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On a conjecture of Green

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Abstract. Green [**5**] conjectured that if M is a closed Riemannian manifold of negative sectional curvature such that the mean curvatures of the horospheres through each point depend only on the point, then V is a locally symmetric space of rank one. He proved this in dimension two. In this paper we prove that under Green's assumption, M must be asymptotically harmonic and that the geodesic flow on M is C^∞ conjugate to that of a locally symmetric space of rank one. Combining this with the recent rigidity theorem of Besson–Courtois–Gallot [**1**], it follows that Green's conjecture is true for all dimensions.

1.

Let N be a simply connected Riemannian manifold of bounded negative sectional curvature. It is well-known that N can be compactified by an ideal boundary ∂N. For each vector v in the unit tangent bundle SN, there exists a unique horosphere $H(v)$ that passes through the base point of v and is perpendicular to v. Let us denote by $U(v)$ the mean curvature of $H(v)$ at the base point of v. Green proved [**5**] that if $U(v)|_{S_xM} \equiv u(x)$ and $\dim N = 2$, N must be a surface of constant curvature. Then he made the following conjecture.

CONJECTURE (L. Green, [**5**]) *If N is a simply-connected manifold of bounded negative curvature such that the mean curvatures of the horospheres through each point depend only on the point, then N is a rank-one symmetric space.*

This conjecture also appears on Burns and Katok's list [**2**] as problem (5.5.1), where the place of N is taken by a compact manifold M, with good reasons. Let us first recall a few related questions.

A Riemannian manifold is called a harmonic manifold if all geodesic spheres in its universal covering with the lifted metric have mean curvature depending only on the radius. A recent result of Szabo [**9**] proves that any simply connected, compact harmonic manifold must be isometric to a compact symmetric space of rank one. On the other hand, Damek and Ricci [**3**] recently constructed examples of simply connected

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homogeneous spaces of negative curvature that are harmonic, yet not symmetric. This provides a counterexample to Green's conjecture in its full generality. Henceforth we shall assume that Green's conjecture is for closed manifolds.

A natural generalization of a harmonic manifold is an asymptotically harmonic manifold (this terminology is due to Ledrappier). A Riemannian manifold without conjugate points is called asymptotically harmonic if all horospheres of its universal covering have the same constant mean curvature. It is easy to see that harmonic manifolds without conjugate points must be asymptotically harmonic. On the other hand, Green's assumption, that horospheres through each point depend only on the point, looks rather weak. The main purpose of this short paper is to prove the following results.

THEOREM 1. *Let M be a C^∞ compact Riemannian manifold of negative curvature. If the mean curvature function $U(v)$ of the horospheres is C^∞ on the unit tangent bundle, then the geodesic flow of M is C^∞ conjugate to the geodesic flow of a locally symmetric space of rank one.*

THEOREM 2. *Let M be a C^∞ compact Riemannian manifold of negative curvature such that the mean curvatures of the horospheres through each point depend only on the point. Then:*

(i) *the geodesic flow of M is C^∞ conjugate to the geodesic flow of a locally symmetric space of rank one;*

(ii) *M is asymptotically harmonic.*

Theorem 1 generalizes a theorem of Foulon and Labourie [**4**] who obtained the same conclusion under the stronger hypothesis that M is asymptotically harmonic. The observation that Foulon and Labourie's proof carries over to the case where $U(v)$ is merely C^∞ rather than constant, is due to Katok [**6**]. The proof of Theorem 1 is presented in §3. Theorem 2 is proved in §5. It follows from Theorem 1 and a study of the Margulis function which is presented in §4.

Since this paper was written (it appeared as an MSRI preprint in the spring of 1992), Besson, Courtois and Gallot have proved Gromov's entropy rigidity conjecture. As we explain in §6, their result and our Theorem 2 imply Green's conjecture:

THEOREM 3. *Let M be as in Theorem 2. Then M is a locally symmetric space.*

2.

From now on we denote by M a C^∞ compact Riemannian manifold of negative sectional curvature and by $\tilde{M}$ its universal covering. Our first observation is that if the mean curvature function satisfies the Green assumption, $U(v)|_{S_xM} \equiv u(x)$, then $U(v)$ is a C^∞ function. This follows from the following two facts: (1) along each weak stable manifold of the geodesic flow the function $U(v)$ is C^∞; (2) the canonical projection $S\tilde{M} \to \tilde{M}$ restricted to any weak stable manifold is a C^∞-diffeomorphism.

3.

In this section we prove Theorem 1. We first recall the set up of [**4**]. Let X be the geodesic spray that generates the geodesic flow ϕ_t and let $TSM = E^+ \oplus E^- \oplus \mathbb{R}X$ be

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the Anosov splitting associated with the geodesic flow, which is, in general, only Hölder continuous. The subbundle $E^+ \oplus E^-$ is smooth and carries a symplectic form ω which is invariant under the geodesic flow. If we denote by V the distribution on SM which is tangential to the fibres $S_x M$, then E^+, E^- and V are Lagrangian subspaces for ω and are mutually transversal.

Let B be the bundle of all Lagrangian n-planes in $E^+ \oplus E^-$ that are transversal to V ($n = \dim M - 1$). Then each fibre B_v, $v \in SM$, is a Zariski open set in an algebraic variety. Choose a C^∞ complement V_0 of V in $E^+ \oplus E^-$. Let g be the Riemannian metric on M and let $\bar{g}$ be the degenerate metric on SM obtained as the pull back of g via the canonical projection $SM \xrightarrow{P} M$. Then for any n-plane Q of B, $\bar{g}|_Q$ is non-degenerate and we define $\bar{g}_t = (d\phi_t)^* \bar{g}$. Define a function $f_t : B \to \mathbb{R}$ via

$$f_t(Q) = \text{square of the determinant of } \bar{g}_t \text{ with respect to } \bar{g} \text{ on } Q.$$

Notice that f_t is a C^∞ function and each restriction $f_t|_{B_v}$ is an algebraic function. Since the distribution E^+ is invariant under the geodesic flow and the contracting coefficient of the geodesic flow along the E^+ direction with respect to the metric $\bar{g}$ is exactly the mean curvature function $U(v)$, we have

$$f_t(E^+_v) = \exp\left(-\int_0^t U(\phi_s v)\, ds\right).$$

If we denote by F_n the function on B given by

$$F_n = \sum_{k=0}^{n} \left[f_k - \exp\left(-\int_0^k U(\phi_s v)\, ds\right)\right]^2,$$

then E^+ is included in the intersection of the zero sets of the functions F_n for all $n \in \mathbb{N}$:

$$E^+ \subset \bigcap_{n\in\mathbb{N}} F_n^{-1}\{0\}.$$

Moreover, all the functions F_n are C^∞ smooth and the restrictions of F_n to B_v are algebraic functions of uniformly bounded degree for all $v \in SM$. The argument of [**4**] is divided into three steps.

(1) If $U(v) \equiv h$, then $E^+ = \bigcap_{n\in\mathbb{N}} F_n^{-1}\{0\}$.

(2) If $B \to V$ is a C^∞ vector bundle with fibers B_v. Let F_n, $n \in \mathbb{N}$ be a family of C^∞ functions on B such that $F_n|_{B_v}$ is an algebraic function of bounded degree d for all $v \in V$. Let $\sigma : V \to B$ be a continuous section such that

$$\sigma(V) = \bigcap_{n\in\mathbb{N}} F_n^{-1}\{0\},$$

then σ is C^∞ on an open dense subset of V.

(3) If the stable and unstable distributions E^+ and E^- of a topologically transitive Anosov flow on a compact manifold are C^∞ on an open dense subset of M, then they are C^∞ everywhere on M.

The argument of (2) is purely algebraic and the argument of (3) is purely dynamical. The only thing that concerns us is step (1). A careful look tells one that the argument

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of [**4**] for the case $U(v) \equiv h$ can be translated almost verbatim to the general situation. Step (1) is also intuitively obvious because anything else besides E^+ would eventually contribute exponentially to f_t under the geodesic flow. So Theorem 1 in §1 is almost a free gift.

4.

The other observation is that the Margulis function satisfies a simple equation. Let M be a C^∞ compact Riemannian manifold of negative sectional curvature and let $\tilde{M}$ be its universal covering. We denote by $\partial\tilde{M}$ the ideal boundary of $\tilde{M}$. For each point x in M, let $S(x, R)$ be the volume of the geodesic sphere $S_x(R)$, with center x and radius R, under the induced Riemannian metric. Let h be the topological entropy of the geodesic flow of M. Margulis [**7**] proved that the limit

$$\lim_{R\to\infty} \frac{S(x, R)}{e^{hR}}$$

exists for every $x \in \tilde{M}$ and defines a continuous function $c(x)$ on $\tilde{M}$. Since $c(x)$ is invariant under isometries on $\tilde{M}$, it can be projected down to M and we denote it by the same symbol.

It is proved in [**10**] that:

(1) $c(x)$ is a C^∞ function on M;

(2) $c(x)$ is proportional to the total mass of Patterson–Sullivan measure μ_x (see, for example, [**10**] for the definition of Patterson–Sullivan measure), i.e. there exists a constant $A > 0$ such that

$$c(y) = A\mu_y(\partial\tilde{M}) = A\int_{\partial\tilde{M}} e^{-h\rho_{x,\xi}(y)}\, d\mu_x(\xi), \qquad (*)$$

where $\rho_{x,\xi}(\cdot)$ is the Busemann function of $\xi \in \partial\tilde{M}$ normalized so that $\rho_{x,\xi}(x) = 0$ and $\rho_{x,\xi}(\xi) = -\infty$.

Take the Laplacian Δ on both sides of $(*)$ and notice that $\Delta_{y\rho x,\xi}(y) = U(y, \xi)$, where $U(y, \xi)$ is the mean curvature at y of the horosphere $H(y, \xi)$ centered at ξ and passing through y. We get

$$\Delta_y c(y) = Ah\int_{\partial\tilde{M}} (h - U(y, \xi))e^{-h\rho_{x,\xi}(y)}\, d\mu_x(\xi). \qquad (**)$$

If Green's assumption is satisfied by M, i.e. $U(y, \xi) \equiv u(y)$ for all $\xi \in \partial\tilde{M}$, then by $(**)$ we have $\Delta_y c(y) = h(h - u(y))c(y)$. Using the fact that $c(y)^{-1}\Delta c(y) = \|\nabla \log c(y)\|^2 + \Delta(\log c(y))$ we get

$$\|\nabla \log c(y)\|^2 + \Delta(\log c(y)) = h(h - u(y)).$$

Integrate the above equation on M with respect to the Riemannian volume dy and apply Green's formula, to get

$$\int_M \|\nabla \log c(y)\|^2\, dy = h\int_M (h - u(y))\, dy. \qquad (\dagger)$$

By the Pesin formula [**8**], the metric entropy h_m of the geodesic flow with respect to the normalized Liouville measure m satisfies $h_m = \int_{SM} U(v)\, dm(v)$. If $U(v)|_{S_xM} \equiv u(x)$,

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then

$$h_m = \frac{1}{\mathrm{Vol}(M)} \int_M u(x)\, dx. \qquad (\dagger\dagger)$$

Combining (†) and (††) we get the following lemma.

LEMMA 1. *Under Green's assumption, the Margulis function $c(x)$ satisfies*

$$\int_M \|\nabla \log c\|^2 dx = h(h - h_m)\,\mathrm{Vol}(M).$$

5.

Next we explain why Theorem 2(ii) follows from §§3 and 4. It is easy to see that any continuous time-preserving conjugacy ψ between the geodesic flows of two negatively curved compact manifolds M and N must preserve the Bowen–Margulis measure. If ψ is C^1, then it also preserves the Liouville measure. Thus, if the Bowen–Margulis measure and the Liouville measure coincide on SN (this is the case when N is locally symmetric) and ψ is C^1, then the two measures on SM must also coincide. It follows that under Green's assumption, the Bowen–Margulis measure coincides with the Liouville measure and therefore $h = h_m$.

By Lemma 1 we get $\|\nabla \log c\| \equiv 0$ and thus the Margulis function is constant. By the equation $\Delta c = h(h - u)c$ we have that $u(x) \equiv h$ on M, i.e. M is asymptotically harmonic.

6.

Besson *et al* [**1**] recently proved the minimal entropy conjecture of Katok and Gromov. They actually obtained a more general result:

THEOREM. *Let $f : M \to N$ be a continuous map between two closed orientable manifolds of dimension n with non-zero degree. Suppose N carries a locally symmetric Riemannian metric g_0 of negative curvature. Then any Riemannian metric g on M satisfies*

$$h^n(M, g)\,\mathrm{Vol}(M, g) \geq |\deg(f)| h^n(N, g_0)\,\mathrm{Vol}(N, g_0).$$

Moreover, if $n \geq 3$, then equality holds if and only if g is locally symmetric and f is homotopic to a homothetic covering map.

Here the entropy $h(M, g)$ is defined to be the exponential growth of the volume of geodesic balls in the universal cover of M:

$$h(M, g) = \lim_{R \to \infty} \frac{\log \mathrm{Vol}(B(x, R))}{R}.$$

If g is a metric of negative curvature, then $h(M, g)$ coincides with the topological entropy of the geodesic flow. Since any C^1 time preserving conjugacy between the geodesic flows preserves topological entropy and Riemannian volume, Theorem 3 follows from Theorem 2 and the Theorem of Besson–Courtois–Gallot.

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