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Covariant formulation of photon acceleration

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We present a covariant theory of photon acceleration in time-varying plasmas. This is a ray-tracing model for frequency upshift of waves interacting with relativistic plasma perturbations, such as ionization fronts and plasma wakefields. This theory explores the formal analogy between a photon in a plasma and a relativistic particle with a finite rest mass. The covariant ray-tracing theory is applied to the case of photon acceleration by a modulated wakefield. Threshold criteria for transition to chaos are derived.

1. Introduction

The concept of photon acceleration has fairly recently been introduced in plasma physics and has received considerable attention over the last few years.

A number of theoretical papers have approached the different aspects of photon acceleration and have been able to define the scope and boundaries of this new concept. Of particular relevance is the work of Wilks *et al.* (1989), which introduced the term 'photon acceleration' in the specific context of plasma wakefield acceleration, and the work of Mendonça (1979), which introduced the concept of adiabatic frequency shift for two laser beams interacting nonlinearly. More recent theoretical models by Mori (1991) and Mendonça and Oliveira e Silva (1994) established the framework of a more systematic theoretical approach.

In broad terms, we can presently define photon acceleration as a non-resonant wave–wave interaction, leading to an adiabatic frequency shift of wavepackets propagating in a space- and time-varying medium. This concept now includes wave-frequency shift by ionization fronts, nonlinear perturbations and wakefields, as well as by flash-ionization processes (Hizanidis *et al.* 1996; Mendoça and Oliveira e Silva 1996). More recently, a related theory of photon Landau damping has also been considered by Bingham *et al.* (1997).

In the present work we shall consider the covariant formulation of the Hamiltonian equations for a photon propagating in a space- and time-varying plasma. This approach is suggested by the formal analogy between the photon dispersion relation in a plasma and the energy of a charged particle with a finite rest mass. We can then say that, in a plasma, the photon possesses an effective rest mass. This corresponds to the generalization of the Hamiltonian theory

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developed by Mendonça and Oliveira e Silva (1994). The covariant ray-tracing equations and the explicit expression for the corresponding super-Hamiltonian will be derived in Sec. 2.

In order to illustrate the interest of this new approach, in Sec. 3 we shall apply the covariant equations of motion to the particularly important case of photon acceleration by a modulated plasma-wave wakefield. This wakefield can eventually be excited by an intense laser beam propagating in a plasma, and its modulation can be due to an ion acoustic oscillation, which can also be excited by the laser beam. In such a configuration, transition to chaos and the corresponding generation of white light can eventually occur. The stochasticity criterion for the photon trajectories will be derived explicitly. Finally, in Sec. 4, the main physical implications of the present work will be discussed, and our conclusions will be stated.

2. Covariant ray-tracing equations

Let us assume a wave propagating in a space- and time-varying plasma. The local values of the wave vector $\mathbf{k}$ and frequency ω of the wave propagating across this varying medium will then change in space and time, according to the ray-tracing equations valid in the geometric-optics approximation. It is well known (Weinberg 1962; Bernstein and Friedland 1983) that the ray-tracing equations can be written in the canonical form

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \omega}{\partial \mathbf{k}}, \tag{1}$$

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial \omega}{\partial \mathbf{r}}. \tag{2}$$

These ray-tracing equations can also be seen as describing point particles: the photons. We notice that the canonical variables here are the photon position and wave vector, $\mathbf{r}$ and $\mathbf{k}$, while the frequency ω plays the role of the Hamiltonian function. In a cold and unmagnetized plasma we have

$$\omega \equiv h(\mathbf{r}, \mathbf{k}, t) = [k^2c^2 + \omega_p^2(\mathbf{r}, t)]^{1/2}. \tag{3}$$

This expression can also be written as

$$\omega = \gamma \omega_p = m_{\text{eff}}\, \gamma c^2. \tag{4}$$

It is obvious that ω can be seen as the energy of a relativistic particle with momentum $\mathbf{k}$ and an effective rest mass proportional to the plasma frequency: $m_{\text{eff}} = \omega_p/c^2$. The corresponding gamma factor is

$$\gamma = \left[1 - \frac{1}{c^2}\left|\frac{d\mathbf{r}}{dt}\right|^2\right]^{-1/2} = \frac{\omega}{\omega_p}. \tag{5}$$

From (1)–(3), we can easily derive the Newtonian equation of motion for a photon in a plasma:

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega_p \frac{c^2}{\omega^2}\left(\frac{\partial \omega_p}{\partial \mathbf{r}} + \frac{\mathbf{k}}{\omega}\frac{\partial \omega_p}{\partial t}\right). \tag{6}$$

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Alternatively, we could also describe the photon motion by using the Lagrangian function

$$L(\mathbf{r},\mathbf{v},t) = -m_{\rm eff}\frac{c^2}{\gamma} = -\omega_p\left(1-\frac{v^2}{c^2}\right)^{1/2}. \tag{7}$$

In this way, we can obtain the three equivalent versions of the photon equations of motion in a space- and time-varying plasma: the Newtonian equation of motion (6), the Hamiltonian equations (1)–(3) and the Lagrangian equation associated with (7).

Let us now turn to the covariant equations of motion. We first define the four-vectors of position and momentum, such that

$$\hat{r} = (ct,\mathbf{r}), \tag{8}$$

$$\hat{k} = \left(\frac{\omega}{c},\mathbf{k}\right). \tag{9}$$

The squared modulus of the photon four-vector momentum is determined by

$$\hat{k}^2 = k^2 - \frac{\omega^2}{c^2} = -m_{\rm eff}^2 c^2. \tag{10}$$

We can now define a covariant Hamiltonian, or super-Hamiltonian, for the photon in a plasma as

$$\hat{H}(\hat{r},\hat{k}) = \frac{\hat{k}^2}{2m_{\rm eff}} + \frac{1}{2}m_{\rm eff}c^2 \tag{11}$$

This can also be written in the equivalent form

$$\hat{H}(\hat{r},\hat{k}) = \frac{\hat{k}^2c^2}{2\omega_p(\hat{r})} + \frac{1}{2}\omega_p(\hat{r}). \tag{12}$$

The canonical equations of motion are now given by

$$\frac{dr^\alpha}{d\tau} = \frac{\partial\hat{H}}{\partial k_\alpha}, \tag{13}$$

$$\frac{dk_\beta}{d\tau} = -\frac{\partial\hat{H}}{\partial r^\beta}, \tag{14}$$

where τ is the photon proper time. It can easily be recognized that these equations are equivalent to the different versions of the photon equations of motion discussed above. In the eight-dimensional phase space $(\hat{r},\hat{k})$, we can represent each photon trajectory by the following microscopic density (in the Klimontovich sense):

$$\rho_1(\hat{r},\hat{k},\tau) = \delta[\hat{r}-\hat{r}_1(\tau)]\,\delta[\hat{k}-\hat{k}_1(\tau)], \tag{15}$$

where $\hat{r}_1(\tau)$ and $\hat{k}_1(\tau)$ represent a particular solution of the covariant equations of motion. Generalizing to a number N of photons, we obtain

$$\rho(\hat{r},\hat{k},\tau) = \sum_{j=1}^{N}\delta[\hat{r}-\hat{r}_j(\tau)]\,\delta[\hat{k}-\hat{k}_j(\tau)]. \tag{16}$$

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Applying the total proper time derivative to ρ and noting that, in the absence of any damping or amplifying mechanism, the total number of photons N is a constant, we can easily derive the kinetic equation for the photons in the Minkowski phase space:

$$\frac{\partial \rho}{\partial \tau} + [\hat{H}, \rho]_4 = 0, \tag{17}$$

where the covariant version of the Poisson bracket has been used. This kinetic equation can be very useful in describing the time–frequency evolution of very short pulses. This problem will be discussed in a separate work. In the next section we shall concentrate on the dynamical properties of a single photon, as described by the covariant equations of motion (13) and (14).

3. Modulated wakefield

Let us now study the photon motion in a plasma with a modulated wakefield. It is known that a strong laser beam can excite large-amplitude plasma waves that can propagate with relativistic phase velocities. Here we will refer to these strong plasma waves as wakefields. Other kinds of oscillations, for instance those associated with the ion motion, can eventually also be excited in the plasma, leading to time modulation of this wakefield. Amplitude modulation of the driving laser beam would also lead to a modulated wakefield amplitude.

The plasma density perturbations associated with such modulated wakefields can be described by

$$\omega_p^2(\hat{r}) = \omega_{p0}^2[1 + a(r^0)\cos(\hat{q}\cdot\hat{r})], \tag{18}$$

where ω_{p0} denotes the unperturbed plasma frequency, $\hat{q}$ is the four-momentum associated with the electron plasma wakefield, and $a(r^0) \equiv a(t)$ is the modulated amplitude. Let us write

$$\hat{q}\cdot\hat{r} = q^1 r^1 - q^0 r^0 = k' r^1 - \omega' t, \tag{19}$$

where $\omega' \approx \omega_{p0}$ obeys the electron plasma dispersion relation $\omega'^2 = \omega_{p0}^2 + k'^2 S_e^2$. The modulation amplitude $a(r^0)$ can also be written as

$$a(r^0) = A\left[1 + \epsilon\cos\left(\frac{\Omega}{c} r^0\right)\right] = A[1 + \epsilon\cos(\Omega t)], \tag{20}$$

where $\Omega \ll \omega'$ is the modulation frequency.

In this particular model, the super-Hamiltonian $\hat{H}$ is independent of the space components r^2 and r^3, which means that we have two constants of motion, k^2 and k^3. We shall assume a one-dimensional photon propagation such that

$$k^2 = k^3 = 0. \tag{21}$$

We can then write the one-dimensional covariant Hamiltonian as

$$\hat{H}(r^0, r^1, k_0, k_1) = \frac{(k_1)^2 - (k_0)^2}{2\omega_p(r^0, r^1)} c^2 + \frac{1}{2}\omega_p(r^0, r^1). \tag{22}$$

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The corresponding equations of motion are

$$\frac{dr^0}{d\tau} = -\frac{k_0 c^2}{\omega_p(r^0, r^1)}, \tag{23}$$

$$\frac{dr^1}{d\tau} = \frac{k_1 c^2}{\omega_p(r^0, r^1)} \tag{24}$$

and

$$\frac{dk_0}{d\tau} = \frac{\omega_{p0}^2}{4\omega_p(r^0, r^1)}\left[\frac{(k_1)^2 - (k_0)^2}{\omega_p^2(r^0, r^1)}c^2 - 1\right]\left[-A\epsilon\frac{\Omega}{c}\sin\left(\frac{\Omega}{c}r^0\right)\cos(\hat{q}\cdot\hat{r}) + aq^0\sin(\hat{q}\cdot\hat{r})\right], \tag{25}$$

$$\frac{dk_1}{d\tau} = -aq^1\frac{\omega_{p0}^2}{4\omega_p(r^0, r^1)}\left[\frac{(k_1)^2 - (k_0)^2}{\omega_p^2(r^0, r^1)}c^2 - 1\right]\sin(\hat{q}\cdot\hat{r}). \tag{26}$$

It is now useful to introduce dimensionless variables such that

$$x = q^1 r^1 = k' r^1, \tag{27}$$

$$y = q^1 r^0 = \frac{q^0}{\beta} r^0 \tag{28}$$

and

$$v = \frac{k_1 c}{\omega_{p0}}, \tag{29}$$

$$u = \frac{k_0 c}{\omega_{p0}}. \tag{30}$$

In (28) we have used the normalized phase velocity of the wakefield $\beta = \omega'/k'c$. The dimensionless proper time dz and the normalized super-Hamiltonian h will now be defined by

$$dz = k' c\, d\tau. \tag{31}$$

$$h = \frac{\hat{H}}{\omega_{p0}}. \tag{32}$$

We can now state the new equations of motion as

$$\frac{dx}{dz} = \frac{\partial h}{\partial v}, \tag{33}$$

$$\frac{dv}{dz} = -\frac{\partial h}{\partial x} \tag{34}$$

and

$$\frac{dy}{dz} = \frac{\partial h}{\partial u}, \tag{35}$$

$$\frac{du}{dz} = -\frac{\partial h}{\partial y}, \tag{36}$$

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where the explicit form of the normalized Hamiltonian h is

$$h(x,v;y,u) = \frac{1}{2}\frac{v^2-u^2}{\Omega_p(x,y)} + \frac{1}{2}\Omega_p(x,y). \tag{37}$$

Here we have introduced a normalized plasma frequency such that

$$\Omega_p(x,y) = \frac{\omega_p(x,y)}{\omega_{p0}}. \tag{38}$$

According to this definition, we can write

$$\Omega_p^2(x,y) = 1 + A[1+\epsilon\cos(\nu\beta y)]\cos(x-\beta y), \tag{39}$$

where we have used

$$\nu = \frac{\Omega}{\omega} \approx \frac{\Omega}{\omega_{p0}} \ll 1. \tag{40}$$

An equivalent expression for Ω_p^2, which is more convenient for the discussion of nonlinear photon behaviour, is

$$\Omega_p^2(x,y) = 1 + \sum_{j=0,\pm 1} a_j \cos(x-\nu_j\beta y), \tag{41}$$

where the amplitudes a_j and the frequencies ν_j are defined by $a_0 = A$, $a_\pm = \frac{1}{2}A\epsilon$ and $\nu_0 = 1$, $\nu_\pm = 1 \pm \nu$. We see from (37) and (41) that the reduced Hamiltonian h presents three distinct nonlinear resonances in photon phase space. Another important conclusion is that, for large amplitudes a_j, these resonances can eventually overlap, leading to the occurrence of stochastic photon acceleration.

In the absence of amplitude modulation, $\epsilon = 0$, the two satellite resonances disappear, and h reduces to

$$h_0(x,v;y,u) = \frac{1}{2}\frac{v^2-u^2+1+A\cos(x-\beta y)}{[1+A\cos(x-\beta y)]^{1/2}}. \tag{42}$$

By definition, we always have $h = h_0 = 0$. Noting that the denominator in this equation cannot be negative (because $A = \delta n/n_0 \ll 1$ is the relative wakefield amplitude), we obtain

$$u^2 = v^2 + 1 + A\cos(x-\beta y). \tag{43}$$

This simply states, in dimensionless variables, that the photon obeys the local dispersion relation in the space- and time-varying plasma. From (42), we can also see that h_0 only depends on the variables x and y through the quantity $x-\beta y$, which means that

$$\frac{\partial h_0}{\partial x} = -\frac{1}{\beta}\frac{\partial h_0}{\partial y}. \tag{44}$$

Using the canonical equations of motion (34) and (36), we conclude that there is a second constant of motion:

$$I = u + \beta v = \text{const.} \tag{45}$$

This is equivalent to stating that $\omega - kv' = \text{const}$, where $v' = \omega'/k'$ is the velocity of the density perturbations associated with the wakefield. The

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existence of these two independent constants of motion h_0 and I proves that the motion of a photon in an unmodulated wakerfield is integrable, as discussed in a non-covariant formulation by Mendonça and Oliveira e Silva (1994). The position of the resonance in (42) is determined by the condition of stationary phase (in proper time), which can be written as

$$v+\beta u = 0. \tag{46}$$

This is equivalent to stating that, at resonance, the photon group velocity kc^2/ω is equal to the phase velocity of the wakefield ω'/k'. Replacing the resonance condition in (43), we obtain

$$u^2(1-\beta^2) = 1+A\cos(x-\beta y). \tag{47}$$

The centre of this nonlinear resonance can be identified as

$$u_0 = -\frac{1}{(1-\beta^2)^{1/2}} \equiv -\gamma', \tag{48}$$

$$v_0 = \frac{\beta}{(1-\beta^2)^{1/2}}. \tag{49}$$

The corresponding value of the invariant I is

$$I_0 = -(1-\beta^2)^{1/2}. \tag{50}$$

We can also easily recognize that the resonance half-width δu_0 is

$$\delta u_0 = \left(\frac{A}{2}\right)^{1/2} \frac{1}{(1-\beta^2)^{1/2}}. \tag{51}$$

In the presence of modulation, $\epsilon \neq 0$, I is no longer an invariant, and the two satellite resonances appear at

$$u_\pm = -\frac{1}{(1-\nu_\pm^2\,\beta^2)^{1/2}}, \tag{52}$$

$$v_\pm = \frac{\beta\nu_\pm}{(1-\nu_\pm^2\,\beta^2)^{1/2}}. \tag{53}$$

The corresponding half-widths are

$$\delta u_\pm = \left(\frac{a_\pm}{2}\right)^{1/2} \frac{1}{(1-\nu_\pm^2\,\beta^2)^{1/2}}. \tag{54}$$

Applying the well-known overlapping criterium to adjacent resonances, we conclude that transition to stochastic photon acceleration will occur for an amplitude larger than a given threshold:

$$A \geqslant 2\frac{(u_0-u_\pm)^2}{[|u_0|+(\frac{1}{2}\epsilon)^{1/2}\,|u_\pm|]^2}. \tag{55}$$

For $\epsilon \ll 1$ and $\nu \ll 1$, this can be simply stated as

$$A \geqslant \frac{2\nu^2\beta^4}{(1-\beta^2)^2}. \tag{56}$$

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Even for β close to 1, this criterion can be satisfied for moderate values of A if the modulation frequency ν is small. Transition to stochasticity in photon motion is an important qualitative transition, because it leads to generation of radiation with a large spectral width (white light). The spectral width can be estimated as

$$\Delta\omega = 2\omega_{p0}\frac{\nu\beta^2}{(1-\beta^2)^{3/2}}. \tag{57}$$

If $\beta \to 1$, this spectral width can become very significant, but the stochasticity criterion (56) will be more difficult to attain.

4. Conclusions

In this work we have explored the fact that photon propagation in non-stationary plasmas can be described in a rigorous but simple way by the Hamiltonian formulation of ray-tracing theory. Because the photons moving in an unmagnetized plasma have an effective mass that is proportional to the mean plasma frequency, their dynamical behaviour is formally analogous to that of massive relativistic particles. In particular, a covariant formulation of their Hamiltonian equations of motion can be derived.

In order to illustrate the interest of our covariant approach to photon dynamics, we have studied in detail the effect of photon acceleration by an amplitude-modulated plasma wakefield. This problem has strong formal analogies with that of electron acceleration by a modulated electrostatic wave (Ram *et al.* 1986).

In this case transition to stochastic acceleration can eventually occur, leading to generation of white light. A qualitative criterion for the occurrence of such a transition has been derived.

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