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CORES OVER RAMSEY STRUCTURES

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Abstract. We prove that if an ω-categorical structure has an ω-categorical homogeneous Ramsey expansion, then so does its model-complete core.

§1. Introduction.

1.1. Model-complete cores. A countable ω-categorical structure $\mathbb{A}$ is *model-complete* if the automorphisms of $\mathbb{A}$ form a dense subset of the self-embeddings of $\mathbb{A}$ in the topology of pointwise convergence. It is a *core* if all its endomorphisms are self-embeddings. A classical result by Saracino [19] states that every ω-categorical structure $\mathbb{A}$ is *embedding-equivalent* to a model-complete structure: there exists a model-complete structure $\mathbb{A}'$ such that $\mathbb{A}$ embeds into $\mathbb{A}'$ and vice-versa. The structure $\mathbb{A}'$ is moreover unique to isomorphism, and is called the *model-companion* of $\mathbb{A}$. This result was later subsumed by a result of [6], where it is proved that every ω-categorical structure $\mathbb{A}$ is *homomorphically equivalent* to a model-complete core $\mathbb{A}'$, i.e., $\mathbb{A}$ has a homomorphism to $\mathbb{A}'$ and vice-versa. Again, $\mathbb{A}'$ is unique up to isomorphism, and called the *model-complete core of* $\mathbb{A}$. The latter result was inspired by and has gained considerable importance in applications to *constraint satisfaction problems*. These are certain computational problems associated with relational structures which are invariant under homomorphic equivalence. Much of the algebraic theory of such problems only works for model-complete cores—see [4–6].

One weakness of the proof in [6] as well as of the later proof in [2] is that they are non-constructive, and it is often hard to decide what properties are transferred from $\mathbb{A}$ to its model-complete core $\mathbb{A}'$. One important property which *is* known to be preserved is ω-categoricity [6], and so are *homogeneity* and *finite boundedness*, as can be seen easily from the proofs. However, often the structure $\mathbb{A}$ of interest does not have these properties itself, but is only first-order definable in a structure $\mathbb{B}$ which might enjoy them; we then call $\mathbb{A}$ a *first-order reduct* of $\mathbb{B}$. It is interesting to note here that the model-complete core $\mathbb{A}'$ of a structure $\mathbb{A}$ is sensitive to which of the first-order definable relations of $\mathbb{A}$ belong to its signature; in particular, even if $\mathbb{A}$ and $\mathbb{B}$ are first-order reducts of one another, their model-complete cores might differ. And while the above-mentioned properties are inherited by $\mathbb{A}'$ from $\mathbb{A}$ rather straightforwardly, it is less clear whether, say, the model-complete core of a

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first-order reduct of a finitely bounded homogeneous structure is again a first-order reduct of a finitely bounded homogeneous structure. This question is of fundamental importance in applications to constraint satisfaction problems mentioned above, in particular in relation with a complexity dichotomy conjecture of Bodirsky and the second author (see [10]).

QUESTION 1. *Is the class of first-order reducts of a finitely bounded homogeneous structures closed under taking model-complete cores*?

1.2. Ramsey expansions. In this article, we shall answer this question for first-order reducts of finitely bounded homogeneous *Ramsey* structures. This additional assumption might, however, be void since the following problem is open.

QUESTION 2. *Does every finitely bounded homogeneous structure have a finitely bounded homogeneous Ramsey expansion*? *In other words, is every first-order reduct of a finitely bounded homogeneous structure also a first-order reduct of a finitely bounded homogeneous Ramsey structure*?

This question has, in various formulations and with varying scope, been considered by several authors: by Bodirsky and the second author when formulating the dichotomy conjecture, and also in the context of a decidability result for first-order reducts of finitely bounded homogeneous Ramsey structures [11]; and later by Melleray, Van Thé, and Tsankov [16] in the context and language of topological dynamics. In [8], Bodirsky formulates a conjecture claiming a positive answer for homogeneous structures in a finite language, and extensively argues the importance of this conjecture. Nešetřil addressed the question indirectly in the context of the characterisation of Ramsey classes [18]. Hubička and Nešetřil then obtained a positive answer for an impressive number of structures [13, 14]. In [12], Evans, Hubička and Nešetřil gave an example of an ω-categorical structure without ω-categorical Ramsey expansion, but the problem as formulated in Question 2 remains open.

Given the importance of Ramsey expansions, it is natural to investigate how their existence is preserved under various constructions. The stability of the Ramsey property in this sense is the main theme of the survey [8], where one of the claims is that if $\mathbb{A}$ has a homogeneous Ramsey expansion by first-order definable relations, then so do its model-complete core [8, Theorem 3.18] and its model-companion [8, Theorem 3.15] (this turns out to be incorrect, as we are going to show). The following problems were left open; we note that the definition of a Ramsey structure in [8] slightly differs from ours, and that the questions that follow have been reformulated to agree with our terminology.

QUESTION 3 (Questions 7.2 and 7.3 in [8]). *Let $\mathbb{A}$ be a structure.*

(1) *Suppose that $\mathbb{A}$ has a homogeneous Ramsey expansion with a finite signature. Does the model-complete core of $\mathbb{A}$ (resp., its model-companion) have such an expansion*?
(2) *Suppose that $\mathbb{A}$ has an ω-categorical homogeneous Ramsey expansion. Does the model-complete core of $\mathbb{A}$ (resp., its model-companion) have such an expansion*?

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1.3. Results. Our main theorem is the following.

THEOREM 4. *Let $\mathbb{A}$ be a first-order reduct of an ω-categorical homogeneous Ramsey structure $\mathbb{B}$, and let $\mathbb{A}'$ be its model-complete core. Then*:

- *$\mathbb{A}'$ also is a first-order reduct of an ω-categorical homogeneous Ramsey structure $\mathbb{B}'$ that is a substructure of $\mathbb{B}$.*
- *If $\mathbb{B}$ is finitely bounded, then $\mathbb{B}'$ can be chosen to be finitely bounded as well.*

This provides a positive answer to Question 3 for model-complete cores. As a by-product, we obtain the same answer for the variant with model-companions. Indeed, given any ω-categorical structure $\mathbb{A}$, let $\mathbb{C}$ be the expansion of $\mathbb{A}$ by the complement of each of its relations. Let $\mathbb{C}'$ be the model-complete core of $\mathbb{C}$, and let $\mathbb{A}'$ be the reduct of $\mathbb{C}'$ obtained by forgetting the new relations. Then $\mathbb{A}'$ is model-complete since it has the same embeddings as $\mathbb{C}'$. Any homomorphism from $\mathbb{C}$ to $\mathbb{C}'$ has to be an embedding since the complement of every relation has to be preserved, so that there exists an embedding of $\mathbb{A}$ into $\mathbb{A}'$, i.e., $\mathbb{A}'$ is the model-companion of $\mathbb{A}$. Thus, if $\mathbb{A}$ has a homogeneous Ramsey expansion (with finite signature), so does $\mathbb{C}$, which means by Theorem 4 that $\mathbb{C}'$ has a homogeneous Ramsey expansion, and finally this implies that $\mathbb{A}'$ has a homogeneous Ramsey expansion.

In the course of this research we discovered that Theorems 3.15 and 3.18 of [8] mentioned before Question 3 are incorrect, as shown by the following example. Let $E, N, S, <$ be binary relation symbols, and let P be a constant symbol. Let $\mathbb{C}$ be the Fraïssé limit of the class of finite structures in this signature for which S is the complement of an equivalence relation with two classes, E is an undirected graph relation without loops contained in the complement of S and such that the element named by P is not E-related to any other element, N is the complement of E, and $<$ is a linear order. Then $\mathbb{C}$ is homogeneous, and easily seen to be Ramsey (the square of the automorphism group of the random graph with a generic linear order has a continuous homomorphism onto the automorphism group of $\mathbb{C}$, hence the latter is extremely amenable). Let $\mathbb{A}$ be the reduct of $\mathbb{C}$ with the relations $<$, E, N, and S. Since their automorphism groups are equal, $\mathbb{A}$ and $\mathbb{C}$ are first-order reducts of one another. But the model-complete core and the model-companion of $\mathbb{A}$ are isomorphic to the structure which we would have obtained constructing $\mathbb{C}$ without the special element denoted by P; they are homogeneous, but not Ramsey (the latter is a consequence of the fact that their automorphism groups act continuously on the classes of the complement of S without a fixed point, and hence are not extremely amenable).

It follows directly from our theorem by setting $\mathbb{A} = \mathbb{B}$ that if $\mathbb{A}$ is itself homogeneous and Ramsey, then so is its model-complete core (this was already observed in [7]). One can push this further and use our theorem to derive that if $\mathbb{A}$ is an ω-categorical structure which has a homogeneous Ramsey expansion by relations with an existential positive definition in $\mathbb{A}$, then its model-complete core has the same property; this is basically a consequence of the fact that homomorphisms preserve existential positive definitions. The same is true for the model-companion and existential definitions. These observations can arguably be viewed as the best possible correction of the aforementioned results from [8].

Since Theorem 4 is also compatible with the additional condition of finite boundedness, it implies that we obtain a positive answer to Question 1 provided

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Question 2 has a positive answer: that is, we obtain that the class of first-order reducts of finitely bounded homogeneous Ramsey structures is closed under taking model-complete cores.

Besides its theoretical interest, Theorem 4 is also practical in that it gives a concrete way of computing the structures $\mathbb{B}'$ and $\mathbb{A}'$. For example, a corollary of Theorem 4 is that if $\mathbb{A}$ is any first-order reduct of the countable universal homogeneous tournament $\mathbb{T}$ (which is *not* a Ramsey structure itself, but has a homogeneous Ramsey expansion by a linear order), then its model-complete core $\mathbb{A}'$ is either a one-element structure or is itself a first-order reduct of $\mathbb{T}$. Similar results can be obtained using Theorem 4 for first-order reducts of countable homogeneous graphs; the (short) proofs of these results can be found in [17]. Finally, a similar proof shows that if $\mathbb{A}$ is any first-order reduct of the countable universal homogeneous partial order $\mathbb{P}$ and $\mathbb{A}'$ is its model-complete core, then $\mathbb{A}'$ is either a one-element structure, a first-order reduct of the order of the rationals, or a first-order reduct of $\mathbb{P}$. Note that in the second case, if one takes the structure $\mathbb{B}$ to be the universal homogeneous partial order with a linear expansion in order to apply Theorem 4 ($\mathbb{P}$ does not work since it is not Ramsey), then $\mathbb{B}'$ is not isomorphic to $\mathbb{B}$; this case occurs, in particular, if $\mathbb{A}$ is chosen to be $\mathbb{P}$ itself.

It might be interesting to note that our proof does not use Bodirsky's theorem on the existence of model-complete cores. Rather than that, it refines, and perhaps sheds some light on, the second proof of this result in [2]. On the other hand, it is built upon the Ramsey property, and deriving the result using our approach seems rather unpractical.

§2. Preliminaries. We use blackboard boldface letters such as $\mathbb{A}, \mathbb{B}, \ldots$ for relational structures, and the same letters in plain font $A, B, \ldots$ for their domains. Similarly, we write $\mathcal{M}, \mathcal{N}, \ldots$ for transformation semigroups, in particular for permutation groups, and $M, N, \ldots$ for their domains. All domains of relational structures as well as of transformation semigroups are tacitly assumed to be countable.

Let $\mathcal{G}$ be a permutation group. Then $\mathcal{G}$ naturally acts componentwise on G^n, for all $n \geq 1$. Any minimal non-empty invariant set under any such action will be called an *orbit* of $\mathcal{G}$; we will sometimes use the notion *n-orbit* when we wish to specify the action. A permutation group $\mathcal{G}$ is *oligomorphic* if it has finitely many n-orbits for every $n \geq 1$.

The automorphism group $\mathrm{Aut}(\mathbb{A})$ of any relational structure $\mathbb{A}$ is a permutation group, and $\mathbb{A}$ is ω-categorical if $\mathrm{Aut}(\mathbb{A})$ is oligomorphic. The group $\mathrm{Aut}(\mathbb{A})$ is a closed subgroup of the full symmetric group on its domain A with respect to the *topology of pointwise convergence*, i.e., the product topology on A^A where A is taken to be discrete. It is called *extremely amenable* if any continuous action on a compact Hausdorff space has a fixed point.

The endomorphism monoid $\mathrm{End}(\mathbb{A})$ of a relational structure $\mathbb{A}$ is a transformation monoid on its domain A. This monoid also acts naturally componentwise on finite powers of A, and we shall write $t(a)$ for the n-tuple obtained by applying $t \in \mathrm{End}(\mathbb{A})$ to a tuple $a \in A^n$, for any $n \geq 1$. The monoid $\mathrm{End}(\mathbb{A})$ also bears the topology of pointwise convergence, and $\mathbb{A}$ is called a *model-complete core* if $\mathrm{Aut}(\mathbb{A})$ is dense in

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$\mathrm{End}(\mathbb{A})$ with respect to this topology. If $\mathcal{M}$ is a subset of M^M (e.g., a transformation semigroup, or even a permutation group on M), then we write $\overline{\mathcal{M}}$ for the closure of $\mathcal{M}$ in M^M (and not in the symmetric group on M, even if $\mathcal{M}$ is a permutation group!) with respect to the topology of pointwise convergence. Thus, a function $f : M \to M$ is an element of $\overline{\mathcal{M}}$ if for every finite $F \subseteq M$ there exists $g \in \mathcal{M}$ such that g agrees with f on F. A transformation semigroup $\mathcal{M}$ is *closed* if $\mathcal{M} = \overline{\mathcal{M}}$.

If $\mathcal{M}, \mathcal{N}$ are transformation semigroups on the same domain, then $\mathcal{M}$ is *left-invariant* (*right-invariant*) under $\mathcal{N}$ if $n \circ m \in \mathcal{M}$ ($m \circ n \in \mathcal{M}$) for all $n \in \mathcal{N}$ and all $m \in \mathcal{M}$. The semigroup $\mathcal{M}$ is *invariant* under $\mathcal{N}$ if it is left- and right-invariant. A *left-ideal* of $\mathcal{M}$ is a subsemigroup which is left-invariant under $\mathcal{M}$.

Two structures $\mathbb{A}, \mathbb{A}'$ are *homomorphically equivalent* if there exists a homomorphism from $\mathbb{A}$ to $\mathbb{A}'$ and vice-versa. Whenever $\mathbb{A}$ is a relational structure, and $\mathbb{A}', \mathbb{A}''$ are model-complete cores which are homomorphically equivalent to $\mathbb{A}$, then $\mathbb{A}', \mathbb{A}''$ are isomorphic. We can thus speak of the model-complete core of a structure $\mathbb{A}$ if it exists.

A structure $\mathbb{A}$ is *homogeneous* if every isomorphism between finite induced substructures of $\mathbb{A}$ extends to an automorphism of $\mathbb{A}$. If $\mathbb{A}$ is homogeneous, then its *age*, i.e., the class of its finite induced substructures up to isomorphism, has the *amalgamation property* (*AP*): a class $\mathcal{K}$ of structures has the AP if for all $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2 \in \mathcal{K}$ and all embeddings e_1, e_2 of $\mathbb{A}_0$ into $\mathbb{A}_1, \mathbb{A}_2$, respectively, there exist embeddings f_1, f_2 of $\mathbb{A}_1, \mathbb{A}_2$ into a structure $\mathbb{C} \in \mathcal{K}$ such that $f_1 \circ e_1 = f_2 \circ e_2$. Conversely, if the age $\mathcal{K}$ of some countable structure $\mathbb{A}'$ has the AP, then there exists a countable homogeneous structure $\mathbb{A}$ whose age is equal to $\mathcal{K}$. The structure $\mathbb{A}$ is called the *Fraïssé limit of* $\mathcal{K}$.

A class $\mathcal{K}$ of finite structures in a common finite signature is called *finitely bounded* if there exists a finite set $\mathcal{F}$ of structures in that signature such that membership in $\mathcal{K}$ is equivalent to not embedding any member of $\mathcal{F}$. An infinite structure is called finitely bounded if its age is finitely bounded.

Given two structures $\mathbb{S}, \mathbb{F}$, an *isomorphic copy of* $\mathbb{S}$ *in* $\mathbb{F}$ is an embedding from $\mathbb{S}$ to $\mathbb{F}$. A structure $\mathbb{A}$ is *Ramsey* if for all structures $\mathbb{S}, \mathbb{F}$ in its age and all colorings of the isomorphic copies of $\mathbb{S}$ in $\mathbb{A}$ with two colors there exists an isomorphic copy of $\mathbb{F}$ in $\mathbb{A}$ on which the coloring is constant. If $\mathbb{A}$ is homogeneous, then this is the case if and only if $\mathrm{Aut}(\mathbb{A})$ is extremely amenable [15].

§3. The proof.

3.1. Range-rigidity.

DEFINITION 5. Let $\mathcal{G}$ be a permutation group, and let $g : G \to G$ be a function. We call g *range-rigid* with respect to $\mathcal{G}$ if for all $\beta \in \mathcal{G}$ we have $g \in \overline{\{\alpha \circ g \circ \beta \circ g \mid \alpha \in \mathcal{G}\}}$; in other words, every orbit of $\mathcal{G}$ which has a tuple within the range of g is invariant under g.

LEMMA 6. *Let $\mathbb{B}$ be a homogeneous structure, and let $g : B \to B$ be range-rigid with respect to* $\mathrm{Aut}(\mathbb{B})$. *Then the age of the structure induced by the range of g in $\mathbb{B}$ has the AP.*

PROOF. Let $U, V, W \subseteq B$ be so that $g[U], g[V], g[W]$ are finite, and such that the structure $\mathbb{S}_U$ induced by $g[U]$ embeds into the structures $\mathbb{S}_V$ and $\mathbb{S}_W$ induced by $g[V]$

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and $g[W]$, respectively. By the homogeneity of $\mathbb{B}$, we know that these embeddings can be performed by restricting automorphisms $\alpha, \beta \in \mathrm{Aut}(\mathbb{B})$ to $g[U]$. Consider the structure $\mathbb{F}$ induced by $F := g[\alpha^{-1}[g[V]] \cup \beta^{-1}[g[W]]$ in $\mathbb{B}$. Then $\mathbb{F}$ is an amalgam of $\mathbb{S}_V$ and $\mathbb{S}_W$ over $\mathbb{S}_U$ and embeddings given by α, β: the witnessing embeddings of the amalgamation are the restrictions of $g \circ \alpha^{-1}$ and $g \circ \beta^{-1}$ to $g[V]$ and $g[W]$, respectively: the fact that these restrictions are embeddings follows from the range-rigidity of g. ⊣

Definition 7. Let $\mathbb{B}$ be a homogeneous structure, and let $g : B \to B$ be range-rigid with respect to $\mathrm{Aut}(\mathbb{B})$.

- We denote by $\mathbb{B}_g$ the Fraïssé limit of the class of finite structures induced by the range of g (which has the AP, by Lemma 6). By the homogeneity of $\mathbb{B}$, we may assume that $\mathbb{B}_g$ is an induced substructure of $\mathbb{B}$.
- If $\mathbb{A}$ is first-order definable in $\mathbb{B}$ without quantifiers, then we denote by $\mathbb{A}_g$ the substructure induced by the domain of $\mathbb{B}_g$ in $\mathbb{A}$.

We remark that in Definition 7, if $\mathbb{B}$ is ω-categorical, then any first-order reduct $\mathbb{A}$ has a quantifier-free first-order definition in $\mathbb{B}$. Hence, $\mathbb{A}_g$ is well-defined in that case.

In the following three lemmas, we show that $\mathbb{B}_g$ retains the properties of $\mathbb{B}$ that we are interested in.

Lemma 8. *Let $\mathbb{B}$ be a homogeneous structure, and let $g : B \to B$ be range-rigid with respect to* $\mathrm{Aut}(\mathbb{B})$. *If $\mathbb{B}$ is ω-categorical, then so is $\mathbb{B}_g$.*

Proof. Since $\mathbb{B}_g$ is homogeneous, it suffices to prove that for every $n \geq 1$, there are only finitely many atomic formulas with n free variables modulo equivalence. Since $\mathbb{B}_g$ is a substructure of $\mathbb{B}$, if two atomic formulas are equivalent in $\mathbb{B}$ then they are equivalent in $\mathbb{B}_g$. Since $\mathbb{B}$ is homogeneous and ω-categorical, for every $n \geq 1$ there are only finitely many atomic formulas with n free variables in $\mathbb{B}$, and we get the desired result. ⊣

Lemma 9. *Let $\mathbb{B}$ be a homogeneous structure, and let $g : B \to B$ be range-rigid with respect to* $\mathrm{Aut}(\mathbb{B})$. *If $\mathbb{B}$ is finitely bounded, then so is $\mathbb{B}_g$.*

Proof. Let $\mathcal{F}$ be a finite set of forbidden substructures for the age of $\mathbb{B}$. Let $m \geq 1$ be the maximum of the arities of the relations of $\mathbb{B}$. Let $\mathcal{F}'$ consist of all structures in $\mathcal{F}$, plus all structures on the set $\{1, \ldots, m\}$ which are isomorphic to a substructure of $\mathbb{B}$ but not of $\mathbb{B}_g$.

Clearly, if a finite structure in the signature of $\mathbb{B}$ embeds a member of $\mathcal{F}'$, then it cannot be in the age of $\mathbb{B}_g$. Conversely, if such a structure $\mathbb{F}$ does not embed any member of $\mathcal{F}'$, then it embeds into $\mathbb{B}$, hence we may assume it is a substructure of $\mathbb{B}$. Applying g to this substructure, we obtain a structure isomorphic to $\mathbb{F}$, because g preserves all m-orbits which intersect its range, and all m-element substructures of $\mathbb{F}$ belong to such orbits. Hence, g shows that $\mathbb{F}$ embeds into $\mathbb{B}_g$. ⊣

Lemma 10. *Let $\mathbb{B}$ be a homogeneous structure, and let $g : B \to B$ be range-rigid with respect to* $\mathrm{Aut}(\mathbb{B})$. *If $\mathbb{B}$ is Ramsey, then so is $\mathbb{B}_g$.*

Proof. Let $\mathbb{S}, \mathbb{F}$ be finite induced substructures of $\mathbb{B}_g$, and let χ be a coloring of the isomorphic copies of $\mathbb{S}$ in $\mathbb{B}_g$ with two colors. Let $f : g[B] \to B_g$ be a function

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which is an embedding with respect to the relations of $\mathbb{B}$; such an embedding exists by the homogeneity of $\mathbb{B}_g$ and since the age of the structure induced by $g[B]$ in $\mathbb{B}$ is equal to the age of $\mathbb{B}_g$. The coloring χ then induces a coloring χ' of the isomorphic copies of $\mathbb{S}$ in $\mathbb{B}$: any such copy is by composing with $f \circ g$ sent to an isomorphic copy of $\mathbb{S}$ in $\mathbb{B}_g$, by the range-rigidity of g; this second copy is assigned a color by χ. Since $\mathbb{B}$ is Ramsey, there exists an isomorphic copy of $\mathbb{F}$ in $\mathbb{B}$ on which χ' is constant. Composing this copy with $f \circ g$ then yields, again by the range-rigidity of g, a copy of $\mathbb{F}$ in $\mathbb{B}_g$. Clearly, χ is constant on this latter copy. ⊣

LEMMA 11. *Let $\mathbb{B}$ be a homogeneous structure, and let $\mathbb{A}$ be first-order definable in $\mathbb{B}$ without quantifiers. Suppose that $\mathcal{N} \subseteq \mathrm{End}(\mathbb{A})$ is a minimal closed left ideal, and that $g \in \mathcal{N}$ is range-rigid with respect to* $\mathrm{Aut}(\mathbb{B})$. *Then $\mathbb{A}_g$ is the model-complete core of $\mathbb{A}$.*

PROOF. We first prove that the set

$$\mathcal{I} := \{(a,b) \mid \exists n \geq 1 \; \exists e \in \mathrm{End}(\mathbb{A}_g) \;\; (a,b \in (A_g)^n \wedge e(a) = b)\}$$

is a back-and-forth system of partial isomorphisms of $\mathbb{A}_g$. This implies that $\mathbb{A}_g$ is a model-complete core, since we then obtain for every $e \in \mathrm{End}(\mathbb{A}_g)$ and every finite tuple a an automorphism $\alpha \in \mathrm{Aut}(\mathbb{A}_g)$ such that $\alpha(a) = e(a)$. To prove that $\mathcal{I}$ consists of partial isomorphisms of $\mathbb{A}_g$ and has the back-and-forth property, note that it suffices to show the following: for every $e \in \mathrm{End}(\mathbb{A}_g)$ and every finite tuple a of elements in $\mathbb{A}_g$, there exists $e' \in \mathrm{End}(\mathbb{A}_g)$ such that $e'(e(a)) = a$.

To this end, let $f : g[B] \to B_g$ be a function which is an embedding with respect to the relations of $\mathbb{B}$, such an embedding exists since the age of the structure induced by $g[B]$ in $\mathbb{B}$ is the same as the age of $\mathbb{B}_g$. Moreover, one can assume that a lies within the range of $f \circ g^2$. Indeed, also the age of the structure induced by $g^2[B]$ in $\mathbb{B}$ is the same as that of $\mathbb{B}_g$. Thus, there is a tuple b in $g^2[B]$ which induces the same substructure as a. Since f is an embedding with respect to the relations of $\mathbb{B}$, the structures induced by $f(b)$ and a in $\mathbb{B}_g$ are isomorphic, so that by the homogeneity of $\mathbb{B}_g$ there is an automorphism $\alpha \in \mathrm{Aut}(\mathbb{B}_g)$ such that $\alpha(f(b)) = a$. Replacing f by $\alpha \circ f$, we can therefore assume without loss of generality that a lies in the range of $f \circ g^2$. Let b be a tuple such that $a = (f \circ g^2)(b)$. By the fact that g lies in a minimal closed left ideal of $\mathrm{End}(\mathbb{A})$, there exists $h \in \mathrm{End}(\mathbb{A})$ such that

$$h \circ (e \circ f \circ g)(g(b)) = g(b) \,.$$

Hence,

$$a = (f \circ g^2)(b) = (f \circ g)(g(b)) = (f \circ g) \circ h \circ (e \circ f \circ g)(g(b)) = (f \circ g \circ h) \circ e(a) \,.$$

The restriction of the function $f \circ g \circ h \in \mathrm{End}(\mathbb{A})$ to A_g thus bears witness to the statement we wanted to prove.

Since $\mathbb{A}_g$ is a substructure of $\mathbb{A}$, and since $f \circ g$ is a homomorphism from $\mathbb{A}$ to $\mathbb{A}_g$, the structures $\mathbb{A}_g$ and $\mathbb{A}$ are homomorphically equivalent. Whence, $\mathbb{A}_g$ is indeed the model-complete core of $\mathbb{A}$. ⊣

3.2. From extreme amenability to canonicity to range-rigidity. The following definition from [11] is incomparable to range-rigidity, but will allow us to produce range-rigid functions.

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DEFINITION 12. Let $\mathcal{G}$ be a permutation group, and let $g : G \to G$ be a function. We call g *canonical* with respect to $\mathcal{G}$ if for all $\beta \in \mathcal{G}$ we have $g \in \overline{\{\alpha \circ g \circ \beta \mid \alpha \in \mathcal{G}\}}$; in other words, the image of any orbit of $\mathcal{G}$ under g is contained in an orbit.

LEMMA 13 (The canonisation lemma (Lemma 14 in [11] and Theorem 5 in [9])). *Let $\mathcal{G}$ be a closed oligomorphic extremely amenable permutation group, and let $g : B \to B$. Then*

$$\overline{\{\alpha \circ g \circ \beta \mid \alpha, \beta \in \mathcal{G}\}}$$

contains a canonical function with respect to $\mathcal{G}$.

LEMMA 14. *Let $\mathcal{G}$ be a closed oligomorphic permutation group, and let $\mathcal{M} \subseteq G^G$ be a non-empty closed transformation semigroup which is invariant under $\mathcal{G}$ and which contains a canonical function with respect to $\mathcal{G}$. Then $\mathcal{M}$ contains a range-rigid function with respect to $\mathcal{G}$.*

PROOF. Pick any canonical function $g \in \mathcal{M}$. For every $n \geq 1$, there exist $k_n \geq 1$ such that $g^{k_n} \circ g^{k_n}[O] \subseteq g^{k_n}[O]$ for all n-orbits O of $\mathcal{G}$. Let $\sim$ be the equivalence relation on G^G defined by $f \sim f'$ if $f' \in \overline{\{\alpha \circ f \mid \alpha \in \mathcal{G}\}}$. It is known that $G^G/\sim$ is compact (see, e.g., Lemma 4 in [9]). The sequence $([g^{k_n}]_\sim)_{n \geq 1}$ thus has an accumulation point in $G^G/\sim$, which means that there exists a sequence $(\alpha_n)_{n \geq 1}$ of elements of $\mathcal{G}$ such that $(\alpha_n \circ g^{k_n})_{n \geq 1}$ has an accumulation point h in G^G, and $h \in \mathcal{M}$ since $\mathcal{M}$ is closed. We then have that $h^2[O] \subseteq h[O]$ for all orbits O of $\mathcal{G}$. Since h is also canonical with respect to $\mathcal{G}$, this means that h is range-rigid with respect to $\mathcal{G}$. ⊣

LEMMA 15. *Let $\mathcal{M}$ be a closed transformation monoid containing a closed oligomorphic extremely amenable permutation group $\mathcal{G}$. Then $\mathcal{M}$ contains a minimal closed left ideal which contains a range-rigid function with respect to $\mathcal{G}$.*

PROOF. The fact that $\mathcal{M}$ contains a minimal closed left ideal $\mathcal{N}$ can be proved by a standard compactness argument (see [1, 2] for the proof).

Pick any $g \in \mathcal{N}$, and let $\mathcal{S}$ be the smallest non-empty closed transformation semigroup which contains g and which is invariant under $\mathcal{G}$. By Lemma 13, $\mathcal{S}$ contains a canonical function with respect to $\mathcal{G}$, and hence by Lemma 14, $\mathcal{S}$ contains a range-rigid function with respect to $\mathcal{G}$. It is easy to see that any element of $\mathcal{S}$ belongs to a minimal closed left ideal of $\mathcal{M}$. ⊣

3.3. Summary of the proof of Theorem 4.

PROOF OF THEOREM 4. Applying Lemma 15 to the monoid $\mathcal{M} := \mathrm{End}(\mathbb{A})$ and the group $\mathcal{G} := \mathrm{Aut}(\mathbb{B})$, we get that $\mathrm{End}(\mathbb{A})$ contains a minimal closed left ideal which contains a range-rigid function g with respect to $\mathrm{Aut}(\mathbb{B})$. By the ω-categoricity and the homogeneity of $\mathbb{B}$, the first-order reduct $\mathbb{A}$ has a quantifier-free definition in $\mathbb{B}$. Hence, $\mathbb{A}_g$ is well-defined, and by Lemma 11, $\mathbb{A}_g$ is the model-complete core of $\mathbb{A}$. By definition, $\mathbb{A}_g$ has a quantifier-free definition in $\mathbb{B}_g$, and in particular is a first-order reduct thereof. Lemma 8 gives that $\mathbb{B}_g$ is ω-categorical, while Lemma 10 tells us that $\mathbb{B}_g$ is a homogeneous Ramsey structure. Finally, by Lemma 9 it is finitely bounded if $\mathbb{B}$ is. ⊣

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