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ANALYSIS OF MULTI-RESOURCE LOSS SYSTEM WITH STATE-DEPENDENT ARRIVAL AND SERVICE RATES

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In this paper, we study a generalization of the classical multi-dimensional Erlang loss model with state-dependent arrival and service rates, in which customers at arrival occupy random quantities of various resources and release them at departure. Total amount of resources allocated to customers cannot exceed predefined maximum levels. There can be two types of customers: positive customers, which occupy positive quantities of resources, and negative customers, which occupy negative quantities of resources. Negative customers increase the amount of resources available to positive customers and therefore decrease blocking of positive customers caused by lack of resources.

Keywords: multi-class loss system, multiple resources, negative customers, state-dependent arrivals, state-dependent service

1. INTRODUCTION

Servers in the classical multi-dimensional Erlang loss model are resources that can be occupied by customers in batches of various sizes. Arriving customers are lost if the system does not have free resources required for their service. Queuing systems with random resource requirements are a generalization of such queuing systems. Random vectors representing resource requirements can follow either discrete or continuous probability distribution. The fact that the resources held by the customer should be released upon completion of service complicates the stochastic processes describing the behavior of such systems since they have to keep track of the quantities of resources occupied by each customer at service.

The loss system with Poisson arrival process, exponential service times and general probability distribution function of the requested quantity of a single resource was studied in

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Romm and Skitovitch [10]. In Tikhonenko [11], it was found that the stationary probability distribution of the system states is independent of the form of the service time distributions. Generalizations to systems with an arbitrary joint probability distribution of the service time and the requested quantity of resource were considered in Tikhonenko and Klimovich [12] and Tikhonenko [13] and extended to systems with multiple resource types in Naumov and Samuoylov [9].

The notion of positive and negative customers introduced by Gelenbe [5,6] has increased radically the scope of applications of queuing theory Artalejo [1] and Do [14]. Positive customers can represent resource requests whereas negative customers can correspond to decisions to cancel such requests. Work removal in G-networks was generalized to arbitrary sizes in Gelenbe [7], and its counterpart that includes the addition of work was discussed in Fourneau and Gelenbe [3], and Gelenbe and Fourneau [8]. G-networks with multiple classes of customers and multiple types of work addition and removal were first discussed in Fourneau, Gelenbe, and Suros [4]. In this paper, we study a system with a different behavior of negative customers. Here, instead of requests cancelation, negative customers temporary increase the amount of resources available to positive customers. We consider a multi-class loss system with multiple types of resources and state-dependent arrival and service rates. Each customer requires a certain quantity of each resource, which may be zero, positive or negative. First, we obtain the stationary probability distribution of the system states for the general case. Second, we consider a particular case with two types of customers: positive customers requesting non-negative quantities of each resource and negative customers requesting non-positive quantities of resources. In this way, negative customers increase the amount of resources available to positive customers and decrease blocking caused by lack of resources. In the rest of the paper, $\mathcal{R}^n$ denotes the set of real vectors of length n, and $\mathbf{e}_k$ denotes a vector with 1 in the kth position and zeros elsewhere. All stochastic processes are assumed to be right-continuous and to have left limits.

2. DESCRIPTION OF THE SYSTEM

We consider a loss system with K classes of customers. When there are n_k customers of class k in the system, the arrival times of class k customers form a Poisson process with intensity $\lambda_k(n_k)$. An admitted customer begins service immediately, and class k receives service at rate $\mu_k(n_k)$, which is equally shared between the customers of the class. We denote the total capacity of the system by C and for each class k assume $\mu_k(0) = 0$ and $\lambda_k(n) = 0$ for $n \geq C$. Let the system have M types of resources with capacities $R_1, R_2, \ldots, R_M$, and let real vectors $\mathbf{r}_j = (r_{j1}, r_{j2}, \ldots, r_{jM})$ denote resource quantities requested by a customer arriving at time $a_j, j = 1, 2, \ldots$. We assume that vectors of resource requests $\mathbf{r}_j$ are independent of the arrival and service processes, mutually independent, and, for class k customers, are identically distributed with joint cumulative distribution function $F_k(\mathbf{x})$.

The system state at time t can be described by a jump Markov process $X(t) = (\boldsymbol{\xi}(t), \boldsymbol{\theta}(t))$, where vector $\boldsymbol{\xi}(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_K(t))$ represents the number of customers of each class in the system at time t, and $\boldsymbol{\theta}(t) = (\boldsymbol{\theta}_1(t), \boldsymbol{\theta}_2(t), \ldots, \boldsymbol{\theta}_K(t))$ represents the resource quantities allocated to each customer. Subvector $\boldsymbol{\theta}_k(t) = (\boldsymbol{\theta}_{k,1}(t), \boldsymbol{\theta}_{k,2}(t), \ldots, \boldsymbol{\theta}_{k,\xi_k(t)}(t))$ of $\boldsymbol{\theta}(t)$ contains the vectors of resource quantities allocated to class k customers in the order of their arrivals.

The system operates as follows. Let $\mathrm{N}(t) = \xi_1(t) + \xi_2(t) + \cdots + \xi_K(t)$ denote the number of customers in the system, $\boldsymbol{\sigma}_k(t) = \boldsymbol{\theta}_{k,1}(t) + \boldsymbol{\theta}_{k,2}(t) + \cdots + \boldsymbol{\theta}_{k,\xi_k(t)}(t)$ denote the vector of resources occupied by class k customers, and $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_1(t) + \boldsymbol{\sigma}_2(t) + \cdots + \boldsymbol{\sigma}_K(t)$. At time $t = 0$ the system is empty and $\boldsymbol{\sigma}(0) = \mathbf{0}$. The customer arriving at time a_j is admitted to

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the system only if $\mathrm{N}(a_j - 0) < C$ and all required resource quantities can be provided, that is, if $\boldsymbol{\sigma}(a_j - 0) + \mathbf{r}_j \le \mathbf{R}$, where $\mathbf{R} = (R_1, R_2, \ldots, R_M)$. If this is the case, the quantities of resources specified by vector $\mathbf{r}_j = (r_{j1}, r_{j2}, \ldots, r_{jM})$ are allocated to the customer and the process $\boldsymbol{\sigma}(t)$ takes value $\boldsymbol{\sigma}(a_j) = \boldsymbol{\sigma}(a_j - 0) + \mathbf{r}_j$. We say that departure of customer j is enabled at time t if $\boldsymbol{\sigma}(t) - \mathbf{r}_j \le \mathbf{R}$, otherwise the customer departure is disabled. If customer departure is enabled at the time of completion of the service the customer leaves the system, otherwise customer service restarts. After departure of a customer, the resources allocated to it are released and the corresponding vector $\mathbf{r}_i$ is subtracted from the vector of occupied resources $\boldsymbol{\sigma}(t)$. Therefore the vector of occupied resources $\boldsymbol{\sigma}(t)$ always satisfies the constraint $\boldsymbol{\sigma}(t) \le \mathbf{R}$.

Let Z_+^K be the K-fold product of the set of non-negative integers and $n_\bullet$ denotes a sum of components of a vector $(n_1, n_2, \ldots, n_K) \in Z_+^K$. Then set $\mathcal{J}$ of allowable states of the population vector $\boldsymbol{\xi}(t)$ consists of all vectors $\mathbf{n} = (n_1, n_2, \ldots, n_K)$ from Z_+^K satisfying inequalities $0 \le n_\bullet \le C$. Set $\mathrm{M}(\mathbf{n})$ of possible states of vector $\boldsymbol{\theta}(t)$ depends on the population vector $\mathbf{n} = \boldsymbol{\xi}(t)$ and is a subset of the set of real vectors $\mathcal{R}^{Mn_\bullet}$. If $\mathbf{n} = \mathbf{0}$, then set $\mathcal{M}(\mathbf{n})$ is empty, but if $n_\bullet > 0$ it consists of all row vectors $\mathbf{y} = (\mathbf{y}_{1,1}, \ldots, \mathbf{y}_{1,n_1}, \ldots, \mathbf{y}_{K,1}, \ldots, \mathbf{y}_{K,n_K})$ satisfying the inequality $\mathbf{yU} \le \mathbf{R}$, where block matrix $\mathbf{U}$ is given by

$$\mathbf{U} = \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix} \tag{1}$$

and consists of $n_\bullet$ identity matrices of the order M. Consider the stationary probability distribution of process $X(t)$

$$p_0 = \lim_{t\to\infty} \mathbb{P}\{\boldsymbol{\xi}(t) = \mathbf{0}\} \quad P_{\mathbf{n}}(\mathbf{x}) = \lim_{t\to\infty} \mathbb{P}\{\boldsymbol{\xi}(t) = \mathbf{n}, \boldsymbol{\theta}(t) \le \mathbf{x}\} \quad \mathbf{x} \in \mathrm{R}^{Mn_\bullet}, \mathbf{n} \in \mathcal{J}. \tag{2}$$

The next results give the stationary solution to a multi-class loss networks with state-dependent arrivals and service rates. We present the proof in the Appendix at the end of this paper.

THEOREM: *Let $F_k^{(n)}(\mathbf{x})$ be the n-fold convolution of the cumulative distribution function $F_k(\mathbf{x})$ and let functions $\Phi_{\mathbf{n}}(\mathbf{x})$ be defined by*

$$\Phi_{\mathbf{n}}(\mathbf{x}) = \int\limits_{\substack{\mathbf{y}\le\mathbf{x} \\ \mathbf{yU}\le\mathbf{R}}} F_1(d\mathbf{y}_{1,1}) \cdots F_1(d\mathbf{y}_{1,n_1}) \cdots F_K(d\mathbf{y}_{K,1}) \cdots F_K(d\mathbf{y}_{K,n_K}), \quad \mathbf{x} \in \mathcal{R}^{Mn_\bullet}, \ \mathbf{n} \in \mathcal{J}. \tag{3}$$

Then the stationary probability distribution of process $X(t)$ is given by

$$p_0 = \left(1 + \sum_{\mathbf{n}\in J} (F_1^{(n_1)} * F_2^{(n_2)} * \ldots * F_K^{(n_K)})(\mathbf{R}) \prod_{k=1}^{K} \prod_{j=1}^{n_k} \frac{\lambda_k(j-1)}{\mu_k(j)}\right)^{-1}, \tag{4}$$

$$P_{\mathbf{n}}(\mathbf{x}) = p_0 \Phi_{\mathbf{n}}(\mathbf{x}) \prod_{k=1}^{K} \prod_{j=1}^{n_k} \frac{\lambda_k(j-1)}{\mu_k(j)}, \quad \mathbf{x} \in \mathcal{R}^{Mn_\bullet}, \quad \mathbf{n} \in \mathcal{J}. \tag{5}$$

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COROLLARY: *The stationary joint probability distribution for the number of customers in the system* $\boldsymbol{\xi}(t)$ *and vectors* $\boldsymbol{\sigma}_1(t), \ldots, \boldsymbol{\sigma}_K(t)$ *of resources occupied by classes is given by*

$$Q_{\mathbf{n}}(\mathbf{x}_1, \ldots, \mathbf{x}_K) = p_0 \Psi_{\mathbf{n}}(\mathbf{x}_1, \ldots, \mathbf{x}_K) \prod_{k=1}^{K} \prod_{j=1}^{n_k} \frac{\lambda_k(j-1)}{\mu_k(j)}, \quad \mathbf{x}_1, \ldots, \mathbf{x}_K \in \mathcal{R}^M, \quad \mathbf{n} \in \mathcal{J}, \tag{6}$$

where functions $\Psi_{\mathbf{n}}(\mathbf{x}_1, \ldots, \mathbf{x}_K)$ are defined by

$$\Psi_{\mathbf{n}}(\mathbf{x}_1, \ldots, \mathbf{x}_K) \int\limits_{\substack{\mathbf{y}_k \le \mathbf{x}_k, k=1,\ldots,K \\ \mathbf{y}_1 + \ldots + \mathbf{y}_K \le \mathbf{R}}} F_1^{(n_1)}(d\mathbf{y}_1) \cdots F_K^{(n_K)}(d\mathbf{y}_K), \quad \mathbf{x}_1, \ldots, \mathbf{x}_K \in \mathcal{R}^M, \quad \mathbf{n} \in \mathcal{J}. \tag{7}$$

Formulas (4)–(7) allow one to compute different stationary characteristics of the system. For example, it follows from (5) that the loss probability of class k customers is given by

$$B_k = 1 - \frac{\gamma_k}{\lambda_k},$$

where

$$\lambda_k = p_0 \sum_{\mathbf{n} \in J} \lambda_k(n_k)(F_1^{(n_1)} * F_2^{(n_2)} * \ldots * F_K^{(n_K)})(\mathbf{R}) \prod_{i=1}^{K} \prod_{j=1}^{n_i} \frac{\lambda_i(j-1)}{\mu_i(j)}$$

is the mean arrival rate and

$$\begin{aligned} \gamma_k &= p_0 \sum_{\mathbf{n} \in J} \lambda_k(n_k)(F_1^{(n_1)} * \ldots * F_{k-1}^{(n_{k-1})} * F_k^{(n_k+1)} * F_{k+1}^{(n_{k+1})} * \ldots * F_K^{(n_K)}) \\ &\quad \times (\mathbf{R}) \prod_{i=1}^{K} \prod_{j=1}^{n_i} \frac{\lambda_i(j-1)}{\mu_i(j)} \\ &= p_0 \sum_{\mathbf{n} \in J} \mu_k(n_k)(F_1^{(n_1)} * F_2^{(n_2)} * \ldots * F_K^{(n_K)})(\mathbf{R}) \prod_{i=1}^{K} \prod_{j=1}^{n_i} \frac{\lambda_i(j-1)}{\mu_i(j)}. \end{aligned}$$

is the mean acceptance rate of class k customers.

Note that formulas for functions (3) and (7) can be simplified for some particular values of arguments: $\Phi_{\mathbf{n}}(\mathbf{x}) = F(\mathbf{x}_{1,1}) \cdots F(\mathbf{x}_{1,n_1}) \cdots F(\mathbf{x}_{K,1}) \cdots F(\mathbf{x}_{K,n_K})$ if $\mathbf{xU} \le \mathbf{R}$, and $\Psi_{\mathbf{n}}(\mathbf{x}_1, \ldots, \mathbf{x}_K) = F_1^{(n_1)}(\mathbf{x}_1) \cdots F_K^{(n_K)}(\mathbf{x}_K)$ if $\sum_{k=1}^{K} \mathbf{x}_k \le \mathbf{R}$.

3. POSITIVE AND NEGATIVE CUSTOMERS

Now assume that the set of customer classes $\mathcal{K} = \{1, 2, \ldots, K\}$ is the union of two disjoint subsets $\mathcal{K}_-$ and $\mathcal{K}_+$, such that the vectors of resource requests of customers from classes belonging to $\mathcal{K}_-$ are non-positive and the vectors of resource requests of customers from classes belonging to $\mathcal{K}_+$ are non-negative. In this case, there are two types of customers: positive customers, which belong to classes from $\mathcal{K}_+$, and negative customers belonging to a class from $\mathcal{K}_-$. The inequality $\boldsymbol{\sigma}(t) \le \mathbf{R}$ holds for all t, and at the time of arrival of a negative customer we always have $\boldsymbol{\sigma}(a_j - 0) + \mathbf{r}_j \le \mathbf{R}$. Therefore, if at the time of arrival of a negative customer the system is not full, the customer cannot be blocked. Similarly, at the moment t of completion of service of a positive customer we always have $\boldsymbol{\sigma}(t) - \mathbf{r}_j \le \mathbf{R}$.

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Therefore departure of a positive customer cannot be disabled and it always leaves the system upon service completion.

We can rewrite the inequality $\boldsymbol{\sigma}(t) \leq \mathbf{R}$ as

$$\sum_{k \in \mathcal{K}_+} \boldsymbol{\sigma}_k(t) \leq \mathbf{R} - \sum_{k \in \mathcal{K}_-} \boldsymbol{\sigma}_k(t). \tag{8}$$

The left-hand side of (8) is the vector of resource quantities allocated to positive customers. It is bounded above by the sum of the vector of available resources, $\mathbf{R}$, and the vector of resource quantities allocated to negative customers taken with the opposite sign. Therefore, arrivals of negative customers temporary increase the amount of resources available to positive customers. If at the completion of service of a negative customer the resources, which it brought to the system, are used by a positive customer, departure of the negative customer is disabled. It cannot leave the system until its departure becomes enabled after re-servicing.

4. CONCLUSION

In this paper, we address the problem of deriving the stationary probability distribution of multi-class loss systems with multiple resources and state-dependent arrival and service rates. We allow the quantities of resources requested by customers to be positive or negative and observe that requests for a negative quantity of a resource increase the amount of the resource available to customers requesting positive quantities of it. In practice, queuing systems with positive and negative resource demands can be used to model stand-alone access points that can dynamically increase their capacity, or to model heterogeneous network deployed on some area. This way, it is possible to account for the user (or vehicles) dynamics, when new entities can provide their network capacity for nearby consumers to use, while taking it back when they leave the area.

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APPENDIX: PROOF OF THE THEOREM

Proof: The system state can change only at times t_i, when either an arrival occurs or the service of one of the customers is completed. Consider the sojourn of process $X(t)$ in state $(\mathbf{n}, \mathbf{y}_1, \dots, \mathbf{y}_K)$, where $\mathbf{n} = (n_1, n_2, \dots, n_K)$ and $\mathbf{y}_k = (\mathbf{y}_{k,1}, \dots, \mathbf{y}_{k,n_k})$, $k = 1, 2, \dots, K$. Each sojourn time of process $X(t)$ in this state is exponentially distributed with rate

$$\gamma(\mathbf{n}) = \sum_{k=1}^{K} \lambda_k(n_k) + \sum_{k=1}^{K} \mu_k(n_k).$$

At the end of such a sojourn, either, with probability $\lambda_k(n_k)/\gamma(\mathbf{n})$, an arrival of a new customer of class k occurs, or, with probability $\mu_k(n_k)/\gamma(\mathbf{n})$, the service of a class k customer completes. In the latter case, any class k customer may complete its service with probability $1/n_k$. At the moment of arrival of class k customer, process $X(t)$ jumps to state $(\mathbf{n}_k + \mathbf{e}_k, \mathbf{y}_1, \dots, \mathbf{y}_{k-1}, \bar{\mathbf{y}}_k, \mathbf{y}_{k+1}, \dots, \mathbf{y}_K)$, where $\bar{\mathbf{y}}_k = (\mathbf{y}_{k,1}, \dots, \mathbf{y}_{k,n_k}, \mathbf{y}_{k,n_k+1})$. At the moment of departure of class k customer, process $X(t)$ jumps to state $(\mathbf{n}_k - \mathbf{e}_k, \mathbf{y}_1, \dots, \mathbf{y}_{k-1}, \tilde{\mathbf{y}}_k, \mathbf{y}_{k+1}, \dots, \mathbf{y}_K)$, where matrix $\tilde{\mathbf{y}}_k$ is obtained from $\mathbf{y}_k$ by deleting the subvector of resource quantities of one of enabled class k customers.

The above state transitions of process $X(t)$ define unambiguously its infinitesimal transition rates Breuer [2] and lead to the following system of equations for the stationary probability distribution:

$$\begin{aligned}
&p_0 \sum_{k=1}^{K} \lambda_j(0) F_j(\mathbf{R}) = \sum_{k=1}^{K} \mu_j(1) P_{\mathbf{e}_j}(\mathbf{R}) \\
&\lambda_k(1) \int\limits_{\mathbf{y} \le \mathbf{x}} F_k(\mathbf{R} - \mathbf{y}) P_{\mathbf{e}_k}(d\mathbf{y}) + \sum_{\substack{i=1 \\ i \ne k}}^{K} \lambda_i(0) \int\limits_{\mathbf{y} \le \mathbf{x}} F_i(\mathbf{R} - \mathbf{y}) P_{\mathbf{e}_k}(d\mathbf{y}) + \mu_k(1) P_{\mathbf{e}_k}(\mathbf{x}) \\
&\quad = \lambda_k(0) F_k(\mathbf{x}) p_0 + \frac{\mu_k(2)}{2} \Big(\int\limits_{\substack{\mathbf{y} \le \mathbf{x} \\ \mathbf{y} \le \mathbf{R}}} P_{2\mathbf{e}_k}(d\mathbf{y}, d\mathbf{z}) + \int\limits_{\substack{\mathbf{y} \le \mathbf{x} \\ \mathbf{y} \le \mathbf{R}}} P_{2\mathbf{e}_k}(d\mathbf{z}, d\mathbf{y}) \Big) \\
&\qquad + \sum_{i=1}^{k-1} \mu_j(1) \int\limits_{\substack{\mathbf{y} \le \mathbf{x} \\ \mathbf{y} \le \mathbf{R}}} P_{\mathbf{e}_k + \mathbf{e}_i}(d\mathbf{y}, d\mathbf{z}) + \sum_{i=k+1}^{K} \mu_j(1) \int\limits_{\substack{\mathbf{y} \le \mathbf{x} \\ \mathbf{y} \le \mathbf{R}}} P_{\mathbf{e}_k + \mathbf{e}_i}(d\mathbf{z}, d\mathbf{y}), \quad \mathbf{x} \in \mathcal{R}^M, \quad 1 \le k \le K;
\end{aligned} \tag{A.1}$$

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$$\sum_{i=1}^{K}\lambda_i(n_i)\int\limits_{\mathbf{y}\le\mathbf{x}} F_i(\mathbf{R}-\mathbf{y}\mathbf{U})P_{\mathbf{n}}(d\mathbf{y}_{1,1},\dots,d\mathbf{y}_{1,n_1},\dots,d\mathbf{y}_{K,1},\dots,d\mathbf{y}_{K,n_K})$$

$$+\sum_{\substack{i=1\\ n_i\neq 0}}^{K}\frac{\mu_i(n_i)}{n_i}\sum_{j=1}^{n_i}\int\limits_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\mathbf{U}-\mathbf{y}_{i,j}\le\mathbf{R}}} P_{\mathbf{n}}(d\mathbf{y}_{1,1},\dots,d\mathbf{y}_{1,n_1},\dots,d\mathbf{y}_{K,1},\dots,d\mathbf{y}_{K,n_K})$$

$$=\sum_{i=1}^{K}\lambda_i(n_i-1)\int\limits_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\mathbf{U}\le\mathbf{R}}} P_{\mathbf{n}-\mathbf{e}_i}(d\mathbf{y}_{1,1},\dots,d\mathbf{y}_{1,n_1},\dots,d\mathbf{y}_{i,1},\dots,d\mathbf{y}_{i,n_i-1},\dots,$$

$$\times\, d\mathbf{y}_{K,1},\dots,d\mathbf{y}_{K,n_K})F_i(d\mathbf{y}_{i,n_i})$$

$$+\sum_{i=1}^{K}\frac{\mu_i(n_i+1)}{n_i+1}\sum_{j=1}^{n_i+1}\int\limits_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\mathbf{U}\le\mathbf{R}}} P_{\mathbf{n}+\mathbf{e}_i}(d\mathbf{y}_{1,1},\dots,d\mathbf{y}_{1,n_1},\dots,d\mathbf{y}_{i,1},\dots,d\mathbf{y}_{i,j-1},d\mathbf{z},$$

$$\times\, d\mathbf{y}_{i,j},\dots,d\mathbf{y}_{i,n_k},\dots,d\mathbf{y}_{K,1},\dots,d\mathbf{y}_{K,n_K}),\quad \mathbf{x}\in\mathcal{R}^{Mn_\bullet},\ \mathbf{n}\in\mathcal{J},\ 1\le n_\bullet<C; \tag{A.2}$$

$$\sum_{\substack{i=1\\ n_i\neq 0}}^{K}\frac{\mu_i(n_i)}{n_i}\sum_{j=1}^{n_i}\int\limits_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\mathbf{U}-\mathbf{y}_{i,j}\le\mathbf{R}}} P_{\mathbf{n}}(d\mathbf{y}_{1,1},\dots,d\mathbf{y}_{1,n_1},\dots,d\mathbf{y}_{K,1},\dots,d\mathbf{y}_{K,n_K})$$

$$=\sum_{i=1}^{K}\lambda_i(n_i-1)\int\limits_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\mathbf{U}-\mathbf{y}_{i,j}\le\mathbf{R}}} P_{\mathbf{n}-\mathbf{e}_i}(d\mathbf{y}_{1,1},\dots,d\mathbf{y}_{1,n_1},\dots,d\mathbf{y}_{i,1},\dots,d\mathbf{y}_{i,n_i-1},\dots,$$

$$\times\, d\mathbf{y}_{K,1},\dots,d\mathbf{y}_{K,n_K})F_i(d\mathbf{y}_{i,n_i}),\quad \mathbf{x}\in\mathcal{R}^{Mn_\bullet},\ \mathbf{n}\in\mathcal{J},\ n_\bullet=C. \tag{A.3}$$

Now it can be verified by substitution that the solution of equations (A.1)–(A.3) satisfying the normalization condition is given by formulas (4) and (5). ■

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