borel measure to be Carleson measure on the Dirichlet space of the bidisc. We will also prove a necessary and sufficient condition for the boundedness of the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2(\mathbb{D}^2)$, applying a very important result of N.Arcozzi, P.Mozolyako, K-M.Perfekt, and G.Sarfatti, in their seminal work [3] which characterizes the Carleson measures in the Dirichlet space of the bidisc. The main result of this section can be stated only for the unweighted Dirichlet space. The condition we obtain is necessary and sufficient for the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2(\mathbb{D}^2)$ to be bounded but it is not as pleasant as one could wish because it involves a union of Carleson boxes and not one box as in [12, Theorem 10].

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1.2 Definitions and tools

In this first section, we will present some backround material, essential for the interested reader. First and foremost, we recall that the Dirichlet-type space $\mathfrak{D}_a(\mathbb{D})$ is consisted of all holomorphic functions on the disc, which satisfy

$$\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^a dA(z) < +\infty$$

for $0 < a < 1$. These spaces have been studied extensively in several works, some of which are [8] and [2]. The problem of boundedness and compactness of the composition operator in these spaces has been treated in [14]. Now we pass to the setting of the bidisc. The anisotropic Dirichlet-type spaces on the bidisc $\mathbb{D}^2$, denoted by $\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$, where $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$, are defined as follows: Firstly, we remind the interested reader that if $f \in H(\mathbb{D}^2)$ then f has a series expansion as follows:

$$f(z_1, z_2) = \sum_{k\geq 0}\sum_{k\geq 0} a_{k,l} z_1^k z_2^l.$$

We say that a holomorphic function f on the bidisc, belongs to the Dirichlet type spaces if

$$||f||^2_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)} = \sum_{k\geq 0}\sum_{k\geq 0}(k+1)^{2a_1}(l+1)^{2a_2}|a_{k,l}|^2 < \infty.$$

For $0 < a_1, a_2 \leq \frac{1}{2}$ we can obtain an equivalent integral representation of the norm of a function in the Dirichlet-type spaces. Specifically,

$$||f||^2_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)} = |f(0,0)|^2 + \mathfrak{D}_{\vec{a}}(f),$$

where,

$$\begin{aligned}\mathfrak{D}_{\vec{a}}(f) = &\int_{\mathbb{D}} |\partial_{z_1} f(z_1, 0)|^2 dA_{a_1}(z_1) + \int_{\mathbb{D}} |\partial_{z_2} f(0, z_2)|^2 dA_{a_2}(z_2) \\ &+ \int_{\mathbb{D}^2} |\partial_{z_2}\partial_{z_1} f(z_1, z_2)|^2 dA_{a_1}(z_1) dA_{a_2}(z_2), \; i = 1, 2 \end{aligned} \tag{1.1}$$

where

$$dA_{a_i}(z_k) = (1-|z_k|^2)^{1-2a_i} dA(z_k),$$

and $dA(z_k)$ is the Lebesgue measure on the disc, that is $dA(z_k) = \frac{1}{\pi} dx_k dy_k$. For the case where $a_1 = a_2 = 1/2$ we obtain the classical Dirichlet space of the bidisc, as it was defined in [10] by Kaptanoglu. It is an easy fact to check that these spaces are Hilbert spaces with respect to the norm defined above, with the usual, naturally induced inner product. Also, they are invariant under unitary transformations but not invariant under automorphisms of the bidisc, as it is showcased in [10]. These spaces enjoy the property of bounded point evaluation functional. For more on the basic theory of the Dirichlet type spaces on the bidisc one can study [10]. Furthermore, in the seminal work of Knese, Kosinski, Ransford, and Sola, [11] one can find characterizations about the cyclic polynomials on the above mentioned spaces. In the present note we introduce the problem of the boundedness of the composition operator. Until now, no work has been found in the literature regarding this specific problem. As an initial

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step, we characterize the symbols Φ of the form $\Phi(z_1, z_2) = (\varphi_1(z_1), \varphi_2(z_2)), z_1, z_2 \in \mathbb{D}$ (each φ_i, $i = 1, 2$ is a holomorphic self map of the unit disc $\mathbb{D}$) that induce bounded composition operator on $\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$. For these symbols the composition operator $C_\Phi : \mathfrak{D}_{\vec{a}}(\mathbb{D}^2) \to \mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$ is easier to tame. In the more general case, where both co-ordinate functions are mapping the bidisc to the disc, things are more difficult.

The general case, due to the insurmountable amount of calculations seems much more difficult to handle. To explain further, the techniques which allowed Bayart and Kosinski to prove the main results of their papers was the known characterization of the Carleson measures. Taking the corresponding pull-back measure provided them with the basic tool. On the Dirichlet-type spaces case, for general symbols, this is not easy to obtain, as there are derivations which significantly change the nature of the pull-back measure. Instead of the general case, we will consider an alternative problem, and that is of the boundedness of the composition operator between the classical Bergman space $A^2_\beta(\mathbb{D}^2), \beta > -1$, which is consisted of the holomorphic functions in $\mathbb{D}^2$ which satisfy

$$\int_{\mathbb{D}^2} |f(z_1, z_2)|^2 dV_\beta(z_1, z_2) < \infty,$$

where $dV_\beta(z_1, z_2) = dA_\beta(z_1) \times dA_\beta(z_2)$, and the Dirichlet space on the bidisc that we briefly discussed before. One can realise the difficulty and the technicality of the problem by studying the recent works of Bayart [6] and Kosinski [12], respectively, for the Hardy and Bergman spaces of the bidisc and tridisc, respectively.

2 A necessary condition and the case of separated symbols

Throughout this section we consider symbols Φ of the form

$$\Phi(z_1, z_2) = (\varphi_1(z_1), \varphi_2(z_2)), z_1, z_2 \in \mathbb{D},$$

where both φ_1, φ_2 are holomorphic self maps of the unit disc. We shall call these symbols "separated symbols" for convenience. In the sequel we will first prove a new sufficient condition for the boundedness of the composition operator acting on $\mathfrak{D}_a(\mathbb{D})$. In order to establish our results, a useful tool will be the *Generalized Nevanlinna Counting Function*

$$\mathcal{N}_{\varphi,\alpha}(z) = \sum_{\varphi(w)=z} \left(\log \frac{1}{|w|}\right)^{1-2a}, z \in \mathbb{D}\backslash\{\varphi(0)\},$$

where φ is a holomorphic self map of the disc. We will also need a recent characterization of the Dirichlet-type spaces, specifically the following result of Balooch and Wu [5].

Theorem 2.1 *Let $\sigma, \tau \geq -1$ and $\beta \in \mathbb{R}$, such that $\frac{\max(\sigma,\tau)}{2} - 1 < \beta \leq \frac{\sigma+\tau}{2}$. Let $f \in H(\mathbb{D})$. Then:*

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - \overline{w}z|^{2(\beta+2)}} dA_\sigma(z) dA_\tau(w) \asymp \|f\|^2_{\mathcal{D}_{\sigma+\tau-2\beta}(\mathbb{D})}.$$

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The notation "$\asymp$" means that the quantities on both sides are equivalent. For what follows, we denote by

$$k^{\varphi}(z_1, z_2) = \frac{1 - \varphi(z_1)\overline{\varphi(z_2)}}{1 - z_1\overline{z_2}}.$$

The main results of this section are the following.

Theorem 2.2 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic self map of the disc. Assume that $C_\varphi : \mathfrak{D}_a(\mathbb{D}) \to \mathfrak{D}_a(\mathbb{D})$ is bounded, for $a = 2\sigma - 2\beta, \sigma - 1 < \beta \leq \sigma$. Then:*

(1)

$$\sup_{z_1, z_2 \in \mathbb{D}} \frac{|k^{\varphi}(z_1, z_2)|}{|\varphi'(z_1) \cdot \varphi'(z_2)|^{\frac{1}{\beta+2}}} < +\infty$$

(2)

$$||C_\varphi||_{\text{operator}} \leq C \left(\sup_{z_1, z_2 \in \mathbb{D}} \frac{|k^{\varphi}(z_1, z_2)|}{|\varphi'(z_1) \cdot \varphi'(z_2)|^{\frac{1}{\beta+2}}} \right)^{\beta+2},$$

where $|| \cdot ||_{\text{operator}}$ denotes the operator norm.

The next result can also be seen as an application of the change of variables formula, and is the following.

Theorem 2.3 *Let $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ be a separated, holomorphic self map of the bidisc. Then, the composition operator $C_\Phi : \mathfrak{D}_{\vec{a}}(\mathbb{D}^2) \to \mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$ is bounded if and only if:*

$$\sup_{z_i \in \mathbb{D}} \frac{\mathcal{N}_{\varphi_i, a_i}(z_i)}{(1 - |z_i|^2)^{1-2a_i}} < \infty, i = 1, 2, 0 < a_1, a_2 < 1/2.$$

First we will prove Theorem 3.2. In order to be able to showcase the proof, we require only the change-of-variable formula that we mentioned in the introductory section of the article.

Lemma 2.1 *Let $g : \mathbb{D}^2 \to \mathbb{C}$ be a positive integrable function, and $\varphi_1, \varphi_2 : \mathbb{D} \to \mathbb{D}$ holomorphic self maps of the disc. Then:*

$$\begin{aligned} &\int_{\mathbb{D}^2} g(\varphi_1(z_1), \varphi_2(z_2))|\varphi_1'|^2|\varphi_2'|^2 dA_{a_1}(z_1) dA_{a_2}(z_2) \\ &= \int_{\mathbb{D}^2} g(w_1, w_2)\mathcal{N}_{\varphi_1, a_1}(w_1)\mathcal{N}_{\varphi_2, a_2}(w_2) dV(w_1, w_2). \end{aligned} \tag{2.1}$$

Proof Fix $z_2 \in \mathbb{D}$. Then the slice function $g_{z_2}(z_1)$ is integrable and positive in $\mathbb{D}$, hence the change of variable formula holds for the disc and for $w_1 = \varphi(z_1)$. That is:

$$\int_{\mathbb{D}} g(\varphi(z_1), \varphi_2(z_2))|\varphi_1'|^2 dA_{a_1}(z) = \int_{\mathbb{D}} g(w_1, \varphi_2(z_2))\mathcal{N}_{\varphi_1, a_1}(w_1) dA(w_1).$$

Next, we multiply both sides of the above equality by

$$(1 - |z_2|^2)^{1-2a_2}|\phi_2'(z_2)|^2$$

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and we integrate both sides into polar rectangles R_k. We obtain:

$$\int_{R_k}(1-|z_2|^2)^{1-2a_2}|\phi_2'|^2\left(\int_{\mathbb{D}}g(\varphi(z_1),\varphi_2(z_2))|\varphi_1'|^2dA_{a_1}(z_1)\right)dA(z_2)$$
$$=\int_{\mathbb{D}}\mathcal{X}_{R_k}(z_2)(1-|z_2|^2)^{1-2a_2}|\phi_2'|^2\left(\int_{\mathbb{D}}g(w_1,\varphi_2(z_2))\mathcal{N}_{\varphi_1,a_1}(w_1)dA(w_1)\right)dA(z_2), \tag{2.2}$$

where $\mathcal{X}_{R_k}$ denotes the characteristic function of each polar rectangle R_k. Summing both sides in terms of k and applying Fubini's theorem, we obtain the result. ∎

At this point we are ready to proceed with the proof of Theorem 3.2.

Proof Pick β,σ as in the statement of the Theorem. Set

$$g(z_1,z_2)=\frac{|f(z_1)-f(z_2)|^2}{|1-\overline{z_2}z_1|^{2\beta+2}},\ z_1,z_2\in\mathbb{D},$$

which is a positive integrable function on the bidisc. We receive:

$$\int_{\mathbb{D}^2}\frac{|f\circ\varphi(z_1)-f\circ\varphi(z_2)|^2}{|1-\overline{\varphi(z_2)}\varphi(z_1)|^{2(\beta+2)}}|\varphi'(z_1)|^2|\varphi'(z_2)|^2dV_\sigma(z_1,z_2)$$
$$=\int_{\mathbb{D}^2}\frac{|f(w_1)-f(w_2)|^2}{|1-\overline{w_2}w_1|^{2(\beta+2)}}\mathcal{N}_{\varphi,\sigma}(w_1)\mathcal{N}_{\varphi,\sigma}(w_2)dV(w_1,w_2)\le C\|f\|^2_{\mathfrak{D}_p(\mathbb{D})} \tag{2.3}$$

with the last inequality obtained by our assumption about the boundedness of the operator ([14, Theorem 3.1]) along with Theorem 3.1. Now we multiply and divide with $|1-\overline{z_2}z_1|^{2(\beta+2)}$ both the numerator and the denominator inside the left hand side integral and we obtain:

$$\int_{\mathbb{D}^2}\frac{|f\circ\varphi(z_1)-f\circ\varphi(z_2)|^2}{|1-\overline{z_2}z_1|^{2(\beta+2)}}\frac{|\varphi'(z_1)|^2|\varphi'(z_2)|^2}{|k^\varphi(z_1,z_2)|^{2(\beta+2)}}dV_\sigma(z_1,z_2)\le C\|f\|^2_{\mathfrak{D}_p(\mathbb{D})}.$$

Taking the infinum out of the integral, yields:

$$\inf_{z_1,z_2\in\mathbb{D}}\frac{|\varphi'(z_1)|^2|\varphi'(z_2)|^2}{|k^\varphi(z_1,z_2)|^{2(\beta+2)}}\int_{\mathbb{D}^2}\frac{|f\circ\varphi(z_1)-f\circ\varphi(z_2)|^2}{|1-\overline{z_2}z_1|^{2(\beta+2)}}dV_\sigma(z_1,z_2)\le C\|f\|^2_{\mathfrak{D}_p(\mathbb{D})}$$

which, by Theorem 3.1 and the properties of supremum, is equivalent to:

$$\|C_\varphi f\|^2_{\mathfrak{D}_p(\mathbb{D})}\le C\sup_{z_1,z_2\in\mathbb{D}}\frac{|k^\varphi(z_1,z_2)|^{2(\beta+2)}}{|\varphi'(z_1)|^2|\varphi'(z_2)|^2}\|f\|^2_{\mathfrak{D}_p(\mathbb{D})}.$$

Because of our assumption that the composition operator is bounded for all holomorphic functions f, the supremum has to be bounded. This provides us with the result. For the second part, we just observe that

$$\|C_\varphi\|_{\text{operator}}=\inf\{c\ge 0:\|C_\varphi\|_{\mathfrak{D}_p}\le c\|f\|_{\mathfrak{D}_p}\}\le C\left(\sup_{z_1,z_2\in\mathbb{D}}\frac{|k^\varphi(z_1,z_2)|}{|\varphi'(z_1)\cdot\varphi'(z_2)|^{\frac{1}{\beta+2}}}\right)^{\beta+2}$$

which comes straightforwardly by the inequality above. ∎

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Before we are able to prove the second result, we have to give an auxiliary lemma. We state the following lemma for the composition $f \circ \Phi$, where $\Phi = (\varphi_1(z_1), \varphi_2(z_2)), z_1, z_2 \in \mathbb{D}$, is a *separated symbol.*

Lemma 2.2 *Let $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ be a separated symbol. Then the following holds:*

$$\begin{aligned} \|C_\Phi f\|^2_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)} = &\int_{\mathbb{D}} |\partial_{z_1} f(z_1, 0)|^2 \mathcal{N}_{\varphi_1, a_1}(z_1) dA(z_1) \\ &+ \int_{\mathbb{D}} |\partial_{z_2} f(0, z_2)|^2 \mathcal{N}_{\varphi_2, a_2}(z_2) dA(z_2) \\ &+ \int_{\mathbb{D}^2} |\partial_{z_2}\partial_{z_1} f(z_1, z_2)|^2 \mathcal{N}_{\varphi_1, a_1}(z_1) \mathcal{N}_{\varphi_2, a_2}(z_2) dA(z_1) dA(z_2). \end{aligned} \tag{2.4}$$

Proof We apply the one-dimensional change of variables formula for the first two integrals defining the norm, and for the third integral we use the two-dimensional version of the change of variables we proved before. For the 1st term in the integral norm of the composition operator we just apply the well known change-of-variables formula for the function $g(z_1) = |\partial_{z_1} f(z_1, 0)|^2$, and similarly for the second term, we apply it for the function $g(z_2) = |\partial_{z_2} f(0, z_2)|^2$. For the third term we apply Lemma 3.1 for the function $g(z_1, z_2) = |\partial_{z_2}\partial_{z_1} f(z_1, z_2)|^2$. ∎

At this point we are ready for the proof of Theorem 3.3.

Proof (Proof of Theorem 3.3) For the sufficiency part, assume that there exist two positive constants C_1, C_2 such that:

$$\mathcal{N}_{\varphi_1, a_1}(z_1) \leq C_1 (1 - |z_1|^2)^{1-2a_1}$$

and

$$\mathcal{N}_{\varphi_2, a_2}(z_2) \leq C_2 (1 - |z_2|^2)^{1-2a_2}$$

then the following three inequalities hold:

$$\int_{\mathbb{D}} |\partial_{z_1} f(z_1, 0)|^2 \mathcal{N}_{\varphi_1, a_1}(z_1) dA(z_1) \leq C_1 \int_{\mathbb{D}} |\partial_{z_1} f(z_1, 0)|^2 dA_{a_1}(z_1),$$

$$\int_{\mathbb{D}} |\partial_{z_2} f(0, z_2)|^2 \mathcal{N}_{\varphi_2, a_2}(z_2) dA(z_2) \leq C_2 \int_{\mathbb{D}} |\partial_{z_2} f(0, z_2)|^2 dA_{a_2}(z_2),$$

and

$$\begin{aligned} &\int_{\mathbb{D}^2} |\partial_{z_2}\partial z_1 f(z_1, z_2)|^2 \mathcal{N}_{\varphi_1, a_1}(z_1) \mathcal{N}_{\varphi_2, a_2}(z_2) dA(z_1) dA(z_2) \\ &\qquad \leq C_1 C_2 \int_{\mathbb{D}^2} |\partial_{z_1}\partial_{z_2} f(z_1, z_2)|^2 dA_{a_1}(z_1) dA_{a_2}(z_2). \end{aligned} \tag{2.5}$$

Moreover, by the boundedness of the point evaluation functional on $\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$ we obtain a positive constant C_3 such that

$$|f(\varphi_1(0), \varphi_2(0))| \leq C_3 |f(0, 0)|.$$

Hence, we receive:

$$\|C_\Phi f\|_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)} \leq C \|f\|_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)}$$

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and this direction of the proof is finished. For the necessity part, we follow the proof of Theorem 3.1 in [14]. We will write down the proof for convenience of the reader. We assume now that there exists a constant $C > 0$ such that:

$$\|C_\Phi f\|_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)} \leq C\|f\|_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)},$$

for all $f \in \mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$. We consider the following test function:

$$\begin{aligned} f_{\omega_1,\omega_2}(z_1,z_2) &= (1-|\omega_1|^2)^{1+\frac{1-2a_1}{2}} \int_0^{z_1} \frac{d\zeta_1}{(1-\overline{\omega_1}\zeta_1)^{3-2a_1}} \\ &+ (1-|\omega_2|^2)^{1+\frac{1-2a_2}{2}} \int_0^{z_2} \frac{d\zeta_2}{(1-\overline{\omega_2}\zeta_2)^{3-2a_2}} \\ &+ (1-|\omega_1|^2)^{1+\frac{1-2a_1}{2}}(1-|\omega_2|^2)^{1+\frac{1-2a_2}{2}} \int_0^{z_1}\int_0^{z_2} \frac{d\zeta_1 d\zeta_2}{(1-\overline{\omega_1}\zeta_1)^{3-2a_1}(1-\overline{\omega_2}\zeta_2)^{3-2a_2}}, \end{aligned} \tag{2.6}$$

where $\omega_1, \omega_2 \in \mathbb{D}$. By calculating the partial derivatives:

$$\begin{aligned} \|C_\Phi f_{\omega_1,\omega_2}\|^2_{\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)} &= \int_{\mathbb{D}} \frac{(1-|\omega_1|^2)^{3-2a_1}}{|1-\overline{\omega_1}z_1|^{6-4a_1}} \mathcal{N}_{\varphi_1,a_1}(z_1)dA(z_1) \\ &+ \int_{\mathbb{D}} \frac{(1-|\omega_2|^2)^{3-2a_2}}{|1-\overline{\omega_2}z_2|^{6-4a_2}} \mathcal{N}_{\varphi_2,a_2}(z_2)dA(z_2) \\ &+ \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|\omega_1|^2)^{3-2a_1}}{|1-\overline{\omega_1}z_1|^{6-4a_1}} \frac{(1-|\omega_2|^2)^{3-2a_2}}{|1-\overline{\omega_2}z_2|^{6-4a_2}} \mathcal{N}_{\varphi_1,a_1}(z_1)\mathcal{N}_{\varphi_2,a_2}(z_2)dA(z_1)dA(z_2). \end{aligned} \tag{2.7}$$

Now, for $\omega_1, \omega_2 \in \mathbb{D}$ with $|\omega_i| > 1/2$, $i = 1, 2$ let

$$D(\omega_i) = \{z_i \in \mathbb{D} : |z_i - \omega_i| < \frac{1}{2}(1-|\omega_i|^2)\}.$$

By Aleman's inequality that one can find in [1] and by the well known fact that the quantities $|1-\overline{\omega_i}z_i|$ and $(1-|\omega_i|^2)$ are comparable for every $z_i \in D(\omega_i)$, $i = 1, 2$, we get the inequalities:

$$\mathcal{N}_{\varphi_i,a_i}(\omega_i) \leq \frac{4}{(1-|\omega_i|^2)} \int_{D(\omega_i)} \mathcal{N}_{\varphi_i,a_i}(z_i)dA(z_i) \tag{2.8}$$

$$\leq C_i \int_{\mathbb{D}} \frac{(1-|\omega_i|^2)^{3-2a_i}}{|1-\overline{\omega_i}z_i|^{6-4a_i}} \mathcal{N}_{\varphi_i,a_i}(z_i)dA(z_i) \tag{2.9}$$

$$\leq C_i(1-|\omega_i|^2)^{1-2a_i}, \tag{2.10}$$

hence,

$$\sup_{|\omega_i|>1/2} \frac{\mathcal{N}_{\varphi_i,a_i}(\omega_i)}{(1-|\omega_i|^2)^{1-2a_i}} < \infty.$$

By the proof of Aleman's inequality, it is assured that there exist two subharmonic functions $u_{a_i}, i = 1, 2$ such that $\mathcal{N}_{\varphi_i,a_i} \leq u_{a_i}$. By boundedness of upper semi-continuous functions on compact sets, we obtain

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$$\sup_{|\omega_i|\le 1/2} \frac{\mathcal{N}_{\varphi_i,a_i}(\omega_i)}{(1-|\omega_i|^2)^{1-2a_i}} \le 2^{1-2a_i} \sup_{|\omega_i|\le 1/2} u_{a_i}(\omega_i),\ i=1,2$$

and we are done with this direction of the proof as well. ∎

Remark 1 The previous result can be extended in polydisc setting. Consider $\Phi : \mathbb{D}^n \to \mathbb{D}^n$ which is separated, in the sense that $\Phi(z_1,\dots,z_n) = (\varphi_1(z_1),\dots,\varphi_n(z_n))$. Then, C_Φ is bounded in the anisotropic Dirichlet space if and only if the condition of Theorem 2.1 is met for $i = 1,\dots,n$.

3 Composition operators between the Dirichlet and the Bergman space on the bidisc

In this section, we will consider the problem of the boundedness of the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$. We have already stated that for the case of the Bergman space and the Hardy spaces of the bidisc, a significant amount of work has been done in [6, 7], and [12]. The motivation that drove us to consider the following problem is simple. The case of the boundedness of the composition operator in the Dirichlet space of the polydisc has insurmountable amount of calculations on the derivatives, and makes the pull-back measure approach quite difficult to handle. The problem of studying the boundedness of the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$ seems to give more initial results, due to the existence of theorems such as those that we will see in the next two subsections. In the first subsection we will derive a only sufficient condition which works for separated symbols, while on the last subsection we will give a necessary and sufficient condition for all holomorphic symbols $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$.

3.1 A sufficient condition

Our main tool for this section will be the Carleson measures for the Dirichlet space on the bidisc. We denote by $I \times J$ the product of two arcs in the torus $\mathbb{T}$. Let $z = (z_1, z_2) \in \mathbb{D}^2$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ and $\delta = (\delta_1, \delta_2) \in (0,1)^2$. A Carleson box on the bidisc is the following set: $S(I \times J) = \{z \in \mathbb{D}^2 : |z_i - \zeta_i| < \delta_i\}$, $i = 1, 2$. A positive Borel measure in $\mathbb{D}^2$ is called *Carleson measure* on $\mathfrak{D}(\mathbb{D}^2)$ if the *Carleson embedding* is a bounded operator, that is, if there exists a positive constant $C(\mu) > 0$ such that

$$\int_{\mathbb{D}^2} |f|^2 d\mu \le C(\mu)\|f\|_{\mathfrak{D}(\mathbb{D}^2)}.$$

As we will observe, it is not easy to characterize the measures that satisfy the previous inequality. Our main target in this section is to provide a sufficient one-box condition for Carleson measures of the Dirichlet space on the bidisc which is inspired by the techniques of [9]. We will use this condition to obtain a one-box volume condition for the boundedness of the composition operator between the Bergman and the Dirichlet space. For what follows, $K_w(z)$ will denote the reproducing kernel of the Dirichlet spaces on the bidisc, that is:

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$$K_w(z) = \left(C_1 + \log \frac{2}{1-\overline{w_1}z_1}\right)\left(C_2 + \log \frac{2}{1-\overline{w_2}z_2}\right)$$

for $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{D}^2$. The main goal of this section is to prove

Theorem 3.1 *Let $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ be a holomorphic self map of the bidisc. Assume that there exist an increasing function ψ with:*

$$\int_0^{2\pi}\int_0^{2\pi} \frac{\psi(xy)}{xy} dxdy < \infty,$$

and $C > 0$ such that:

$$V_\beta\left(\Phi^{-1}(S(I \times J)\right) \leq C\psi(|I \times J|).$$

Then, the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$ is bounded.

For the proof of the above sufficient condition, we will establish the following lemma.

Lemma 3.1 *Let μ a positive Borel measure on $\mathbb{D}^2$. Then, μ is a Carleson measure on $\mathfrak{D}(\mathbb{D}^2)$, if and only if*

$$\int_{\mathbb{D}^2}\int_{\mathbb{D}^2} \Re K_w(z)|g(z)||g(w)|d\mu(z)d\mu(w) \leq C\|g\|_{L^2(\mathbb{D}^2, d\mu)},$$

for any $g \in L^2(\mathbb{D}^2, d\mu)$.

Proof The proof can be found in [3]. Also the above lemma holds for all Reproducing Kernel Hilbert Spaces defined in an arbitrary polydisc $\mathbb{D}^n$, $n > 2$. For more details the interested reader can check [4] ∎

Lemma 3.2 *Let μ a positive Borel measure on $\mathbb{D}^2$. If:*

$$\sup_{w\in\mathbb{D}^2} \int_{\mathbb{D}^2} \Re K_w(z) d\mu(z) < +\infty$$

then μ is a Carleson measure for $\mathfrak{D}(\mathbb{D}^2)$.

Proof By *Cauchy–Schwarz* inequality and the symmetry property of the kernel function, we obtain:

$$\int_{\mathbb{D}^2}\int_{\mathbb{D}^2} \Re K_w(z)|g(z)||g(w)|d\mu(z)d\mu(w) \leq \int_{\mathbb{D}^2}\int_{\mathbb{D}^2} \Re K_z(w)|g(w)|^2 d\mu(z)d\mu(w)$$

and the result follows immediately. ∎

Now we will state the two dimensional version of [9, Theorem 1.1] which will be the key for the establishment of the sufficient condition that we stated at the starting point of this subsection.

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Lemma 3.3 *Let μ be two finite positive Borel measures on $\mathbb{D}^2$. Let ψ be a non decreasing functions with:*

$$\int_0^{2\pi}\int_0^{2\pi}\frac{\psi(xy)}{xy}dxdy<\infty.$$

If

$$\mu(S(I\times J))\leq C\psi(|I\times J|)$$

for I, J two arcs on $\mathbb{T}$, then, the measure μ is a Carleson measure for $\mathfrak{D}(\mathbb{D}^2)$.

Proof We will follow the idea of the proof of Theorem 1.1 on [9]. We will bound the integral of the real part of the reproducing kernel. For the measure μ it is true that:

$$\int_{\mathbb{D}^2}\Re K_w(z)d\mu(z)=\int_{\mathbb{D}^2}\left(C_1+\log\frac{2}{|1-\overline{w_1}z_1|}\right)\left(C_2+\log\frac{2}{|1-\overline{w_2}z_2|}\right)d\mu(z_1,z_2).$$

The right-hand side integral can be re-written now as follows:

$$\begin{aligned}&\int_{\mathbb{D}^2}\left(C_1+\log\frac{2}{|1-\overline{w_1}z_1|}\right)\left(C_2+\log\frac{2}{|1-\overline{w_2}z_2|}\right)d\mu(z_1,z_2)\\ &=\int_{\mathbb{D}^2}\left(\int_{|1-\overline{w_1}z_1|=t_1}^{2}\int_{|1-\overline{w_2}z_2|=t_2}^{2}\frac{1}{t_1}\frac{1}{t_2}dt_1dt_2\right)d\mu(z_1,z_2).\end{aligned}\tag{3.1}$$

Now, the product of the sets $\{|1-\overline{w_1}z_1|\leq t_1\}\times\{|1-\overline{w_2}z_2|\leq t_2\}$ is contained inside $S(I\times J)$ where $I\times J$ is the Cartesian product of two arcs, with $|I\times J|=64t_1t_2$. So now it is easy to observe that

$$\mu(\{(z_1,z_2)\in\mathbb{D}^2:|1-\overline{w_1}z_1|\leq t_1,|1-\overline{w_2}z_2|\leq t_2\})\leq C\psi(64t_1t_2).$$

Hence,

$$\int_{\mathbb{D}^2}\Re K_w(z)d\mu(z)\leq C\int_0^8\int_0^8\frac{\psi(t_1t_2)}{t_1t_2}dt_1dt_2<+\infty.$$ ∎

At this point we are ready for the proof of Theorem 4.3.

Proof The composition operator $C_\Phi:\mathfrak{D}(\mathbb{D}^2)\to A^2_\beta(\mathbb{D}^2)$ is bounded if and only if:

$$\int_{\mathbb{D}^2}|f(z_1,z_2)|^2dV_\beta\circ\Phi^{-1}(z_1,z_2)\leq C\int_{\mathbb{D}^2}|\partial_{z_2}\partial_{z_1}[z_1z_2f(z_1,z_2)]|^2dV(z_1,z_2),$$

which is equivalent to the measure $V_\beta\circ\Phi^{-1}$ being a Carleson measure for $\mathfrak{D}(\mathbb{D}^2)$. By our assumption and Lemma 4.3 this is the case, and the proof is now completed ∎

Remark 2 The above arguments and lemmata work for any polydisc $\mathbb{D}^n$, $n\geq 2$. One can find this fact on [13].

3.2 A necessary and sufficient condition

In this section, we give a necessary and sufficient condition for the boundedness of $C_\Phi:\mathfrak{D}(\mathbb{D}^2)\to A^2_\beta(\mathbb{D}^2)$. In order to do that we have to give some background

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theorems. For what follows, we assume that E is a compact subset of the bitorus $\mathbb{T}^2$. The *Bessel capacity of E* with parameter 1/2 is defined in the following manner. Consider $\theta = (\theta_1, \theta_2)$ and $\eta = (\eta_1, \eta_2)$ such that $\theta_1, \theta_2, \eta_1, \eta_2 \in [-\pi, \pi)$. We consider the 2-dimensional extension of the 1/2-Bessel kernel on the bi-torus as

$$k_{\mathbb{T}^2,1/2}(\theta, \eta) = |\theta_1 - \eta_1|^{-1/2}|\theta_2 - \eta_2|^{-1/2}.$$

The above kernel can be extended to a convolution operator as follows:

$$k_{\mathbb{T}^2,1/2}\psi(\theta) = \int_{\mathbb{T}^2} k_{\mathbb{T}^2,1/2}(\theta - \eta)d\psi(\eta).$$

Now let $E \subset \mathbb{T}^2$ be a closed subset of the bi-torus. The Bessel 1/2-capacity of E is:

$$\mathrm{Cap}^{1/2}_{\mathbb{D}^2}(E) = \inf\{||h||^2_{L^2(\mathbb{T}^2)} : h \geq 0, k_{\mathbb{T}^2,1/2}h \geq 1, on E\}.$$

In their seminal work [3], Arcozzi, Mozolyako, Perfekt, and Sarfatti proved the following theorem for the classical Dirichlet space on the bidisc which characterises the Carleson measures in this space, generalizing the result of Stegenga on the disc setting.

Theorem 3.2 *Let $\mu \geq 0$ be a Borel measure on $\overline{\mathbb{D}^2}$. Then the following are equivalent:*

(1) *There exists a constant $C_1 > 0$ such that:*

$$\int_{\overline{\mathbb{D}^2}} |f|^2 d\mu \leq C_1||f||_{\mathfrak{D}(\mathbb{D}^2)}.$$

(2) *There exists a constant $C_2 > 0$ such that $\forall n \geq 1$ and for all choices of arcs $J^1_1, \ldots, J^1_n, J^2_1, \ldots, J^2_n$ on $\mathbb{T}$ we have that:*

$$\mu\left(\bigcup_{k=1}^{n} S(J^1_k \times J^2_k)\right) \leq C_2 \mathrm{Cap}^{1/2}_{\mathbb{D}^2}\left(\bigcup_{k=1}^{n} J^1_k \times J^2_k\right).$$

The constants C_1, C_2 are comparable independently of μ.

We remind that $S(J^1_k \times J^2_k)$ denotes the cartesian product of Carleson boxes associated with the arcs $J^1_k \times J^2_k$, generalizing the concept of the usual one-dimensional Carleson boxes on the bidisc case. Utilizing the previously mentioned tools, we can obtain a necessary and sufficient condition for the boundedness of $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$. We obtain the following.

Theorem 3.3 *Let $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ holomorphic. Then $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$ for $\beta > -1$, is bounded, if and only if there exists a constant $C > 0$ such that $\forall n \geq 1$ and for all choices of arcs $I^1_1, \ldots, I^1_n, I^2_1, \ldots, I^2_n$ on $\mathbb{T}$ we have that:*

$$V_\beta\left(\Phi^{-1}\left(\bigcup_{k=1}^{n} S(I^1_k \times I^2_k)\right)\right) \leq C\mathrm{Cap}^{1/2}_{\mathbb{D}^2}\left(\bigcup_{k=1}^{n} I^1_k \times I^2_k\right).$$

Proof The composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$ is bounded if and only if

$$\int_{\mathbb{D}^2} |f \circ \Phi(z_1, z_2)|^2 dV_\beta(z_1, z_2) \leq C \int_{\mathbb{D}^2} |\partial_{z_2}\partial_{z_1}[z_1 z_2 f(z_1, z_2)]|^2 dV(z_1, z_2).$$

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By taking the pull-back measure this condition turns into the following:

$$\int_{\mathbb{D}^2} |f(z_1,z_2)|^2 dV_\beta \circ \Phi^{-1}(z_1,z_2) \le C \int_{\mathbb{D}^2} |\partial_{z_2}\partial_{z_1}[z_1 z_2 f(z_1,z_2)]|^2 dV(z_1,z_2).$$

But observing the above condition, this is equivalent to the measure $d\mu = dV_\beta \circ \Phi^{-1}$ being a Carleson measure for the Dirichlet space on the bidisc. So by applying Theorem 4.2, we obtain that the composition operator $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$ is bounded if and only if

$$V_\beta\left(\Phi^{-1}\left(\bigcup_{k=1}^{n} S(I_k^1 \times I_k^2)\right)\right) \le C \mathrm{Cap}^{1/2}_{\mathbb{D}^2}\left(\bigcup_{k=1}^{n} I_k^1 \times I_k^2\right)$$

for all choices of arcs $I_1^1, \ldots, I_n^1, I_1^2, \ldots, I_n^2$ on $\mathbb{T}$, for all $n \ge 1$. ∎

Remark 3 In the inequality above, we can replace the 1/2-Bessel capacity with the logarithmic capacity generated by the kernel function:

$$K_w(z) = \left(C_1 + \log \frac{2}{1-\overline{w_1}z_1}\right)\left(C_2 + \log \frac{2}{1-\overline{w_2}z_2}\right)$$

for $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{D}^2$. This fact is pointed out in [3].

Remark 4 The capacity of the the product of arcs can be replaced by the capacity of the corresponding 2-dimensional Carleson box.

4 Further discussion

Observing the work that has been done for the case of Bergman and Hardy spaces on the bidisc by Bayart and Kosinski in [6, 7, 12] we can say that these types of volume conditions can help us provide characterizations of the symbols that induce bounded composition operators between Bergman and Dirichlet spaces. It would be more pleasant to work only on the Dirichlet-type space on the bidisc, but as we mentioned before, the involvement of the derivative is what makes things much more complicated. The necessary and sufficient condition we provide here for the boundedness of $C_\Phi : \mathfrak{D}(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$ is still quite difficult to work with because it does not involve only one Carleson box but rather a finite union of them. The sufficient condition on the other hand has the positive aspect of being stated only for one Carleson box. Another interesting observation we should point out here is that Theorem 3.3 is true only for the bidisc, as the analog of Theorem 3.2 is not proved for the poly-disc but only for the bidisc setting. Last but not least, for the case of the anisotropic Dirichlet-type spaces, we sadly do not yet have the extension of Theorem 3.2 for all $a_i < 1/2$, and this is the main reason that Theorem 3.2 is only stated for the unweighted Dirichlet space. If Theorem 3.2 happens to be true for some set of parameters a_i then we can characterize the composition operators that are bounded between the Bergman and Dirichlet-type spaces for the anisotropic setting for the said set of parameters.

Another interesting topic to be discussed, is the extension of the above techniques to other domains. If a domain $\Omega \subset \mathbb{C}^n$ is a product domain, then the separated

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symbols case seems to provide similar results as the ones obtained in our work, but in Bergman spaces of the said domain Ω.

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