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LIKELIHOOD RATIO ORDERING OF PARALLEL SYSTEMS WITH HETEROGENEOUS SCALED COMPONENTS

Jiantian Wang
School of Mathematical Sciences, Kean University, New Jersey 07083, USA
E-mail: jwang@kean.edu

This paper considers stochastic comparison of parallel systems in terms of likelihood ratio order under scale models. We introduce a new order, the so-called q-larger order, and show that under certain conditions, the q-larger order between the scale vectors can imply the likelihood ratio order of parallel systems. Applications are given to the generalized gamma scale family.

Keywords: likelihood ratio order, order statistic, parallel system, stochastic comparison

1. INTRODUCTION

Parallel systems are the building blocks for some complex coherent systems. In reliability analysis, statistical inference and other areas, it is needed to compare the reliability of two parallel systems in terms of some stochastic orders, such as, likelihood ratio order, hazard rate order, reversed hazard rate order, and the usual stochastic order. Among these orders, the likelihood ratio order can be regarded as the most stringent one, since it implies all these other orders.

So far, there is an extensive literature on stochastic comparisons. See, for example, Pledger and Proschan [12], Dykstra, Kochar, and Rojo [7], Khaledi and Kochar [8], Kundu, Nanda, and Hu [10], Da, Ding, and Li [6], and a nice comprehensive survey paper of Balakrishnan and Zhao [2]. As we can see, the most existing stochastic comparison results are about the usual stochastic order or reversed hazard rate order or hazard rate order. The results in terms of likelihood ratio order are relatively fewer. Among them are those of Dykstra *et al.* [7], Kochar and Xu [9], Zhao and Balakrishnan [16], Zhao and Balakrishnan [17], Zhao and Zhang [19], and some others. Most of those results are about the parallel systems with exponential components.

For a parallel system consisting of components whose lifetimes are $X_1, \ldots, X_n$, its lifetime is $X_{n:n} = \max\{X_1, \ldots, X_n\}$. In this paper, we assume X_i, $i = 1, \ldots, n$, are independent. We say these variables belong to a scale family with scale vector $(\lambda_1, \ldots, \lambda_n)$, if, for $i = 1, \ldots, n$, the variable X_i has distribution function $F(\lambda_i x)$, where $F(x)$, the baseline distribution function of the family, is absolutely continuous. Typically, when $F(x) = 1 - e^{-x}$,

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the scale family is the usual exponential family. By symmetry, we assume $0 < \lambda_1 \le \cdots \le \lambda_n$ in the sequel.

Under exponential scale model, for the case of $n = 2$, let $X_{2:2} = \max\{X_1, X_2\}$, $Y_{2:2} = \max\{Y_1, Y_2\}$, with $X_i \sim \exp(\lambda_i)$ and $Y_i \sim \exp(\mu_i)$, $i = 1, 2$. Dykstra *et al.* [7] proved that

$$(\lambda_1, \lambda_2) \overset{m}{\succ} (\mu_1, \mu_2) \Longrightarrow X_{2:2} \ge_{lr} Y_{2:2}, \tag{1}$$

here $\overset{m}{\succ}$ stands for majorization order, and $\ge_{lr}$ for likelihood ratio order. Zhao and Balakrishnan [16] revealed that, when $\lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2$, and $\lambda_1 + \lambda_2 \le \mu_1 + \mu_2$, then,

$$X_{2:2} \ge_{lr} Y_{2:2}. \tag{2}$$

From the definition of majorization order, we can easily see that the result Eq. (2) extends that of Eq. (1).

Denote the triangle with vertices (λ_1, λ_2), (λ_2, λ_2), and $(\bar{\lambda}, \bar{\lambda})$ as Δ_λ. Notice, triangle Δ_λ is formed by the lines starting from point (λ_1, λ_2) with directions $\vec{u} = a(1, 0) + b(1, -1)$, where $a, b \ge 0$, along with the symmetry condition $y \ge x$. We generalize such a triangle region to high dimension. Let $\vec{\epsilon_1} = (1, 0, \ldots, 0)$, and for $k = 2, \ldots, n$, $\vec{\epsilon_k} = (1, 0, \ldots, -1, \ldots, 0)$, that is, the vector whose first element is 1, kth element is -1, while all others are 0. For two vectors $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$, if there exist $a_k \ge 0$, such that $\vec{\mu} - \vec{\lambda} = \sum_{k=1}^{n} a_k \vec{\epsilon_k}$, we then say, $\vec{\lambda}$ is q-larger than $\vec{\mu}$, and denote it by $\vec{\lambda} \overset{q}{\succ} \vec{\mu}$. With this notation, the result Eq. (2) can be restated as, for parallel systems consisting of $n = 2$ exponential components with hazard rate vectors (or scale vectors) (λ_1, λ_2) and (μ_1, μ_2), then,

$$(\lambda_1, \lambda_2) \overset{q}{\succ} (\mu_1, \mu_2) \Longrightarrow X_{2:2} \ge_{lr} Y_{2:2}. \tag{2}'$$

As it has been pointed out by Boland, El-Neweihi, and Proschan [5], the result Eq. (1) can not be extended directly from $n = 2$ to the case of $n \ge 3$. Until now, excepting some special cases, such as the one of Kochar and Xu [9], the one of Torrado and Kochar [15], and the one of Zhao, Hu, and Zhang [18], no stochastic comparison result in terms of likelihood ratio order for two parallel systems with general n components has been available.

This paper focuses on the stochastic comparison of parallel systems with general n components under scale models. In this paper, we show that under certain conditions,

$$(\lambda_1, \ldots, \lambda_n) \overset{q}{\succ} (\mu_1, \ldots, \mu_n) \Longrightarrow X_{n:n} \ge_{lr} Y_{n:n}, \tag{3}$$

where $X_{n:n} = \max\{X_1, \ldots, X_n\}$ with $\{X_1, \ldots, X_n\}$ belonging to a scale family with scale vector $(\lambda_1, \ldots, \lambda_n)$, and $Y_{n:n} = \max\{Y_1, \ldots, Y_n\}$ with $\{Y_1, \ldots, Y_n\}$ belonging to the same scale family with scale vector $(\mu_1, \ldots, \mu_n)$.

We consider a special scale family, the so-called generalized gamma scale family, which includes the exponential family as its special case. We find that in most cases, the generalized gamma family satisfies those conditions. Since the generalized gamma scale family includes exponential, Weibull, and gamma variables as its members, we thus extend most existing results in the literature, including that of Dykstra *et al.* [7] and that of Zhao and Balakrishnan [16].

The paper is organized as follows. Section 2 gives the required notations and definitions, and lists some lemmas. Section 3 provides the proofs of the main results. In Section 4, we examine the generalized gamma scale family, and establish some comparison results with exponential, Weibull, and some gamma components.

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2. PRELIMINARY

Let X be a nonnegative continuous random variable with distribution function F_X, survival function $\bar{F}_X = 1 - F_X$, and density function f_X. The hazard function and the reversed hazard function of X are defined as $h_X = f_X/\bar{F}_X$ and $r_X = f_X/F_X$, respectively. For two nonnegative continuous random variables X and Y, we say X is larger than Y in the usual stochastic order (denoted by $X \geq_{st} Y$), if $\bar{F}_X(t) \geq \bar{F}_Y(t), t \geq 0$; X is larger than Y in hazard rate order (denoted by $X \geq_{hr} Y$), if $h_X(t) \leq h_Y(t), t \geq 0$; X is larger than Y in reversed hazard rate order (denoted by $X \geq_{rh} Y$), if $r_X(t) \geq r_Y(t), t \geq 0$; X is larger than Y in likelihood ratio order (denoted by $X \geq_{lr} Y$), if the ratio $f_X(t)/f_Y(t)$ is increasing in $t > 0$. It is well known that the likelihood ratio order implies the other orders. For more details on stochastic orders, see Shaked and Shanthikumar [13].

For later use, we need the following two lemmas, which are due to Barlow and Proschan [3] and Bon and Păltănea [4], respectively.

LEMMA 2.1: *Let X be a nonnegative random variable and let $\mu_1 = E(X)$ and $\mu_2 = E(X^2)$. If X has increasing (decreasing) hazard rate, then $2\mu_1^2 \geq (\leq)\mu_2$.*

LEMMA 2.2: *Let $\phi : R_+^n \to R_+$ be a symmetric and continuously differentiable mapping. If, for any $\vec{y} = (y_1, \ldots, y_n) \in R_+^n$ with $y_p = \min y_i$, and $y_q = \max y_i$,*

$$(y_p - y_q)\left(\frac{\partial \phi}{\partial y_p} - \frac{\partial \phi}{\partial y_q}\right) < 0, \quad \text{for} \quad y_p \neq y_q,$$

then the following inequality holds:

$$\phi(y_1, \ldots, y_n) \leq \phi(\bar{y}, \ldots, \bar{y}),$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$, is the arithmetic mean of $y = (y_1, \ldots, y_n)$.

3. MAIN RESULTS

THEOREM 3.1: *Suppose the independent variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ belong to a scale family with scale vector $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$, respectively. Let $r(x)$ be the baseline reversed hazard rate function of the family. Denote $b(x) = xr(x)$, $d(x) = xr'(x)/r(x)$. Then, when $b'(x) \leq 0$, $b''(x) \geq 0$, $d(x) \leq 0$, $d'(x) \leq 0$, and $[b(x)d'(x)]' \geq 0$,*

$$(\lambda_1, \ldots, \lambda_n) \overset{q}{\succ} (\mu_1, \ldots, \mu_n) \Longrightarrow X_{n:n} \geq_{lr} Y_{n:n}. \tag{4}$$

PROOF: Denote $f(x)$ and $r(x) = f(x)/F(x)$ be the baseline density function and the baseline reversed hazard rate function of the family, respectively. The distribution function of $X_{n:n}$ is,

$$F(\lambda; t) = P(X_{n:n} < t) = \prod_{i=1}^n P(X_i < t) = \prod_{i=1}^n F(\lambda_i t),$$

and its density function is

$$f(\lambda; t) = F(\lambda; t)\psi(\lambda; t),$$

where

$$\psi(\lambda; t) = \sum_{i=1}^n \frac{\lambda_i f(\lambda_i t)}{F(\lambda_i t)}.$$

So, the reversed hazard function of $X_{n:n}$ is $r(\lambda; t) = \psi(\lambda; t) = \sum_{i=1}^n \lambda_i r(\lambda_i t)$.

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Consider function

$$\Psi(x_1,\ldots,x_n)=\sum_{i=1}^{n}x_i r(x_i),\quad 0<x_1\le x_2\le\cdots\le x_n.$$

From $b'(x)\le 0$ and $b''(x)\ge 0$, we know,

$$\begin{aligned}\bigtriangledown_{\epsilon_1}\Psi&=\frac{\partial\Psi}{\partial x_1}=b'(x_1)\le 0,\\ \bigtriangledown_{\epsilon_k}\Psi&=\frac{\partial\Psi}{\partial x_1}-\frac{\partial\Psi}{\partial x_k}=b'(x_1)-b'(x_k)\le 0,\end{aligned}$$

where $\vec{\epsilon_1}=(1,0,\ldots,0)$, and $\vec{\epsilon_k}=(1,0,\ldots,-1,\ldots,0)$. Thus, in the directions ϵ_k, the function Ψ is decreasing. Therefore, when $(\lambda_1,\ldots,\lambda_n)\overset{q}{\succ}(\mu_1,\ldots,\mu_n)$, $r(\lambda;t)\ge r(\mu;t)$. Hence, we have shown,

$$(\lambda_1,\ldots,\lambda_n)\overset{q}{\succ}(\mu_1,\ldots,\mu_n)\Longrightarrow X_{n:n}\ge_{rh}Y_{n:n}.$$

Denote the ratio of the reversed hazard rate function of $X_{n:n}$ over that of $Y_{n:n}$ as $\psi(t)$. From Theorem 1.C.4 (b) of Shaked and Shanthikumar [13], in order to prove the theorem, it is enough to show the function $\psi(t)$ is increasing in $t>0$. For our convenience, we denote $A\overset{\text{sgn}}{=}B$ if the signs of A and B are the same. Since $\psi(t)=r(\lambda;t)/r(\mu;t)$, so,

$$\psi'(t)\overset{\text{sgn}}{=}\frac{r_t'(\lambda;t)}{r(\lambda;t)}-\frac{r_t'(\mu;t)}{r(\mu;t)},$$

where $r_t'(\lambda;t)=\sum_{i=1}^{n}\lambda_i^2 r'(\lambda_i t)$, and similar to $r_t'(\mu;t)$.

Introduce the function

$$\Phi(x_1,\ldots,x_n)=\sum_{i=1}^{n}x_i^2 r'(x_i)\Big/\sum_{i=1}^{n}x_i r(x_i),\quad 0<x_1\le x_2\le\cdots\le x_n.$$

Denote $c(x)=x^2r'(x)$, then, $d(x)=xr'(x)/r(x)=c(x)/b(x)$. Let $N=\sum_{i=1}^{n}x_i^2r'(x_i)=\sum_{i=1}^{n}c(x_i)$, $D=\sum_{i=1}^{n}x_i r(x_i)=\sum_{i=1}^{n}b(x_i)$. So, $\Phi=N/D$, and

$$\begin{aligned}\frac{\partial\Phi}{\partial x_i}&\overset{\text{sgn}}{=}\frac{\partial N}{\partial x_i}D-N\frac{\partial D}{\partial x_i}\\ &=c'(x_i)D-b'(x_i)N\\ &=\big[b'(x_i)d(x_i)+b(x_i)d'(x_i)\big]\sum_{i=1}^{n}b(x_i)-b'(x_i)\sum_{i=1}^{n}b(x_i)d(x_i).\end{aligned}$$

For vector $\vec{\epsilon_1}$, since $b'(x)\le 0$ and $d(x)$ is decreasing, then,

$$\begin{aligned}\bigtriangledown_{\vec{\epsilon_1}}\Phi&=\frac{\partial\Phi}{\partial x_1}\\ &=\big[b'(x_1)d(x_1)+b(x_1)d'(x_1)\big]\sum_{i=1}^{n}b(x_i)-b'(x_1)\sum_{i=1}^{n}b(x_i)d(x_i)\\ &\le\big\{b'(x_1)d(x_1)+b(x_1)d'(x_1)-b'(x_1)d(x_1)\big\}\sum_{i=1}^{n}b(x_i)\\ &\overset{\text{sgn}}{=}b(x_1)d'(x_1)\le 0.\end{aligned}$$

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Since $b'(x) \le 0$, $b''(x) \ge 0$, $d(x) \le 0$, $d'(x) \le 0$, we have, for $k = 2, \ldots, n$,

$$\begin{aligned}
\bigtriangledown_{\epsilon_k}\Phi &= \frac{\partial\Phi}{\partial x_1} - \frac{\partial\Phi}{\partial x_k} \\
&= \big[b'(x_1)d(x_1) + b(x_1)d'(x_1) - b'(x_k)d(x_k) - b(x_k)d'(x_k)\big]\sum_{i=1}^{n} b(x_i) \\
&\quad - [b'(x_1) - b'(x_k)]\sum_{i=1}^{n} b(x_i)d(x_i) \\
&\le \big[b'(x_1)d(x_1) + b(x_1)d'(x_1) - b'(x_k)d(x_k) - b(x_k)d'(x_k)\big]\sum_{i=1}^{n} b(x_i) \\
&\quad - [b'(x_1) - b'(x_k)]d(x_1)\sum_{i=1}^{n} b(x_i) \\
&\overset{\text{sgn}}{=} b'(x_1)d(x_1) + b(x_1)d'(x_1) - b'(x_k)d(x_k) - b(x_k)d'(x_k) - b'(x_1)d(x_1) + b'(x_k)d(x_1) \\
&= b(x_1)d'(x_1) - b'(x_k)d(x_k) - b(x_k)d'(x_k) + b'(x_k)d(x_1) \\
&= b(x_1)d'(x_1) - b(x_k)d'(x_k) - b'(x_k)\big[d(x_k) - d(x_1)\big] \\
&\le b(x_1)d'(x_1) - b(x_k)d'(x_k).
\end{aligned}$$

Since the function $b(x)d'(x)$ is increasing, we obtain $\bigtriangledown_{\vec{\epsilon_k}}\Phi \le 0$.

Now, $(\lambda_1, \ldots, \lambda_n) \overset{q}{\succ} (\mu_1, \ldots, \mu_n)$ implies $\vec{\mu} - \vec{\lambda} = \sum_{k=1}^{n} a_k\vec{\epsilon_k}$, $a_k \ge 0$, so, the function Φ is decreasing in the direction $\vec{\mu} - \vec{\lambda}$. Therefore,

$$\begin{aligned}
\psi'(t) &\overset{\text{sgn}}{=} \frac{r_t'(\lambda;t)}{r(\lambda;t)} - \frac{r_t'(\mu;t)}{r(\mu;t)} \\
&= \Phi(\lambda_1 t, \ldots, \lambda_n t) - \Phi(\mu_1 t, \ldots, \mu_n t) \\
&\ge 0.
\end{aligned}$$

This shows $X_{n:n} \ge_{lr} Y_{n:n}$.

From the assumption $0 < \lambda_1 \le \cdots \le \lambda_n$ and the fact $\bigtriangledown_{\epsilon_n}\Phi \le 0$, by applying Lemma 2.2, we readily have the following corollary. ■

COROLLARY 3.2: *Suppose the independent variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ belong to a scale family with scale vector $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$, respectively. Let $F(x)$ be the baseline distribution function with reversed hazard rate $r(x)$. Let $b(x) = xr(x)$, $d(x) = xr'(x)/r(x)$. If $b'(x) \le 0$, $b''(x) \ge 0$, $d(x) \le 0$, $d'(x) \le 0$ and $[b(x)d'(x)]' \ge 0$, then, when $\mu_1 = \cdots = \mu_n = \bar{\lambda} = \frac{1}{n}\sum\lambda_i$,*

$$X_{n:n} \ge_{lr} Y_{n:n}.$$

4. THE GENERALIZED GAMMA SCALE MODEL

Stacy [14] introduced the so-called generalized gamma random variables, which have the density functions

$$f(p, q; x) = \frac{p}{\Gamma(q/p)}x^{q-1}e^{-x^p}, \quad x > 0,$$

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with positive parameters p and q. Those generalized gamma variables includes the exponential variables, gamma variables and Weibull variables as the special cases. Specifically, when $p = q = 1$, $p = q$, and $p = 1$, the generalized gamma variables become the exponential variables, Weibull variables, and gamma variables, respectively. We refer to the scale family consisting of these generalized gamma random variables as generalized gamma scale family.

Under generalized gamma scale model, the baseline distribution function is

$$F(p,q;x) = \frac{p}{\Gamma(q/p)} \int_0^x t^{q-1} e^{-t^p} dt.$$

So,

$$b(x) = xr(x) = \frac{xf(p,q;x)}{F(p,q;x)} = \frac{x^q e^{-x^p}}{\int_0^x t^{q-1} e^{-t^p} dt}.$$

By substitution, we obtain $b(x) = 1/\eta(x)$, where

$$\eta(x) = \frac{1}{p} \int_0^1 (1-v)^{q-p/p} e^{x^p v} dv.$$

Define

$$E_i(p,q;x) = \int_0^1 (1-v)^{\frac{q-p}{p}} e^{x^p v} v^i dv, \quad i = 1,2,\ldots$$

So, $\eta'(x) = x^{p-1} E_1(p,q;x)$, $\eta''(x) = p(x^{p-1})^2 E_2(p,q;x) + (p-1)x^{p-2} E_1(p,q;x)$. Denote $\mu_i = E(W^i)$, $i = 1,2$, where W is the random variable with density function

$$f(v) = K(1-v)^{q-p/p} e^{x^p v}, \quad 0 < v < 1,$$

with positive constants p, q, x, and $K^{-1} = \int_0^1 (1-v)^{\frac{q-p}{p}} e^{x^p v} dv$. We have,

$$\frac{\eta'}{\eta} = px^{p-1}\mu_1, \quad \frac{\eta''}{\eta} = p^2 x^{2p-2}\mu_2 + p(p-1)x^{p-2}\mu_1.$$

Clearly, $0 < E(W) < 1$. So, $b'(x) = -\eta^{-2}\eta' = -\eta^{-1} px^p E(W) \le 0$.

For $d(x) = xr'(x)/r(x) = -(1 + x\eta^{-1}\eta')$, we know $d(x) \le 0$, and

$$\begin{aligned}
d'(x) &= -\eta^{-1}\eta' + x\eta^{-2}\eta'^2 - x\eta^{-1}\eta'' \\
&= -\frac{\eta'}{\eta} + x\Big[\frac{\eta'^2}{\eta^2} - \frac{\eta''}{\eta}\Big] \\
&= -px^{p-1}E(W) + x(px^{p-1})^2\big[E^2(W) - E(W^2)\big] - p(p-1)x^{p-1}E(W) \\
&= -px^{p-1}E(W) - x(px^{p-1})^2 Var(W) - p(p-1)x^{p-1}E(W) \\
&= -p^2 x^{p-1}E(W) - x(px^{p-1})^2 Var(W) \\
&\le 0.
\end{aligned}$$

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For $b''(x)$, we have,

$$\begin{aligned} b''(x) &= 2\eta^{-3}\eta'^2 - \eta^{-2}\eta'' \\ &= \eta^{-1}\Big[2\frac{\eta'^2}{\eta^2} - \frac{\eta''}{\eta}\Big] \\ &= \eta^{-1}(px^{p-1})^2[2E^2(W) - E(W^2)] - \eta^{-1}p(p-1)x^{p-2}E(W), \end{aligned}$$

The hazard rate function of W is

$$h(v) = \frac{(1-v)^{q-p/p}e^{x^p v}}{\int_v^1 (1-t)^{q-p/p}e^{x^p t}dt}.$$

For $q \le p$, it can be easily see that the function $h(v)$ is increasing, since both $(1-v)^{q-p/p}$ and $e^{x^p v}$ are increasing. For $q > p$, we have

$$\ln'' f(v) = -\frac{q-p}{p}\frac{1}{(1-v)^2} \le 0.$$

So, in this case, W is log-concave. By a well-known implication, see, Bagnoli and Bergstrom [1], and Misra and Misra [11], W has increasing hazard rate. From Lemma 2.1, $2E^2(W) - E(W^2) \ge 0$. Clearly, $0 < E(W) < 1$, and therefore, for $0 < p \le 1$, $b''(x) \ge 0$.

It turns out that to prove the function $b(x)d'(x)$ is increasing for general cases is not so easy. Here we just check some typical cases.

From

$$\begin{aligned} \int_0^1 e^{\alpha v}dv &= \frac{1}{\alpha}[e^\alpha - 1], \\ \int_0^1 ve^{\alpha v}dv &= \frac{1}{\alpha^2}[\alpha e^\alpha - e^\alpha + 1], \\ \int_0^1 v^2e^{\alpha v}dv &= \frac{1}{\alpha^3}[\alpha^2e^\alpha - 2\alpha e^\alpha + 2e^\alpha - 2], \end{aligned}$$

we obtain that, in the case of $q = p$, $\eta(x) = \dfrac{e^u - 1}{pu}$, $d(x) = -p\dfrac{ue^u}{e^u - 1} - (1-p)$, and

$$b(x)d'(x) = -p^2u'_x\frac{ue^u(e^u - 1 - u)}{(e^u - 1)^3} = -p^2u'_x\theta_1(u),$$

with $u = x^p$.

In the case of $q = 2p$,

$$\eta(x) = \frac{1}{p}\int_0^1 (1-v)e^{x^p v}dv = \frac{e^u - 1 - u}{pu^2}, \quad u = x^p,$$

so, $b(x) = 1/\eta(x) = pu^2/(e^u - 1 - u)$, $d(x) = -p\dfrac{u(e^u - 1)}{e^u - 1 - u} - (1 - 2p)$, and

$$b(x)d'(x) = -p^2u'_x\frac{u^2(e^{2u} - u^2e^u - 2e^u + 1)}{(e^u - 1 - u)^3} = -p^2u'_x\theta_2(u).$$

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In the case of $q = 3p$,

$$\eta(x) = \frac{1}{p}\int_0^1 (1-v)^2 e^{x^p v} dv = \frac{2e^u - 2 - 2u - u^2}{pu^3},$$

and thus,

$$\begin{aligned} b(x)d'(x) &= -p^2 u'_x \frac{2u^3(2e^{2u} - u^3e^u + u^2e^u - 4ue^u - 4e^u + u^2 + 4u + 2)}{(2e^u - 2 - 2u - u^2)^3} \\ &= -p^3 \frac{2u^{4-\frac{1}{p}}(2e^{2u} - u^3e^u + u^2e^u - 4ue^u - 4e^u + u^2 + 4u + 2)}{(2e^u - 2 - 2u - u^2)^3} \\ &= -p^3\theta_3(u), \end{aligned}$$

with $u = x^p$.

As we can easily confirm, the functions $\theta_1(u)$ and $\theta_2(u)$ are decreasing with respect to u. Clearly, when $0 < p \le 1$, $u''_x \le 0$, hence $b(x)d'(x)$ is increasing in the cases of $q = p$ and $q = 2p$. For function $\theta_3(u)$, we find, when $0 < p < 1$, it is decreasing with respect to u. We thus have proved the following corollary.

COROLLARY 4.1: *Suppose the independent variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ belong to the generalized gamma scale family with scale vector $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$, respectively. Then, when $0 < p \le 1$, and $q = p$; or, $0 < p \le 1$ and $q = 2p$; or, $0 < p < 1$, and $q = 3p$;*

$$(\lambda_1, \ldots, \lambda_n) \overset{q}{\succ} (\mu_1, \ldots, \mu_n) \Longrightarrow X_{n:n} \ge_{lr} Y_{n:n}. \tag{5}$$

Since when $p = q = 1$ and $p = q$, the generalized gamma variables become the exponential variables, Weibull variables, respectively, so we have,

COROLLARY 4.2: *Suppose the independent variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ follow exponential distributions with scale vector $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$; and Weibull distributions with scale vector $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$ and $0 < p \le 1$, respectively, then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{q}{\succ} (\mu_1, \ldots, \mu_n) \Longrightarrow X_{n:n} \ge_{lr} Y_{n:n}.$$

Even though when $p = 1$, $\theta_3(u)$ is not strictly decreasing, the result Eq. (5), we believe, will still hold. We guess that the result Eq. (5) will hold for all parameters p, q, satisfying $0 < p \le 1$, $q > 0$. Since a rigorous proof has not been available, we leave this as a conjecture.

Conjecture. *Assume the random variables $X_1, \ldots, X_n$, and $Y_1, \ldots, Y_n$ follow the generalized gamma scale model with scale vectors $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$, respectively. Then, for $0 < p \le 1, q > 0$,*

$$(\lambda_1, \ldots, \lambda_n) \overset{q}{\succ} (\mu_1, \ldots, \mu_n) \Longrightarrow X_{n:n} \ge_{lr} Y_{n:n}.$$

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