J. London Math. Soc. (2) 72 (2005) 717–730 © 2005 London Mathematical Society
doi:10.1112/S0024610705006903

UNCERTAINTY PRINCIPLES FOR THE AMBIGUITY FUNCTION

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Abstract

We extend uncertainty principles which are valid for the Fourier transform to the setting of the ambiguity function. A general result is established for annihilating sets: strongly/weakly annihilating sets for the Fourier transform yield such sets for the ambiguity function, extending a result known for sets of finite measure. We apply this to sublevel sets of nondegenerate quadratic forms. Our main result is a sharp version of Beurling's uncertainty principle for the ambiguity function.

1. *Introduction*

An uncertainty principle for the Fourier transform is a way to quantify to which extent a function and its Fourier transform cannot simultaneously be sharply localized. We take the Fourier transform on the Euclidean space $\mathbb{R}^d$ to be normalized as

$$\widehat{f}(\xi) = \int_{R^d} f(x) e^{-2i\pi\langle x,\xi\rangle} dx.$$

There are various forms of the uncertainty principle. See for example the book of Havin and Jöricke [**5**]. In time-frequency analysis, another tool of investigation is the *Wigner distribution*. Its definition is very similar to the definition of the *windowed Fourier transform*, also known as the *short time Fourier transform*, and to the so-called *radar-ambiguity function* (or just ambiguity function). The definition of this last one is as follows.

DEFINITION 1.1. Let u, v be two functions of $L^2(\mathbb{R}^d)$. The *radar-ambiguity function* associated to u and v is defined for $x, y \in \mathbb{R}^d$ by

$$A(u,v)(x,y) = \int_{\mathbb{R}^d} u\left(t + \frac{x}{2}\right)\overline{v\left(t - \frac{x}{2}\right)} e^{-2i\pi\langle t,y\rangle} dt.$$

The aim of this paper is to transfer some uncertainty principles of Fourier analysis to the setting of the ambiguity function. Actually many uncertainty principles are already known to have their counterpart for time frequency representations, and it is a general belief that it should be the case for any uncertainty principle. See [**3**] and [**11**] for examples of this sort.

In this paper we focus on uncertainty principles of Hardy type, and on annihilating sets for the Fourier transform. See [**5**] for a general overview of these theorems. A generalization of the classical result of Hardy, due to Beurling (see the paper of Hörmander [**6**]), was established in the general setting of $\mathbb{R}^d$ in [**2**].

Received 19 April 2004; revised 17 February 2005.

2000 *Mathematics Subject Classification* 42B10, 33C45, 94A12.

https://doi.org/10.1112/S0024610705006903 Published online by Cambridge University Press

There the authors apply the result to the ambiguity function and obtain Beurling's theorem for $A(u,v)$. Our contribution here consists in proving the following sharper version. This is the main result.

THEOREM 1.2. *Let $u, v \in L^2(\mathbb{R}^d)$. If there exists an $N \geqslant 0$ such that*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|A(u,v)(x,y)|}{(1+\|x\|+\|y\|)^N} e^{\pi|\langle x,y\rangle|}\, dx\, dy < \infty, \tag{1.1}$$

then either u or v is identically zero, or both can be written as

$$\begin{aligned} u(x) &= P(x)e^{-\langle Ax,x\rangle - 2i\pi\langle w,x\rangle}, \\ v(x) &= Q(x)e^{-\langle Ax,x\rangle - 2i\pi\langle w,x\rangle}, \end{aligned} \tag{1.2}$$

with P and Q polynomials whose degrees satisfy $\deg(P)+\deg(Q) < N-d$, $w \in \mathbb{C}^d$, and A a real positive symmetric matrix. The converse is also true. In particular, for $N \leqslant d$, u or v are identically vanishing.

In [**2**], we considered a weaker version, with L^2 instead of L^1 norms, and the argument in the exponential was $2\pi(\|x^2\|+\|y^2\|)$, which is bigger than $\pi|\langle x,y\rangle|$.

The following form of Hardy's theorem is a corollary. It generalizes the results in [**4**].

COROLLARY 1.3. *Let $u, v \in L^2(\mathbb{R}^d)\backslash\{0\}$. Suppose that*

$$|A(u,v)(x,y)| \leqslant C(1+\|x\|+\|y\|)^N e^{-\pi|\langle x,y\rangle|}$$

for all $x, y \in \mathbb{R}^d$. Then u and v can be written as in (1.2) *with $\deg(P)+\deg(Q) \leqslant N$.*

Theorem 1.2 also implies restrictions on the support of ambiguity functions. It is known that $A(u,v)$ cannot be supported by a set of finite measure unless u or v vanishes (see [**8**, **9**, **11**], and also [**3**]). Theorem 1.2 enables us to find another class of sets with this property, which are of infinite Lebesgue measure.

THEOREM 1.4. *Let $C > 0$. The radar-ambiguity function associated to two non-vanishing square integrable functions cannot be supported by the set*

$$E(C) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d; |\langle x,y\rangle| < C\}.$$

Note that these sets are good candidates for uncertainty principles, because if C is small enough, $E(C)$ will be seen to be an ε-thin set in the sense of [**10**], and thus yields strong annihilation uncertainty principles for the Fourier transform. Actually one can take any nondegenerate quadratic form instead of $2\langle x,y\rangle$. Note also that the idea of considering sublevel sets of nondegenerate quadratic forms in the uncertainty principle is already present in [**10**].

Here we will prove that a strong annihilation uncertainty principle for the Fourier transform implies a similar result for the ambiguity function. It was already established for sets of finite measure on $\mathbb{R}^2$ in [**11**] (the proof works also on $\mathbb{R}^d$). We will deduce from this another form of Heisenberg's inequality. See [**2**] for a sharp version of the usual Heisenberg inequality for $A(u,v)$. This new Heisenberg inequality can be stated in the following way.

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THEOREM 1.5. *There exists a constant $\varepsilon > 0$ such that for all functions $u, v \in L^2(\mathbb{R}^d)$,*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle x, y\rangle^2 |A(u,v)(x,y)|^2 \, dx \, dy \geqslant \varepsilon \|u\|^2 \|v\|^2.$$

Instead of $\langle x, y\rangle$, the version present in [**2**] deals with the bigger quantity $\|x\|^2 + \|y\|^2$. We are not able to get the sharpest constant. Notice that Theorem 1.4 is a corollary of this inequality for C small enough ($C < \sqrt{\varepsilon}$).

The paper is organized as follows. In the next section, we consider the support properties for the ambiguity function. We will prove the transfer theorem for strongly annihilating sets which is mentioned above. We then apply it to the sets $E(C)$ for small values of C, and more generally to the sets $Z(E) = \{q(x,y) \in E\}$, where q is any quadratic form on $\mathbb{R}^{2d}$, and $E \subset \mathbb{R}$ a set of small Lebesgue measure. We use this to prove Theorem 1.5. The second part of the paper is devoted to the proof of the main result (Theorem 1.2). The proof will also enable us to get a bilinear form of Beurling's theorem obtained in [**2**] (Corollary 3.7).

2. *Heisenberg's inequality and strong annihilation conditions for $A(u,v)$*

2.1. *Strong annihilation conditions*

Following the terminology of Havin and Jöricke [**5**], we call the pair of measurable sets $E, \Sigma \subset \mathbb{R}^d$ a *weakly annihilating pair*, or *weak a-pair*, if for every $f \in L^2(\mathbb{R}^d)$, supported in E, and such that its Fourier transform $\widehat{f}$ is supported in Σ, we have $f = 0$. A *strongly annihilating pair*, or *strong a-pair*, satisfies moreover the inequality

$$\|f\|^2 \leqslant C \left(\int_{\mathbb{R}^d \setminus E} |f(x)|^2 \, dx + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, d\xi \right),$$

with $C = C(E, \Sigma)$ depending only on E and Σ. In [**3**], the author gives his conviction that every uncertainty principle for the usual Fourier transform should have its analogue for the ambiguity function. Our result will prove this fact for the strong annihilation property.

If $\Sigma \subset \mathbb{R}^d \times \mathbb{R}^d$, the set $\tilde{\Sigma}$ will be $\{(\xi, \eta); (-\eta, \xi) \in \Sigma\}$.

THEOREM 2.1. *Let $\Sigma \subset \mathbb{R}^d \times \mathbb{R}^d$ be a set such that $(\Sigma, \tilde{\Sigma})$ is a strong a-pair. Then there exists a constant $C(\Sigma)$ depending only on the set such that for every $u, v \in L^2(\mathbb{R}^d)$,*

$$\|u\|^2 \|v\|^2 \leqslant C(\Sigma) \iint_{\mathbb{R}^{2d} \setminus \Sigma} |A(u,v)(x,y)|^2 \, dx \, dy.$$

For the proof, we will use a fundamental symmetry relation of the ambiguity function. The general formulation can be found in [**2**]. It states that if $u, v, w \in L^2(\mathbb{R}^d)$, then the Fourier transform of

$$F(x,y) = A(u,v)(x,y)\overline{A(v,w)(x,y)}$$

is $\widehat{F}(\xi, \eta) = F(-\eta, \xi)$. We will use a particular case given in [**3**]. For our purposes, it can be reformulated as follows.

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PROPOSITION 2.2. *Let $a, b \in \mathbb{R}^d$. Let $u, v \in L^2(\mathbb{R}^d)$. Then the function*

$$F_{a,b}(x,y) = A(u,v)(x-a, y-b)\overline{A(v,u)(x+a, y+b)}$$

satisfies the relation $\widehat{F_{a,b}}(\xi,\eta) = F_{a,b}(-\eta,\xi)$.

Proof of Theorem 2.1. Let

$$G_{a,b}(x,y) = F_{a,b}(x+a, y+b)e^{2i\pi(\langle a,y\rangle - \langle b,x\rangle)},$$

so that

$$\widehat{G_{a,b}}(\xi,\eta) = \widehat{F_{a,b}}(\xi+b, \eta-a)e^{2i\pi\langle a,\xi\rangle + 2i\pi\langle b,\eta\rangle}.$$

Since $(\Sigma, \tilde{\Sigma})$ is a strong a-pair, we have

$$\begin{aligned}
\|G_{a,b}\|^2_{L^2(\mathbb{R}^{2d})} &\leqslant C(\Sigma)\left(\iint_{\mathbb{R}^{2d}\setminus\Sigma} |G_{a,b}|^2 + \iint_{\mathbb{R}^{2d}\setminus\tilde{\Sigma}} |\widehat{G_{a,b}}|^2\right) \\
&= C(\Sigma)\left(\iint_{\mathbb{R}^{2d}\setminus\Sigma} |A(u,v)(x,y)|^2 |A(v,u)(x+2a, y+2b)|^2\, dx\, dy\right. \\
&\quad \left. + \iint_{\mathbb{R}^{2d}\setminus\tilde{\Sigma}} |A(u,v)(-\eta,\xi)|^2 |A(v,u)(-\eta+2a, \xi+2b)|^2\, d\xi\, d\eta\right) \\
&= 2C(\Sigma)\iint_{\mathbb{R}^{2d}\setminus\Sigma} |A(u,v)(x,y)|^2 |A(v,u)(x+2a, y+2b)|^2\, dx\, dy.
\end{aligned}$$

Next, we integrate with respect to $(a,b) \in \mathbb{R}^{2d}$. We get

$$\|u\|^4\|v\|^4 \leqslant \frac{2C(\Sigma)}{4^d}\|u\|^2\|v\|^2 \iint_{\mathbb{R}^{2d}\setminus\Sigma} |A(u,v)(x,y)|^2\, dx\, dy,$$

as we wanted. □

REMARK 2.3. Proposition 2.2 can also be used to prove that if $(\Sigma, \tilde{\Sigma})$ is a weak a-pair then Σ is weakly annihilating for the ambiguity function. The proof is straightforward.

A classical result, due to Amrhein and Berthier (see [**5**]), states that any two sets of finite measure in $\mathbb{R}^d$ are strongly annihilating. Theorem 2.1 implies that this is the case for the ambiguity function. This was first proved in [**11**]. The weak annihilating property was proved first in [**7**, **9**].

For example the result applies to the sets $\Sigma = \{\|(x,y)\| < C\}$. In the next section we will apply Theorem 2.1 to the more general sets $Z(C)$ given by $Z(C) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d; |q(x,y)| < C\}$, where q is any nondegenerate quadratic form. Those sets have infinite Lebesgue measure in general.

2.2. *Thin hyperbolas and variants*

We recall the uncertainty principle proved in [**10**]. A set E in $\mathbb{R}^d$ is called ε-thin (where $0 < \varepsilon < 1$) if for all points $x \in \mathbb{R}^n$,

$$\frac{|E \cap B(x)|}{|B(x)|} < \varepsilon,$$

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where $B(x)$ denotes the ball centred at x of radius $\min(1, 1/\|x\|)$. The result is that any two ε-thin sets form a strong a-pair, provided that ε is small enough.

Following a remark in [**10**], examples of ε-thin sets are given by sublevel sets of nondegenerate quadratic forms. We give the proof here, since the proof is not in [**10**], and our constant, $|E|\ln(1/|E|)$, is better than the one given in [**10**] ($\sqrt{|E|}$).

PROPOSITION 2.4. *Let $q(x)$ be a nondegenerate quadratic form on $\mathbb{R}^d$, $d > 1$, and E a set of finite measure in $\mathbb{R}$. If $|E|$ is small enough, then the set*

$$Z(E) = \{x \in \mathbb{R}^d; q(x) \in E\}$$

is η-thin with $\eta = C_0|E||\ln(|E|)|$, and C_0 a constant depending only on q.

Proof. By a change of variables we can assume that

$$q(x) = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_d^2.$$

We can assume that $E \subset \mathbb{R}^+$. If $k = 0$ or $k = d$, then q is positive or negative definite. In this case the result is obvious, with a constant $|E|$. Thus we may assume that $0 < k < d$. Fix a point $x \in \mathbb{R}^d$ and put $R = \|x\|$. We call B the ball centred at x, and of radius $\min(1, R^{-1})$.

First, when $R \geqslant 2d^{1/4}$, we show that

$$|Z(E) \cap B| \leqslant C_1|E||B|. \tag{2.1}$$

Since $\|x\| = R$, one can suppose that one of the coordinates, say x_1, is positive and satisfies $R \geqslant x_1 \geqslant R/\sqrt{d}$. It is then elementary to show that $(Y_1, \ldots, Y_d) = (q(y), y_2, \ldots, y_d)$ is a system of coordinates inside the ball B. The Jacobian, equal to $2|y_1|$, is comparable to R (with absolute constants) since $R \geqslant y_1 \geqslant (R/\sqrt{d}) - R^{-1} \geqslant R/2\sqrt{d}$. The new coordinates satisfy $Y_1 \in E$, and $|Y_i - x_i| < R^{-1}$, so inequality (2.1) follows at once.

It remains to prove that when $R \leqslant 2d^{1/4}$,

$$|Z(E) \cap B| \leqslant C_2|E|\ln(1/|E|). \tag{2.2}$$

Let $y' = (y_1, \ldots, y_k)$ and $y'' = (y_{k+1}, \ldots, y_d)$. We first prove the inequality for the set $Z^+(E) = \{y \in Z(E); \|y'\| \geqslant \|y''\|\}$. We split $Z^+(E)$ into two parts. For the one with $\|y''\| \leqslant |E|$, (2.2) is clearly valid. When $\|y''\| > |E|$, we have, by the definition of q,

$$\|y'\|^2 \in \|y''\|^2 + E.$$

For fixed y'', the volume in y' is controlled by

$$\int_{r^2 \in \|y''\|^2 + E} dr = \int_{u \in \|y''\|^2 + E} \frac{du}{2\sqrt{u}} \leqslant C_3 \frac{|E|}{\|y''\|}.$$

Hence

$$|Z^+(E) \cap B| \leqslant C_3 \int_{|E| \leqslant \|y''\| \leqslant 2d^{1/4}} \frac{|E|}{\|y''\|}\, dy'',$$

which is bounded by a constant times $|E|$ when $k < d-1$ and by $|E|\ln(1/|E|)$ when $k = d-1$. We also define $Z^-(E)$ and obtain the same estimate. This concludes the proof. □

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REMARK 2.5. With minor changes, the proof actually gives the better bound $|E|$ when $1 < k < d-1$. The bound $|E|\ln(1/|E|)$ is attained for the quadratic form $2xy$ on $\mathbb{R}^2$ with $E = [-\varepsilon, \varepsilon]$. The result is false when q is degenerate.

We have the following corollaries, which give Theorem 1.4 for small values of C and Theorem 1.5.

COROLLARY 2.6. *Let $q(x,y)$ be a nondegenerate quadratic form on $\mathbb{R}^d \times \mathbb{R}^d$. Define the sets*

$$Z(\varepsilon) = \{(x,y); |q(x,y)| < \varepsilon\}.$$

If ε is small enough, then there exists a constant $C(\varepsilon) > 0$ such that for all $u, v \in L^2(\mathbb{R}^d)$,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus Z(\varepsilon)} |A(u,v)|^2 \geqslant C(\varepsilon)\|u\|^2\|v\|^2.$$

REMARK 2.7. Corollary 2.6 is false when q is degenerate. If we take functions u, v with compact support, then $A(u,v)$ is supported in a strip $\Sigma = \{(x,y); \|x\|^2 < C\}$.

COROLLARY 2.8. *Let $q(x,y)$ be a nondegenerate quadratic form on $\mathbb{R}^d \times \mathbb{R}^d$, and $n \geqslant 0$. Then there exists a constant $C(n) > 0$ such that for all $u, v \in L^2(\mathbb{R}^d)$,*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |q(x,y)|^n |A(u,v)(x,y)|^2 \, dx\, dy \geqslant C(n)\|u\|^2\|v\|^2.$$

Proof of Corollaries 2.6 *and* 2.8. Proposition 2.4 implies that the sets $Z(\varepsilon)$ and $\widetilde{Z(\varepsilon)}$ are thin if ε is small enough, since the forms q and $\tilde{q}(\xi,\eta) = q(-\eta,\xi)$ are nondegenerate. By [**10**], the pair $(Z(\varepsilon), \widetilde{Z(\varepsilon)})$ is a strong a-pair, and Theorem 2.1 yields

$$\|u\|^2\|v\|^2 \leqslant C(\varepsilon) \iint_{|q(x,y)|>\varepsilon} |A(u,v)(x,y)|^2 \, dx\, dy,$$

with a constant $C(\varepsilon)$ independent of u, v. Corollary 2.8 is a direct consequence of this inequality. This also proves Theorem 1.4 for small values of C. The result for general C will follow from our main result (see the next section). □

REMARK 2.9. We do not know the value of the optimal constant in this inequality. However it can be shown, but we will not do it here, that there are functions u, v minimizing this Heisenberg inequality when $q(x,y) = 2\langle x, y\rangle$. This may be done using standard techniques for minimizing functionals.

In [**3**], the author gives examples of uncertainty principles for the Fourier transform which have their analogues for the ambiguity function. In most cases it consists in replacing the quantity $f(x)\widehat{f}(y)$ by $A(u,v)(x,y)$. It may be interesting to mention that Theorem 1.5 has an analogue for the Fourier transform. It is a special form of Heisenberg's inequality.

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THEOREM 2.10. *For any function $f \in L^2(\mathbb{R}^d)$, we have*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle x, y\rangle^2 |f(x)\widehat{f}(y)|^2 \, dx\, dy \geqslant \frac{d}{16\pi^2} \|f\|^4_{L^2(\mathbb{R}^d)},$$

with equality if and only if $f(x) = Ce^{-\pi\langle Ax,x\rangle}$, and A any real positive definite matrix.

REMARK 2.11. In [**5**] one can find general versions of Heisenberg's inequality. Usually they are stated in terms of the moments of order 2 of $|f|^2$. Then the equality cases are particular Gaussian functions, radial if we take $\|x\|^2\|y\|^2$ instead of $\langle x, y\rangle$. Here this inequality, which is stronger, has the advantage of providing a characterization of any Gaussian function.

Proof of Theorem 2.10. We suppose for simplicity that f is real, in the Schwartz class. Then we have

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} \langle x, y\rangle^2 |f(x)\widehat{f}(y)|^2 \, dx\, dy &= (4\pi^2)^{-1} \sum_{i=1}^d \int_{\mathbb{R}^d} x_i^2 |f(x)|^2 \, dx \int_{\mathbb{R}^d} \left|\frac{\partial f}{\partial t_i}(t)\right|^2 dt \\ &\geqslant (4\pi^2)^{-1} \sum_{i=1}^d \left(\int_{\mathbb{R}^d} x_i f(x) \frac{\partial f}{\partial x_i}(x)\, dx\right)^2 = \frac{d}{16\pi^2}\|f\|^4.\end{aligned}$$

It is easily seen that equality holds only for Gaussian functions. □

3. *Proof of Beurling's uncertainty principle*

Throughout this section and the next one, the notation $I_N(u, v)$ will denote the quantity (1.1). For any function Φ and any number M, we also write

$$I_M(\Phi) = \iint_{\mathbb{R}^{2d}} \frac{|\Phi(x, y)|}{(1 + \|x\| + \|y\|)^M} e^{2\pi|\langle x,y\rangle|} \, dx\, dy. \tag{3.1}$$

Here we proceed with the proof of Theorem 1.2. Let $u, v \in L^2(\mathbb{R}^d)$, $u, v \neq 0$, satisfying (1.1). In order to simplify the proof, we want to restrict to functions u, v that are smooth (in fact restrictions to $\mathbb{R}^d$ of entire functions of order 2 in $\mathbb{C}^d$). The next proposition allows us to do so.

PROPOSITION 3.1. *If $u, v \in L^2(\mathbb{R}^d)$ satisfy* (1.1)*, then the functions u_1, v_1 such that $\widehat{u_1}(\xi) = e^{-\pi\|\xi\|^2} u(\xi)$ and $\widehat{v_1}(\xi) = e^{-\pi\|\xi\|^2} v(\xi)$ can be extended to entire functions of order 2 on $\mathbb{C}^d$. Moreover, we still have*

$$\iint_{\mathbb{R}^{2d}} \frac{|A(u_1, v_1)(x, y)|}{(1 + \|x\| + \|y\|)^N} e^{\pi|\langle x,y\rangle|} \, dx\, dy < \infty. \tag{3.2}$$

Proof. It is immediate that u_1 and v_1 extend to entire functions of order 2. Further, we know that $A(u_1, v_1)(x, y) = A(\widehat{u}_1, \widehat{v}_1)(-y, -x)$ (see [**1**, **2**]). Hence a

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direct computation gives

$$A(u_1,v_1)(x,y) = e^{-(\pi/2)\|y\|^2}\int_{\mathbb{R}^d} u\left(s-\frac{y}{2}\right)\overline{v\left(s+\frac{y}{2}\right)}e^{-2\pi\|s\|^2}e^{2i\pi\langle s,x\rangle}\,ds$$
$$= \frac{1}{\sqrt{2}}e^{-(\pi/2)\|y\|^2}\int_{\mathbb{R}^d} A(u,v)(-y,t)e^{-(\pi/2)\|x+t\|^2}\,dt.$$

Then, by Fubini's theorem,

$$I_N(u_1,v_1) \leqslant 2^{-1/2}\iiint_{\mathbb{R}^{3d}}\frac{|A(u,v)(-y,t)|}{(1+\|x\|+\|y\|)^N}e^{-(\pi/2)(\|y\|^2+\|t+x\|^2)}e^{2\pi|\langle x,y\rangle|}\,dy\,dx\,dt$$
$$\leqslant C\iint_{\mathbb{R}^{2d}}|A(u,v)(y,t)|B(y,t)\,dx\,dt,$$

with B defined by

$$B(y,t) = \int_{\mathbb{R}^d}\frac{e^{-(\pi/2)(\|y\|^2+\|t+x\|^2)}e^{\pi|\langle x,y\rangle|}}{(1+\|x\|+\|y\|)^N}\,dx.$$

Let us now check that

$$B(y,t) \leqslant C_N\frac{e^{\pi|\langle y,t\rangle|}}{(1+\|y\|+\|t\|)^N}, \tag{3.3}$$

from which the claim follows immediately. Actually,

$$B(y,t) \leqslant e^{\pi|\langle y,t\rangle|}\int_{\mathbb{R}^d}\frac{e^{-(\pi/2)\|y-x-t\|^2}+e^{-(\pi/2)\|y+x+t\|^2}}{(1+\|x\|+\|y\|)^N}\,dx, \tag{3.4}$$

since $\|y\|^2+\|x+t\|^2-2|\langle x,y\rangle|$ is equal to $\|y-x-t\|^2+2\langle y,t\rangle$ when $\langle x,y\rangle>0$, and to $\|y+x+t\|^2-2\langle y,t\rangle$ when $\langle x,y\rangle<0$. Finally, observe that the integral in the right-hand side of (3.4) is a convolution of a Gaussian function with an inverse polynomial. Hence the polynomial decay is preserved,

$$\int_{\mathbb{R}^d}\frac{e^{-(\pi/2)\|y-x-t\|^2}+e^{-(\pi/2)\|y+x+t\|^2}}{(1+\|x\|+\|y\|)^N}dx \leqslant C(1+\|y\|+\|t\|)^{-N},$$

and (3.3) is proved. □

It is sufficient to prove that u_1, v_1 are Hermite functions of the type given in (1.2). Indeed, u and v are then of the same type, and satisfy the other assumptions of Theorem 1.2. Changing the notations, we can assume that u and v are entire functions of order 2, apart from the other assumptions. In order to proceed with the proof, we introduce the parameter b, which is a real vector, and consider the family of functions

$$F_b(x,y) := A(u,v)(x,y-b)\overline{A(v,u)(x,y+b)}. \tag{3.5}$$

The key point of the proof is that the functions F_b satisfy the following relations.

LEMMA 3.2. *For almost every $b\in\mathbb{R}^d$, we have*

$$\iint_{\mathbb{R}^{2d}}\frac{|F_b(x,y)|}{(1+\|x\|+\|y\|)^{2N}}e^{2\pi|\langle x,y\rangle|}\,dx\,dy < +\infty$$
$$\iint_{\mathbb{R}^{2d}}\frac{|\widehat{F_b}(\xi,\eta)|}{(1+\|\xi\|+\|\eta\|)^{2N}}e^{2\pi|\langle \xi,\eta\rangle|}\,d\xi\,d\eta < +\infty. \tag{3.6}$$

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Proof. By Proposition 2.2, we have $\widehat{F_b}(\xi,\eta) = F_b(-\eta,\xi)$, so the second integral is equal to the first one. Note that

$$2\langle x,y\rangle = \langle x, y-b\rangle + \langle x, y+b\rangle.$$

Using the fact that

$$(1+\|b\|)(1+\|x\|+\|y\|) \geqslant (1+\|x\|+\|y+b\|)^{1/2}(1+\|x\|+\|y-b\|)^{1/2}$$

and Fubini's theorem, we get

$$\int_{\mathbb{R}^d} (1+\|b\|)^{-2N} I_{2N}(F_b)\, db \leqslant I_N(u,v)^2 < \infty,$$

where I_{2N} was defined by (3.1). In particular, $I_{2N}(F_b) < \infty$ for almost every $b \in \mathbb{R}^d$. Since $I_{2N}(F_b) = I_{2N}(\widehat{F_b})$, we obtain (3.6). □

REMARK 3.3. Let us add some remarks on functions satisfying (3.6). It expresses that F_b and $\widehat{F_b}$ decay rapidly in some directions at infinity, but not in all directions, as in the statement of Beurling's uncertainty principle in [**2**]. Thus general solutions to (3.6) are not expected to be L^1 functions (although our particular functions F_b are). Beurling's uncertainty principle in [**2**] characterizes the functions $f(x)\widehat{f}(y)$ satisfying (3.6) as Hermite functions. Proposition 3.6 will actually allow us to get a bilinear form of Beurling's theorem in [**2**], with a characterization of tensor products $f(x)g(y)$ satisfying (3.6). Again, the solutions will be Hermite functions, see Corollary 3.7. In a forthcoming paper we fully characterize the functions satisfying (3.6), and even the distributions, in dimension 2. Here the conclusion of Proposition 3.6 will suffice for our purpose.

Now we investigate some properties of the functions F satisfying (3.6).

3.1. *Some properties of functions F satisfying* (3.6)

In this section we study general solutions of (3.6). We want at first to regularize them.

DEFINITION 3.4. The *right regularization* of F is defined by

$$\mathcal{R}(F)(x,y) := \int_{\mathbb{R}^d} F(x,t) e^{-\pi\|x\|^2 - \pi\|y-t\|^2}\, dt.$$

The *left regularization* is defined by

$$\mathcal{L}(F)(x,y) := \int_{\mathbb{R}^d} F(s,y) e^{-\pi\|y\|^2 - \pi\|x-s\|^2}\, ds.$$

PROPOSITION 3.5. *Left and right regularizations leave* (3.6) *stable and are conjugated with respect to the Fourier transform*:

$$\mathcal{R}(\widehat{F}) = \widehat{\mathcal{L}(F)}.$$

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Proof. The commutation relation comes from Parseval's inequality. Then write

$$I_{2N}(\mathcal{L}(F)) \leqslant \iiint_{\mathbb{R}^{3d}} |F(s,y)| \frac{e^{-\pi\|s-x\|^2-\pi\|y\|^2+2\pi|\langle x,y\rangle|}}{(1+\|x\|+\|y\|)^{2N}}\,dx\,dy\,ds$$
$$= \iint_{\mathbb{R}^{2d}} |F(s,y)|B(s,y)\,ds\,dy.$$

Just as in the proof of Proposition 3.1, we have

$$B(s,y) \leqslant C\frac{e^{2\pi|\langle s,y\rangle|}}{(1+\|s\|+\|y\|)^{2N}},$$

which allows us to conclude the proof. □

The idea of using regularization comes initially from [**2**]. There, given a solution of (3.6) of the form $F(x,y) = f(x)\widehat{f}(y)$, we considered the function f_1 defined by $\widehat{f_1}(\xi) = \widehat{f}(\xi)e^{-\pi\|\xi\|^2}$, and proved that the new function $F_1(x,y) = f_1(x)\widehat{f_1}(y)$ also satisfies (3.6). We see that

$$F_1(x,y) = \int f(s)\widehat{f}(y)e^{-\pi\|y\|^2-\pi\|s-x\|^2}\,ds,$$

so Definition 3.4 is the natural choice for the regularizations in our more general context. Following ideas in [**2**], where we studied the entire function $f_1(z)f_1(iz)$, we are led to consider the entire function $G(z_1,z_2)$ on $\mathbb{C}^{2d}$ defined by

$$G(z_1,z_2) = \iint_{\mathbb{R}^2} F(x,y)e^{-\pi\|y\|^2-2i\pi\langle z_1,y\rangle}e^{-\pi\|x+z_2\|^2}\,dx\,dy. \tag{3.7}$$

The quantity $G(z_1,-z_2)$ is actually the Fourier transform, with respect to the y-variable, of the left regularization of F defined above. With this notation, it is easily seen that

$$\begin{aligned} G(z,r) &= \int_{\mathbb{R}^d} \mathcal{L}(F)(-r,\eta)e^{-2i\pi\langle z,\eta\rangle}d\eta \qquad \text{for } z\in\mathbb{C}^d,\ r\in\mathbb{R}^d \\ G(r,z) &= \int_{\mathbb{R}^d} \mathcal{R}(\widehat{F})(\xi,r)e^{-2i\pi\langle z,\xi\rangle}d\xi \qquad \text{for } z\in\mathbb{C}^d,\ r\in\mathbb{R}^d. \end{aligned} \tag{3.8}$$

We want first to show that $G(z,iz)$ is a polynomial in $z\in\mathbb{C}^d$.

PROPOSITION 3.6. *If F satisfies* (3.6)*, then $G(z,iz)$ is a polynomial in $z\in\mathbb{C}^d$.*

Proof. Consider for $z\in\mathbb{C}^d$ and $0<\alpha<\pi/2$

$$\theta_\alpha(z) = \int_0^{z_1}\dots\int_0^{z_d} G(e^{-i\alpha}w, iw)\,dw.$$

In order to deal with functions of one variable, fix a direction $\xi\in\mathbb{R}^d$. We consider $\theta_{\alpha,\xi}(z) = \theta_\alpha(z\xi)$, $z\in\mathbb{C}$. Because of the definition of G, the function $\theta_{\alpha,\xi}$ is an entire function of order 2. We obtain polynomial bounds for $\theta_{\alpha,\xi}$ on the axes $i\mathbb{R}$

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and $e^{i\alpha}\mathbb{R}$ as follows. For $R > 0$ for example, using (3.8),

$$\begin{aligned}|\theta_{\alpha,\xi}(iR)| &\leqslant \int_0^{R\xi_1} \dots \int_0^{R\xi_d} \int_{\mathbb{R}} |\mathcal{L}(F)(w,\eta)| e^{2\pi|\langle w,\eta\rangle|}\, d\eta\, dw \\ &\leqslant \int_0^{R\xi_1} \dots \int_0^{R\xi_d} \int_{\|\eta\|>8\|w\|} + \int_0^{R\xi_1} \dots \int_0^{R\xi_d} \int_{\|\eta\|\leqslant 8\|w\|} \\ &= A + B.\end{aligned}$$

Let us consider the terms A and B separately. We have

$$A \leqslant \int_{\mathbb{R}^d} \int_{\|\eta\|>8\|w\|} |\mathcal{L}(F)(w,\eta)| e^{\pi/4\|\eta\|^2}\, dw\, d\eta \leqslant C'.$$

This follows from the definition of the left regularization (Definition 3.4), and the fact that $(1+\|x\|+\|y\|)^{-2N} F(x,y) \in L^1(\mathbb{R}^{2d})$.

To get a bound for B, we use the fact that $\mathcal{L}(F)$ also satisfies (3.6).

$$\begin{aligned}B &\leqslant C(1+R)^{2N} \int_0^{R\xi_1} \dots \int_0^{R\xi_d} \int_{\|\eta\|\leqslant 8\|w\|} \frac{|\mathcal{L}(F)(w,\eta)|}{(1+|w|+|\eta|)^N} e^{2\pi|\langle w,\eta\rangle|}\, dr\, d\xi \\ &\leqslant C'(1+R)^{2N}.\end{aligned}$$

Thus we have obtained a polynomial bound of order $2N$ for $\theta_{\alpha,\xi}$ on the imaginary axis. In a similar way, using the second part of (3.8), we get the same bound (independent of α) on the $e^{i\alpha}\mathbb{R}$ axis. When applying the celebrated Phragmèn–Lindelöf principle for the entire functions of order 2, and following the same arguments as in [**2**], we conclude that $G(z,iz)$ is a polynomial of degree $\leqslant 2N-d$. □

Here we give an immediate corollary of Proposition 3.6. It actually already enables us to get a bilinear form of Beurling's uncertainty principle in [**2**].

COROLLARY 3.7. *Let $f, g \in L^2(\mathbb{R}^d)$. Suppose that*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)g(y)| + |\widehat{f}(x)\widehat{g}(y)|}{(1+\|x\|+\|y\|)^N} e^{2\pi|\langle x,y\rangle|}\, dx\, dy < \infty.$$

Then either one of the functions is zero, or they both have the form

$$f(x) = P(x)e^{-\pi\langle Ax,x\rangle}, \quad g(y) = Q(y)e^{-\pi\langle By,y\rangle},$$

with real positive symmetric matrices A, B, and polynomials P, Q whose degrees satisfy $\deg(P)+\deg(Q) < N-d$. The converse is true. In particular if $N \leqslant d$, f or g is zero.

Proof. We may assume that $f, g \neq 0$. Let us apply Proposition 3.6 to $F(x,y) = f(x)g(y)$. The function G defined by (3.7) may be written as $f_1(z_2)g_1(z_1)$, with entire functions f_1, g_1 on $\mathbb{C}^d$, defined by the formulae $\widehat{g_1}(\xi) = e^{-\pi\|\xi\|^2}\widehat{g}(-\xi)$ and $\widehat{f_1}(\xi) = e^{-\pi\|\xi\|^2}\widehat{f}(-\xi)$. Then $f_1(z)g_1(iz)$ is a polynomial in z. By similar arguments as in [**2**], we conclude that f_1 and g_1 are Hermite functions, and so are f and g. The conditions on the degree are checked by plugging the value of f and g into the double integral. □

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3.2. *End of the proof of Theorem* 1.2

We apply Proposition 3.6 to F_b. Let G_b be the corresponding function defined by (3.7), with $F = F_b$.

We can rewrite $G_b(z_1, z_2)$ as

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-\pi\|x+z_2\|^2} \int_{\mathbb{R}^d} A(u_b, v)(x,y)\overline{A(v,u_b)(x,y)} e^{-\pi\|y\|^2 - 2i\pi\langle y, z_1\rangle}\, dy\, dx \\ = \iiint_{\mathbb{R}^{3d}} e^{-\pi\|x+z_2\|^2} u\left(s+\frac{x}{2}\right) e^{2i\pi\langle b, s+x/2\rangle} \overline{v\left(s-\frac{x}{2}\right)} e^{-\pi\|t+z_1-s\|^2} \\ \times \overline{v\left(t+\frac{x}{2}\right)} u\left(t-\frac{x}{2}\right) e^{2i\pi\langle b, t-x/2\rangle}\, ds\, dt\, dx.\end{aligned}$$

Let us use the change of variables

$$\alpha = \frac{t-s-x}{2}, \quad \beta = \frac{t-s+x}{2}, \quad \gamma = s+t, \quad z_1 = \frac{1+i}{2}z, \quad z_2 = \frac{i-1}{2}z = iz_1.$$

Then

$$\begin{aligned}G\left(\frac{z+iz}{2}, \frac{iz-z}{2}\right) &= \int_{\mathbb{R}^d} \iint_{\mathbb{R}^{2d}} u\left(\frac{\gamma}{2}-\alpha\right) \overline{v\left(\frac{\gamma}{2}-\beta\right) v\left(\frac{\gamma}{2}+\beta\right)} u\left(\frac{\gamma}{2}+\alpha\right) \\ &\quad \times e^{-2\pi(\|\alpha\|^2+\|\beta\|^2) - 2i\pi\langle\beta, z\rangle - 2\pi\langle\alpha, z\rangle}\, d\alpha\, d\beta\, e^{2i\pi\langle b,\gamma\rangle}\, d\gamma \\ &= \int \phi_{u,\gamma}(z)\phi_{\overline{v},\gamma}(iz) e^{2i\pi\langle b,\gamma\rangle}\, d\gamma,\end{aligned}$$

with $\phi_{u,\gamma}$ equal to the entire function of order 2

$$\phi_{u,\gamma}(z) = \int_{\mathbb{R}} u\left(\frac{\gamma}{2}-\alpha\right) u\left(\frac{\gamma}{2}+\alpha\right) e^{-2\pi\|\alpha\|^2 - 2\pi\langle\alpha, z\rangle}\, d\alpha,$$

and $\phi_{\overline{v},\gamma}$ defined accordingly. We get from Proposition 3.6 that $G((z+iz)/2, (iz-z)/2)$ is a polynomial of degree at most $2N-d$. Hence

$$\int \frac{d^{2N}}{dz^{2N}} \phi_{u,\gamma}(z\xi)\phi_{\overline{v},\gamma}(iz\xi) e^{2i\pi\langle b,\gamma\rangle}\, d\gamma = 0$$

for all ξ, almost every b, and all $z \in \mathbb{C}$. The integrand is zero, and in particular $\phi_{u,\gamma}(z\xi)\phi_{\overline{v},\gamma}(iz\xi)$ is a polynomial for almost every value of the parameter γ. It follows that $\phi_{u,\gamma}(z)$, restricted to every line, is either identically zero, or has at most $2N$ zeros. Hence (see [**2**, Proposition 2.3])

$$\phi_{u,\gamma}(z) = P_\gamma(z) e^{\langle A_\gamma z, z\rangle + i\langle b_\gamma, z\rangle}$$

for every complex vector z. Here A_γ is a complex symmetric matrix, b_γ is a complex vector, and P_γ is a polynomial.

Since $\phi_{u,\gamma}(iz)$ is the Fourier transform of an L^2 function, the real part of A_γ is positive definite. Taking the Fourier transform of $\phi_{u,\gamma}(i.)$, we get

$$u\left(\frac{\gamma}{2}-\right) u\left(\frac{\gamma}{2}+z\right) = Q_\gamma(z) e^{-\langle A'_\gamma z, z\rangle + i\langle b'_\gamma, z\rangle},$$

with Q_γ a polynomial, A'_γ a matrix, and b'_γ a complex vector.

We turn to the final step. If we fix a particular value of γ in this formula, we see that the restriction of u to every line passing through γ is either identically vanishing, or has at most N zeros. Since it is an entire function of order 2, it

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follows that

$$u(x) = P(x)e^{-\langle Ax,x\rangle - 2i\pi\langle w_1,x\rangle}$$

for all $x \in \mathbb{R}$. The matrix A is complex symmetric, with positive real part, w_1 is complex, and P is a polynomial of degree at most N. By the same argument, the function v has the form

$$v(x) = Q(x)e^{-\langle Bx,x\rangle - 2i\pi\langle w_2,x\rangle}.$$

It remains to show that $A = B$ is real, and also that $w_1 = w_2$.

With these notations, $\phi_{u,\gamma}(z)$ has the form

$$P_{\gamma,w_1}(z)e^{(\pi/2)\langle(\mathrm{Id}+A)^{-1}z,z\rangle}$$

and $\phi_{\overline{v},\gamma}(iz)$ has the form

$$Q_{\gamma,w_1}(z)e^{(\pi/2)\langle(\mathrm{Id}+\overline{B})^{-1}z,z\rangle}.$$

Their product is a polynomial, so $A = \overline{B}$. Now if we explicitly write the radar-ambiguity function associated to u and v, we see that it is of the form

$$A(u,v)(x,y) = R(x,y)e^{-(\pi/2)\langle A^{-1}(y+w_1-\overline{w_2}),y+w_1-\overline{w_2}\rangle}e^{-(\pi/2)\langle Ax,x\rangle - i\pi\langle x,w_1+\overline{w_2}\rangle}$$

(R is a polynomial of degree $\deg(P) + \deg(Q)$). For $I_N(u,v)$ to be finite, it is necessary that for every x, y,

$$\mathrm{Re}\left(\frac{\langle A^{-1}y,y\rangle}{2} + \frac{\langle Ax,x\rangle}{2}\right) \geqslant \pi|\langle x,y\rangle|.$$

This implies that A is real positive, and so $A = \overline{B} = B$. The linear terms in the exponential then have to be purely imaginary, which gives exactly $w_1 = w_2$.

If we just plug this expression of $A(u,v)$ into $I_N(u,v)$, we see that the condition $\deg(P) + \deg(Q) < N - d$ is necessary and sufficient for $I_N(u,v)$ to be finite. We have completed the proof of Theorem 1.2.

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