# SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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The equations to be considered are of the form

$$\sum_i \sum_k l_{jk}^{(i)} X_k^{(i)} = F_j,$$

where $X_k$ is to be determined as a function of $t$, $X_k^{(i)}$ is its $i$th derivative with respect to $t$, $F_j$ is a given analytic function of $t$, $l_{jk}^{(i)}$ a given constant, and $i = 0, 1, \ldots, n$; $j, k = 1, \ldots, m$, and they represent a system of simultaneous linear differential equations of order $n$ in $m$ functions $X_k\,(k = 1, \ldots, m)$.

If the matrix $l^{(0)}$ of the coefficients $l_{jk}^{(0)}$ of the terms of the highest order is regular, the system can be represented by an equation of the form

$$\mathbf{X}^{(n)} + \mathbf{a}_1 \mathbf{X}^{(n-1)} + \ldots + \mathbf{a}_{n-1} \mathbf{X}^{(1)} + \mathbf{a}_n \mathbf{X} = \mathbf{Y},$$

where $\mathbf{X}$ is a vector whose elements are the functions which are to be determined, $\mathbf{X}^{(i)}$ is the vector whose elements are the $i$th derivatives of the elements of $\mathbf{X}$, $\mathbf{Y}$ is a vector whose elements are given functions, and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are given constant square matrices; and it is taken that $\mathbf{X}, \mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n-1)}$ have given values $\mathbf{X}_0, \mathbf{X}_0^{(1)}, \ldots, \mathbf{X}_0^{(n-1)}$ at $t = 0$.

If

$$\mathbf{U} = \begin{pmatrix} \mathbf{X}^{(n-1)} \\ \vdots \\ \mathbf{X}^{(1)} \\ \mathbf{X} \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} \mathbf{X}_0^{(n-1)} \\ \vdots \\ \mathbf{X}_0^{(1} \\ \mathbf{X}_0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ -\mathbf{1} & \ldots & \mathbf{0} & \mathbf{0} \\ \ldots\ldots & \ldots & \ldots\ldots & \ldots\ldots \\ \mathbf{0} & \ldots & -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mathbf{Y} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

then this equation is equivalent to the equation

$$\dot{\mathbf{U}} + \mathbf{c}\mathbf{U} = \mathbf{V},$$

where $\mathbf{U} = \mathbf{U}_0$ at $t = 0$; whence the solution is obtained in the form

$$\mathbf{U} = e^{-\mathbf{c}t}\left\{U_0 + \int_0^t e^{\mathbf{c}\theta}\,\mathbf{V}\,d\theta\right\},$$

where $e^{\mathbf{c}t}$ has its definition as a matrix whose elements are analytic functions of $t$.

Thus

$$e^{\mathbf{c}t} = \sum_\gamma \mathbf{R}_\gamma \sum_{0 \leqslant s < i_\gamma} \frac{t^s}{s!} e^{\gamma t} (\mathbf{c} - \gamma \mathbf{1})^s,$$

where $\gamma$ ranges over the characteristic values of $\mathbf{c}$, $i_\gamma$ is the index of $\gamma$ as a characteristic value of $\mathbf{c}$, this being the multiplicity of $\gamma$ as a root of the minimum equation of $\mathbf{c}$, and where $\mathbf{R}_\gamma$ is the resolvent projector of $\mathbf{c}$ for the characteristic value $\gamma$, for which there is the expression

$$\mathbf{R}_\gamma = \prod_{\mu \neq \gamma} (\mathbf{c} - \mu \mathbf{1})^{i_\mu} / \sum_\lambda \prod_{\nu \neq \lambda} (\mathbf{c} - \nu \mathbf{1})^{i_\nu},$$



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giving $\mathbf{R}_\gamma$ as a rational function of $\mathbf{c}$ formed directly from the indices of the characteristic values of $\mathbf{c}$.

These formulae for $e^{\mathbf{c}t}$ and $\mathbf{R}_\gamma$ give the same values if the indices $i_\gamma$ are replaced by any greater exponents, for example, the multiplicities $m_\gamma$ in the characteristic equation, or the order $N = \sum_\gamma m_\gamma$. This is shown in Afriat (1), and it can also be deduced from the statement in Penrose ((7), footnote), that matrices $\mathbf{R}_\gamma$ which satisfy $(\mathbf{c}-\gamma\mathbf{1})^N\mathbf{R}_\gamma = 0$, $\sum_\gamma \mathbf{R}_\gamma = \mathbf{1}$ are unique.

Equivalent to the given definition for $e^{\mathbf{c}t}$, which is as a polynomial in $\mathbf{c}$ whose coefficients are analytic functions of $t$, there is the definition by power series, $\sum_{s=0}^{\infty} \frac{(\mathbf{c}t)^s}{s!}$, and in terms of contour integrals, $\frac{1}{2\pi i}\oint_{\Gamma_\mathbf{c}} \frac{e^{\zeta t}\mathbf{1}\,d\zeta}{\zeta\mathbf{1}-\mathbf{c}}$, where the integral is taken on any simple contour $\Gamma_\mathbf{c}$ which encloses all the characteristic values of $\mathbf{c}$ in its interior. For $\mathbf{R}_\gamma$ there is the alternative definition $\frac{1}{2\pi i}\oint_\gamma \frac{\mathbf{1}\,d\zeta}{\zeta\mathbf{1}-\mathbf{c}}$, where the integral is taken on any simple contour which encloses just the characteristic value $\gamma$ of $\mathbf{c}$ in its interior. For discussions of functions of matrices, reference may be made to Hamburger and Grimshaw (5), Ferrar (2), Halmos (4) and MacDuffee (6).

The function $e^{\mathbf{c}t}$ has the properties

$$e^{\mathbf{0}} = \mathbf{1}, \quad e^{\mathbf{c}(s+t)} = e^{\mathbf{c}s}e^{\mathbf{c}t},$$

and

$$\frac{d}{dt}(e^{\mathbf{c}t}) = \lim_{s\to t}\frac{e^{\mathbf{c}s}-e^{\mathbf{c}t}}{s-t} = \mathbf{c}\,e^{\mathbf{c}t}.$$

In general, however, $e^{\mathbf{a}+\mathbf{b}} \neq e^{\mathbf{a}}e^{\mathbf{b}}$, unless $\mathbf{ba} = \mathbf{ab}$.

It follows that

$$\begin{aligned}\frac{d}{dt}(e^{\mathbf{c}t}\mathbf{U}) &= \lim_{\delta t\to 0}\frac{\delta(e^{\mathbf{c}t}\mathbf{U})}{\delta t}\\ &= \lim_{\delta t\to 0}\frac{e^{\mathbf{c}t+\mathbf{c}\delta t}(\mathbf{U}+\delta\mathbf{U})-e^{\mathbf{c}t}\mathbf{U}}{\delta t}\\ &= \lim_{\delta t\to 0} e^{\mathbf{c}t}\left\{e^{\mathbf{c}\delta t}\frac{\delta\mathbf{U}}{\delta t}+\frac{(e^{\mathbf{c}\delta t}-\mathbf{1})\mathbf{U}}{\delta t}\right\}\\ &= e^{\mathbf{c}t}(\dot{\mathbf{U}}+\mathbf{c}\mathbf{U}),\end{aligned}$$

and hence $e^{\mathbf{c}t}$ is obtained as an integrating factor of the equation considered. Also, if $\mathbf{k} = e^{\mathbf{c}t}$ and $t/\tau = n$ is an integer, then

$$e^{\mathbf{c}t} = \overbrace{\mathbf{k}\ldots\mathbf{k}}^{n} = \mathbf{k}^n.$$

Accordingly, the values of $e^{\mathbf{c}t}$, for $t$ running through successive integral multiples of some interval $\tau$, are calculated by repeated multiplications with the matrix $\mathbf{k}$.

For the homogeneous linear system, which corresponds to the case $\mathbf{Y} = \mathbf{0}$ and, equivalently, $\mathbf{V} = \mathbf{0}$, the solution is

$$\mathbf{U} = e^{-\mathbf{c}t}\mathbf{U}_0.$$

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If it is required to calculate this solution at successive intervals of $t$ of length all equal to $\tau$, so that $t = n\tau$ is the value of $t$ at the $n$th interval, the technique would be to calculate $k = e^{-\mathbf{c}\tau}$, which is explicitly given as a rational function of $\mathbf{c}$, and then to calculate successive powers of $\mathbf{k}$, to obtain the value $\mathbf{U}_n$ of $\mathbf{U}$ at the $n$th interval as

$$\mathbf{U}_n = \mathbf{k}^n \mathbf{U}_0.$$

This is a procedure of computation which is well adapted for automatic methods. The asymptotic behaviour of this solution is decided by the characteristic values of $\mathbf{c}$, or, equivalently, by the characteristic values of $\mathbf{k}$, these being mutually determined by the relation $\kappa = e^{-\gamma\tau}$, with multiplicities determined correspondingly.

From the identity

$$\begin{vmatrix} \mathbf{a}_1 - \gamma\mathbf{1} & \mathbf{a}_2 & \dots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ -\mathbf{1} & -\gamma\mathbf{1} & \dots & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & -\mathbf{1} & -\gamma\mathbf{1} \end{vmatrix} = \left| (-1)^n (\gamma^n \mathbf{1} - \dots + (-1)^n \mathbf{a}_n) \right|$$

(proved directly by elementary operations on the columns of the characteristic determinant of $\mathbf{c}$, which leave its value unchanged, followed by a Laplace expansion), it follows that the characteristic values $\gamma$ of $\mathbf{c}$ are the roots of the equation

$$|\gamma^n \mathbf{1} - \gamma^{n-1}\mathbf{a}_1 + \dots + (-1)^n \mathbf{a}_n| = 0.$$

The above conclusions, and some immediate consequences of them, are stated together in the following theorem:

THEOREM. *The equation*

$$\mathbf{X}^{(n)} + \mathbf{a}_1 \mathbf{X}^{(n-1)} + \dots + \mathbf{a}_{n-1}\mathbf{X}^{(1)} + \mathbf{a}_n \mathbf{X} = \mathbf{Y},$$

*with* $\mathbf{X}^{(i)} = \mathbf{X}_0^{(i)}$ $(i = 0, 1, \dots, n-1)$ *at* $t = 0$, *is equivalent to the equation*

$$\dot{\mathbf{U}} + \mathbf{c}\mathbf{U} = \mathbf{V}$$

*with* $\mathbf{U} = \mathbf{U}_0$ *at* $t = 0$, *and has the solution*

$$\mathbf{U} = e^{-\mathbf{c}t}\left\{\mathbf{U}_0 + \int_0^t e^{\mathbf{c}\theta}\,\mathbf{V}\,d\theta\right\},$$

*where*

$$\mathbf{U} = \begin{pmatrix} \mathbf{X}^{(n-1)} \\ \vdots \\ \mathbf{X}^{(1)} \\ \mathbf{X} \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} \mathbf{X}_0^{(n-1)} \\ \vdots \\ \mathbf{X}_0^{(1)} \\ \mathbf{X}_0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ -\mathbf{1} & \dots & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \dots & -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mathbf{Y} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

*and, for any indices* $j_\gamma$ *such that* $\prod_\gamma (\mathbf{c} - \gamma\mathbf{1})^{j_\gamma} = \mathbf{0}$,

$$e^{\mathbf{c}t} = \sum_\gamma \mathbf{R}_\gamma \sum_{0 \leqslant s < j_\gamma} \frac{t^s}{s!} e^{\gamma t} (\mathbf{c} - \gamma\mathbf{1})^s, \quad \mathbf{R}_\gamma = \prod_{\mu \neq \gamma} (\mathbf{c} - \mu\mathbf{1})^{j_\mu} / \sum_\alpha \prod_{\beta \neq \alpha} (\mathbf{c} - \beta\mathbf{1})^{j_\beta}.$$

*For constant* $\mathbf{V} = \mathbf{V}_0$ *the solution is*

$$\mathbf{U} = e^{-\mathbf{c}t}(\mathbf{U}_0 + \mathbf{c}^{-1}\mathbf{V}_0) + \mathbf{c}^{-1}\mathbf{V}_0,$$



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*provided* $\mathbf{c}$ *is regular, for which a necessary and sufficient condition is that* $\mathbf{a}_n$ *be regular; and if* $\mathbf{V}_0 = \mathbf{0}$ *the solution is* $\mathbf{U} = e^{-\mathbf{c}t}\mathbf{U}_0$. *If* $t/\tau = n$ *is an integer, then*

$$e^{-\mathbf{c}t} = \overbrace{\mathbf{k} \dots \mathbf{k}}^{n} = \mathbf{k}^n,$$

*where* $\mathbf{k} = e^{-\mathbf{c}\tau}$. *The characteristic values* $\gamma$ *of* $\mathbf{c}$ *are the roots of the equation*

$$|\gamma^n \mathbf{1} - \gamma^{n-1}\mathbf{a}_1 + \dots + (-1)^n \mathbf{a}_n| = 0.$$

Another method of solution is obtained by the method of Laplace transforms. A discussion of homogeneous systems is given in Poole (8). The method given here has been found appropriate for some questions in mathematical economics. Thus it gives the required form of discussion for equations which are obtained to describe the development of an economy, analysing the tendencies towards growth and oscillation.

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