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Mean values of arithmetic functions and application to sums of powers

BY RÉGIS DE LA BRETÈCHE
Université Paris Cité, Sorbonne Université, CNRS,
Institut Universitaire de France,
Institut de mathématiques de Jussieu-Paris Rive Gauche, F-75013 *Paris, France.*
e-mail: regis.delabreteche@imj-prg.fr

AND GÉRALD TENENBAUM
Institut Élie Cartan Université de Lorraine, BP 70239,
54506 *Vandœuvre-lès-Nancy Cedex, France.*
e-mail: gerald.tenenbaum@univ-lorraine.fr

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Abstract

We provide new upper bounds for sums of certain arithmetic functions in many variables at polynomial arguments and, exploiting recent progress on the mean-value of the Erdős—Hooley Δ-function, we derive lower bounds for the cardinality of those integers not exceeding a given limit that are expressible as certain sums of powers.

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1. *Introduction*

The Erdős-Hooley Δ-function is defined by the formulae

$$\Delta(n,u) := |\{d : d \mid n,\ \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|, \quad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u) \qquad (n \in \mathbb{N}^*).$$

Recent, spectacular progress on the mean-value of the Δ-function has been obtained in [9, 17], and refined in [4]. This invites one to revisit some of the applications linked to this

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quantity. Put

$$S(x) := \sum_{n\leqslant x} \Delta(n), \quad \mathfrak{S}(x) := \frac{1}{\log x} \sum_{n\leqslant x} \frac{\Delta(n)}{n},$$

so that (see [12, theorem. 61])

$$S(x) \ll x\mathfrak{S}(x) \qquad (x \geqslant 2).^{1}$$

After the works of Erdős [8], Hall—Tenenbaum [11, 12], Hooley [15], Tenenbaum [28], Koukoulopoulos-Tao [17], Ford-Koukoulopoulos-Tao [9], and La Bretèche-Tenenbaum [4], we are now equipped with the bounds[2]

$$(\log_2 x)^{1+\eta+o(1)} \ll \mathfrak{S}(x) \ll (\log_2 x)^{5/2} \qquad (x \geqslant 3), \tag{1·1}$$

where $\eta \approx 0.353327$ is the exponent appearing in the new lower bound for the normal order of $\Delta(n)$ by Ford, Green and Koukoulopoulos [7].

The fact that the asymptotic order of magnitude of $\mathfrak{S}(x)$ is a power of $\log_2 x$ makes it pertinent to refine the estimates relating to the various applications, as the number of representations of an integer as a sum of powers, the Diophantine approximation of an irrational number by a rational with square denominator, or Waring's problem.

Two other applications are also worth mentioning. First note that [19, corollary 8] readily yields that, for any given $\varepsilon > 0$, and uniformly for $x \geqslant 3$, $1 \leqslant |a| \leqslant x$, $x^{\varepsilon} < y \leqslant x$, we have

$$\sum_{x<p\leqslant x+y} \Delta(|p+a|) \ll \frac{|a|y\mathfrak{S}(x)}{\varphi(|a|)\log x} \ll \frac{|a|y(\log_2 x)^{5/2}}{\varphi(|a|)\log x},$$

where the letter p denotes a prime number. Second, observe that the upper bound of (1·1) furnishes, for any fixed $\varepsilon > 0$ and uniformly for $1 \leqslant a \leqslant q \leqslant x^{2-\varepsilon}$, $(a,q) = 1$, $x \geqslant 3$, the estimate

$$\sum_{\substack{n,m\leqslant x\\ mn\equiv a \,(\mathrm{mod}\, q)}} 1 \ll \frac{x^2}{q}(\log_2 x)^{5/2}. \tag{1·2}$$

Indeed, if $N \leqslant x^{1-\varepsilon/2}$, a trivial bound, using $\tau(n) = n^{o(1)}$ suffices, and if $x^{1-\varepsilon/2} < N \leqslant x$, we have

$$\sum_{\substack{m\leqslant x\\ N<n\leqslant 2N\\ mn\equiv a \,(\mathrm{mod}\, q)}} 1 \leqslant \sum_{0\leqslant k\leqslant 2Nx/q} \Delta(kq+a) \ll \frac{Nx}{q}\mathfrak{S}(x),$$

where the latter bound results from a special case of [19, theorem 1]. This implies (1·2) by summing over N.

We intend here to return to the problem of sums of powers, also addressed in [17].

Given $\boldsymbol{c} := \{c_j\}_{0\leqslant j\leqslant t}$, $\boldsymbol{\ell} := \{\ell_j\}_{0\leqslant j\leqslant t} \in (\mathbb{N}^*)^{t+1}$, consider the number

$$r(n) = r(n; \boldsymbol{c}, \boldsymbol{\ell})$$

[1] It is actually easy to show that $S(x) \asymp x\mathfrak{S}(x)$ $(x \geqslant 2)$, but we shall not need the lower bound here.

[2] Here and throughout, we let $\log_k$ denote the kth iterated of the logarithm.

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of representations of a natural integer n in the form

$$n = \sum_{0\leqslant j\leqslant t} c_j m_j^{\ell_j} \qquad (\boldsymbol{m}\in\mathbb{N}^{t+1}).$$

In all the sequel, we shall assume without loss of generality that $\{\ell_j\}_{0\leqslant j\leqslant t}$ is non-decreasing.

Under the condition $\sum_{0\leqslant j\leqslant t} 1/\ell_j = 1$, we trivially have

$$V_0(x;\boldsymbol{c},\boldsymbol{\ell}) := \{n\leqslant x : r(n)\geqslant 1\} \leqslant x \qquad (x\geqslant 1).$$

The exact order of magnitude of $V_0(x) = V_0(x;\boldsymbol{c},\boldsymbol{\ell})$ is only known in the case $\boldsymbol{\ell} = (2,2)$. A famous conjecture (see, for instance, [1], or [31]) is that $V_0(x)\asymp x$ for $\boldsymbol{\ell} = (3,3,3)$, $\boldsymbol{c} = (1,1,1)$. The case $t = 2$, $\boldsymbol{c} = (1,1,1)$, $\boldsymbol{\ell} = (2,4,4)$ was studied by Hooley [15], Tenenbaum [29], Robert [21], and Koukoulopoulos–Tao [17]. Robert [21] also considered the case $\boldsymbol{\ell} = (2,3,6)$.

We obtain the following result.

THEOREM 1·1. *Let $t\geqslant 2$, $\boldsymbol{c}\in(\mathbb{N}^*)^{t+1}$, $\boldsymbol{\ell} = \{\ell_j\}_{0\leqslant j\leqslant t}$, $\ell_0 = 2$, $\ell_1\in\{3,4\}$, $\sum_{1\leqslant j\leqslant t} 1/\ell_j = 1/2$. Let $s := \max\{j\in[1,t-1] : \ell_{t-j+1} = \ldots = \ell_t\}$. Assume*:

(i) $1\leqslant s\leqslant 3$;

(ii) $\ell_t\geqslant 26$ *if* $s = 3$;

(iii) $\sum_{j\geqslant r} 1/\ell_j \leqslant 1/\ell_{r-1}$ $(1\leqslant r\leqslant t-s+1)$.

Then, we have

$$V_0(x;\boldsymbol{c},\boldsymbol{\ell}) \gg x/\mathfrak{S}(x) \gg x/(\log_2 x)^{5/2}. \tag{1·3}$$

We have not sought to specify uniformity in respect to the coefficients c_j in this statement.

Observe that the following sets fulfill the assumptions:

$t = 2$, $\boldsymbol{\ell}\in\{(2,3,6),\ (2,4,4)\}$;

$t = 3$, $\boldsymbol{\ell}\in\{(2,3,7,42),\ (2,3,8,24),\ (2,3,9,18),\ (2,3,10,15),\ (2,3,12,12),\ (2,4,8,8)\}$;

$t = 4$, $\boldsymbol{\ell}\in\{(2,3,7,84,84),\ (2,3,8,48,48),\ (2,4,8,16,16),\ (2,3,12,24,24)\}$;

$t = 5$, $\boldsymbol{\ell}\in\{(2,3,12,36,36,36),\ (2,3,10,45,45,45),\ (2,4,8,16,32,32)\}$;

this list being non exhaustive. New sets can indeed be constructed by increasing the cardinality t. For instance, when $\boldsymbol{\ell}$ belongs to the above list, we can define $\boldsymbol{\ell}^+\in(\mathbb{N}^*)^{t+2}$ by setting $\ell_j^+ := \ell_j$ for $0\leqslant j\leqslant t-1$, $\ell_t^+ := 2\ell_t$, $\ell_{t+1}^+ = 2\ell_t$. Yet another possibility is to define $\boldsymbol{\ell}^*\in(\mathbb{N}^*)^{t+3}$ with $\ell_t^* = \ell_{t+1}^* = \ell_{t+2}^* := 3\ell_t$ provided $\ell_t\geqslant 9$. Thus, $\boldsymbol{\ell} = (2,3,6)$ induces (2,3,12,12) and (2, 3, 18, 18, 18).

The article [17] comes back on the case $\boldsymbol{\ell} = (2,4,4)$—see, however, the remark following the statement of Proposition 5·1 *infra*.

The cases with $t\geqslant 3$ in Theorem 1·1 are new. The corresponding results rely on the bound

$$\left|\left\{(m_{t-2},m_{t-1},m_t,n_{t-2},n_{t-1},n_t) : \sum_{t-2\leqslant j\leqslant t} c_j m_j^{\ell_t} = \sum_{t-2\leqslant j\leqslant t} c_j n_j^{\ell_t} \leqslant x\right\}\right| \ll x^{3/\ell_t}, \tag{1·4}$$

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established by Salberger provided $\ell_t \geqslant 26$—see the proof of Proposition 5·2 *infra* for precise references. In private communication [**24**], Salberger informed us that, as a direct consequence of his works [**22**, **23**], the bound (1·4) persists under the weaker condition $\ell_t \geqslant 16$. Assuming this, we get that (1·3) also holds for $\boldsymbol{\ell} = (2, 3, 18, 18, 18)$ and $\boldsymbol{\ell} = (2, 4, 8, 24, 24, 24)$. The starting point of the proof of Theorem 1·1 is classically the Cauchy-Schwarz inequality

$$V_1(x)^2 \leqslant V_0(x)V_2(x), \tag{1·5}$$

where we have set

$$V_j(x) = V_j(x; \boldsymbol{c}, \boldsymbol{\ell}) := \sum_{n \leqslant x} r(n)^j \qquad (j = 1, 2). \tag{1·6}$$

We shall deduce Theorem 1·1 from a general estimate regarding $j = 2$.

2. *Notation*

Some notation must be introduced in order to state our results.

Given $k \in \mathbb{N}^*$, $\boldsymbol{a} = (a_j)_{1 \leqslant j \leqslant k}, \boldsymbol{b} = (b_j)_{1 \leqslant j \leqslant k} \in \mathbb{N}^{*k}$, we put

$$\mathfrak{s}\boldsymbol{a} := \sum_{1 \leqslant j \leqslant k} a_j, \quad \wp\boldsymbol{a} := a_1 \cdots a_k, \quad \boldsymbol{ab} = (a_1 b_1, \dots, a_k b_k).$$

For each value of the parameters $A \geqslant 1$, $B \geqslant 1$, $\varepsilon > 0$, we designate by $\mathcal{M}_k(A, B, \varepsilon)$ the class of those non-negative arithmetic functions $F : \mathbb{N}^{*k} \to \mathbb{R}^+$, satisfying the condition

$$F(\boldsymbol{ab}) \leqslant \min\left\{A^{\Omega(\wp\boldsymbol{a})}, B(\wp\boldsymbol{a})^{\varepsilon}\right\} F(\boldsymbol{b}) \tag{2·1}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^{*k}$ such that $(\wp\boldsymbol{a}, \wp\boldsymbol{b}) = 1$. By convention, the functions $F \in \mathcal{M}_k(A, B, \varepsilon)$ are extended to $\mathbb{N}^k$ by setting $F(\boldsymbol{a}) = 0$ if $\min_j a_j = 0$. When $F \neq 0$, we define

$$G(\boldsymbol{a}) = G_F(\boldsymbol{a}) := \max_{\substack{\boldsymbol{b} \in \mathbb{N}^{*k} \\ (\wp\boldsymbol{a}, \wp\boldsymbol{b}) = 1 \\ F(\boldsymbol{b}) \neq 0}} \frac{F(\boldsymbol{ab})}{F(\boldsymbol{b})} \qquad \big(\boldsymbol{a} \in \mathbb{N}^{*k}\big). \tag{2·2}$$

Given a family $\{Q_j\}_{j=1}^k \in \mathbb{Z}[X_1, \dots, X_t]^k$ of polynomials in t variables, we put

$$Q = \prod_{1 \leqslant j \leqslant k} Q_j = \prod_{1 \leqslant h \leqslant r} R_h^{\gamma_h} \in \mathbb{Z}[X_1, \dots, X_t], \quad g := \deg Q, \tag{2·3}$$

where the R_h are irreducible in $\mathbb{Z}[X_1, \dots, X_t]$. We may then write, canonically,

$$Q_j = \prod_{1 \leqslant h \leqslant r} R_h^{\gamma_{jh}} \qquad (1 \leqslant j \leqslant k),$$

where $\gamma_{jh} \geqslant 0$ for all j, h, so that

$$\gamma_h = \sum_{1 \leqslant j \leqslant k} \gamma_{jh} \qquad (1 \leqslant h \leqslant r), \qquad \sum_{1 \leqslant h \leqslant r} \gamma_h \deg R_h = g.$$

Recall that a polynomial is said to be primitive if the greatest common divisor of its coefficients is 1. In all the sequel we assume the form Q to be primitive, which implies that the same holds for all the Q_j.

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For $T \in \mathbb{Z}[X_1, \dots, X_t]$, we write

$$\varrho_T^+(s) := \sum_{\substack{\boldsymbol{\xi} \in [1,s]^t \\ T(\boldsymbol{\xi}) \equiv 0 \pmod{s}}} 1 \qquad (s \geqslant 1). \tag{2·4}$$

Letting $\kappa(s) := \prod_{p \mid s} p$ denote the squarefree kernel of a natural integer s, we further put

$$\mathcal{K}(\boldsymbol{s}) := \operatorname{lcm}(s_j \kappa(s_j))_{1 \leqslant j \leqslant r} = [s_1 \kappa(s_1), \dots, s_r \kappa(s_r)] \qquad \big(\boldsymbol{s} = (s_1, \dots, s_r) \in \mathbb{N}^{*r}\big). \tag{2·5}$$

Given $\boldsymbol{R} \in \mathbb{Z}[X_1, \dots, X_t]^r$, we may then define two arithmetic functions of the variable $\boldsymbol{s} = (s_1, \dots, s_r) \in \mathbb{N}^{*r}$ by

$$\varrho_{\boldsymbol{R}}^+(\boldsymbol{s}) := \sum_{\substack{\boldsymbol{\xi} \in [1, \wp \boldsymbol{s}]^t \\ R_h(\boldsymbol{\xi}) \equiv 0 \pmod{s_h}\ (1 \leqslant h \leqslant r)}} 1, \tag{2·6}$$

$$\varrho_{\boldsymbol{R}}^{\#}(\boldsymbol{s}) := \sum_{\substack{\boldsymbol{\xi} \in [1, \mathcal{K}(\boldsymbol{s})]^t \\ s_h \| R_h(\boldsymbol{\xi})\ (1 \leqslant h \leqslant r) \\ (R_h(\boldsymbol{\xi})/s_h, \wp \boldsymbol{s}) = 1}} 1, \tag{2·7}$$

where the symbol $a \| b$ means that the conditions $a \mid b$ and $(a, b/a) = 1$ hold simultaneously. It is worthwhile to note that the above definition of $\varrho_{\boldsymbol{R}}^{\#}$ is more restrictive than that of [**3**]. The grounds for this alteration are described in [**14**].

For any $\boldsymbol{s} = (s_1, \dots, s_r) \in \mathbb{N}^{*r}$, let us put

$$s_j' := \prod_{1 \leqslant h \leqslant r} s_h^{\gamma_{jh}} \quad (1 \leqslant j \leqslant k), \qquad \boldsymbol{s}' := (s_1', \dots, s_k'), \qquad s'' := \wp \boldsymbol{s}' = \prod_{1 \leqslant j \leqslant k} s_j' = \prod_{1 \leqslant h \leqslant r} s_h^{\gamma_h}.$$

Thus, given $\boldsymbol{m} = (m_1, \dots, m_t) \in \mathbb{N}^{*t}$, and writing $s_h = R_h(\boldsymbol{m})$ $(1 \leqslant h \leqslant r)$, we have

$$s_j' = Q_j(\boldsymbol{m}) \quad (1 \leqslant j \leqslant k), \qquad s'' = Q(\boldsymbol{m}).$$

With these pieces of notation, to any function F in $\mathcal{M}_k(A, B, \varepsilon)$ we may associate a further function $\widehat{F}$ in $\mathcal{M}_r(A^g, B, g\varepsilon)$ defined by

$$\widehat{F}(\boldsymbol{s}) = F(\boldsymbol{s}').$$

We then designate by $\widehat{G} := G_{\widehat{F}}$ the function associated to $\widehat{F}$ by (2·2) with $k = r$ and, for $1 \leqslant h \leqslant r$, we let $\widehat{G}_h : \mathbb{N}^* \to \mathbb{R}^+$ denote the composition of $\widehat{G}$ with the hth coordinate.

3. *Statement of results*

The key tool of this work consists in an estimate of the quantity

$$\begin{aligned} S &:= \sum_{x_j < n_j \leqslant x_j + y_j\ (1 \leqslant j \leqslant t)} F\big(|Q_1(\boldsymbol{n})|, \dots, |Q_k(\boldsymbol{n})|\big) \\ &= \sum_{x_j < n_j \leqslant x_j + y_j\ (1 \leqslant j \leqslant t)} \widehat{F}\big(R_1(\boldsymbol{n}), \dots, R_r(\boldsymbol{n})\big). \end{aligned}$$

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The following result is obtained. It is the t-dimensional counterpart of that of [13]. We let $\|Q\|$ denote the maximum of the absolute values of the coefficients of a real form Q.

THEOREM 3·1. *Let $k, t \in \mathbb{N}^*$, and let $\{Q_j\}_{j=1}^k \in \mathbb{Z}[X_1, \dots, X_t]^k$ be a family of primitive polynomials. Define $Q \in \mathbb{Z}[X_1, \dots, X_t]$, $\{R_h\}_{1\leqslant h\leqslant r} \in \mathbb{Z}[X_1, \dots, X_t]^r$, $g \in \mathbb{N}$ by (2·3), ϱ_Q^+ by (2·4), and $\varrho_{\boldsymbol{R}}^\#$ by (2·7). For all*

$$\alpha \in]0,1], \quad \beta \in]0,1[, \quad A \geqslant 1, \quad B \geqslant 1, \quad 0 \leqslant \varepsilon \leqslant \alpha\beta/\{50g^2(\beta g+1)\}, \tag{3·1}$$

and uniformly under the conditions

$$\begin{aligned} &F \in \mathcal{M}_k(A,B,\varepsilon), \quad \boldsymbol{x} = (x_1,\dots,x_t) \in \mathbb{N}^{*t}, \quad \boldsymbol{y} = (y_1,\dots,y_t) \in \mathbb{N}^{*t}, \\ &x := \min_{1\leqslant j\leqslant t} x_j \geqslant c\Big\{\max_{1\leqslant j\leqslant t} x_j + \|Q\|\Big\}^\beta, \quad x_j^\alpha \leqslant y_j \leqslant x_j \quad (1\leqslant j\leqslant t), \end{aligned} \tag{3·2}$$

we have

$$\sum_{\substack{\boldsymbol{n}\in\mathbb{N}^{*t}\\ x_j-y_j<n_j\leqslant x_j (1\leqslant j\leqslant t)}} F\big(|Q_1(\boldsymbol{n})|,\dots,|Q_k(\boldsymbol{n})|\big) \ll \wp\boldsymbol{y} E_{\boldsymbol{R}}(\mathfrak{s}\boldsymbol{x}) \prod_{g<p\leqslant x}\Big(1-\frac{\varrho_Q^+(p)}{p^t}\Big), \tag{3·3}$$

where c and the implicit constant depend at most upon g, α, β, A, B and where we have set

$$E_{\boldsymbol{R}}(v) := \sum_{\substack{\boldsymbol{s}\in\mathbb{N}^{*r}\\ \wp\boldsymbol{s}\leqslant v}} \widehat{F}(\boldsymbol{s}) \frac{\varrho_{\boldsymbol{R}}^\#(\boldsymbol{s})}{\mathcal{K}(\boldsymbol{s})^t} \qquad (v \geqslant 1). \tag{3·4}$$

Remarks.

(i) We have

$$\frac{\varrho_{\boldsymbol{R}}^\#(\boldsymbol{s})}{\mathcal{K}(\boldsymbol{s})^t} \leqslant \frac{\varrho_{\boldsymbol{R}}^+(\boldsymbol{s})}{(\wp\boldsymbol{s})^t} \qquad \big(\boldsymbol{s}=(s_1,\dots,s_r)\in\mathbb{N}^{*r}\big), \tag{3·5}$$

however the sum $E_{\boldsymbol{R}}(x)$ may turn out to be smaller than that which follows from this upper bound. For instance, if $k=1$, $t=1$, $Q(X)=Q_1(X)=X(X+\ell)$, we have $\varrho_{\boldsymbol{R}}^\#(p)=0$ as soon as $p\mid\ell$, while $\varrho_{\boldsymbol{R}}^+(p)=1$. As noted in [13] and [14], this can be significant when it comes to obtaining results uniform with respect to ℓ.

(ii) A weaker version of (3·3) for $k=1$ has recently been established [6, theorem 1·15], where, in definition (3·4), the left-hand side of (3·5) is replaced by its majorant. Note that the dependence of our bounds with respect to the coefficients of Q is effective.

(iii) In the case $t=1$, Theorem 3·1 is due to Henriot [13], [14], extending [26], then [18] and [19]. When the Q_j are binary forms in $\mathbb{Z}[X_1,X_2]$, Theorem 3·1 has been proved in [3] with a slightly different definition of $\varrho^\#$, refining and extending the estimates of [2].

(iv) When F is multiplicative and satisfies an assumption such that $F(\boldsymbol{n}) \geqslant \eta^{\Omega(\wp\boldsymbol{n})}$ for some $\eta>0$, it is possible to establish lower bounds for the the left-hand side of (3·3) of the same order of magnitude as the right-hand side. We refer to [13] for this aspect of the question.

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4. *Proof of Theorem 3·1*

Recall a consequence of the Schwartz-Zippel lemma—see [27, lemma 1], or [32, theorem 1].

PROPOSITION 4·1. *Let $t \geqslant 1$ and let $Q \in \mathbb{Z}[X_1, \dots, X_t]$ be a primitive polynomial of degree g. For all prime numbers p, we have*

$$\varrho_Q^+(p) \leqslant g p^{t-1}. \tag{4·1}$$

In particular, for all integers $\nu \geqslant 1$, we have $\varrho_Q^+(p^\nu) \leqslant g p^{\nu t-1} < p^{\nu t}$ provided $p > g$.

Let $c(Q)$ denote the content of the polynomial Q of t variables and let $v_p(n)$ denote the p-adic valuation of an integer n. The following lemma is an effective version of [6, lemma 2·8]—see also [20, lemma 4·10].

LEMMA 4·2. *Let $t \geqslant 1$ and let $Q \in \mathbb{Z}[X_1, \dots, X_t]$ be a polynomial of degree g. For all prime numbers p and all integers $\nu \geqslant 1$, we have*

$$\varrho_Q^+(p^\nu) \leqslant g^t (\nu+1)^{t-1} p^{\nu(t-1/g)+v_p(c(Q))/g}. \tag{4·2}$$

Proof. Consider first the case $t = 1$. When $(c(Q), p) = 1$, inequality (4·2) has been proved by Stewart [25]. If $\gamma = v_p(c(Q)) \geqslant 1$, we have

$$\varrho_Q^+(p^\nu) = p^\gamma \varrho_{Q^*}^+(p^{\nu-\gamma}) \leqslant g p^{\nu(1-1/g)} p^{v_p(c(Q))/g},$$

with $Q^* := Q/p^\gamma$. Thus (4·2) holds for $t = 1$.

We now proceed by induction on t. Let x_1 be such that Q depends on x_1 and write

$$Q(x_1, x_2, \dots, x_t) = \sum_{0 \leqslant j \leqslant g} Q_j(x_2, \dots, x_t) x_1^j$$

with $Q_j \in \mathbb{Z}[X_2, \dots, X_t]$ and $g_j := \deg Q_j \leqslant g - j$. By the case $t = 1$, we have

$$\varrho_Q^+(p^\nu) \leqslant g p^{\nu(1-1/g)} \sum_{0 \leqslant \gamma \leqslant \nu} p^{\gamma/g + (t-1)(\nu-\gamma)} N(\gamma, p),$$

with $N(\gamma, p) := |\{\boldsymbol{x} \in (\mathbb{Z}/p^\gamma\mathbb{Z})^{t-1} : Q_j(\boldsymbol{x}) \equiv 0 \pmod{p^\gamma}\ (0 \leqslant j \leqslant g)\}|$.

Now the induction hypothesis yields

$$\begin{aligned} N(\gamma, p) &\leqslant \min_{0 \leqslant j \leqslant g} |\{\boldsymbol{x} \in (\mathbb{Z}/p^\gamma\mathbb{Z})^{t-1} : Q_j(\boldsymbol{x}) \equiv 0 \pmod{p^\gamma}\}| \\ &\leqslant \min_{0 \leqslant j \leqslant g} g_j^{t-1} (\nu+1)^{t-2} p^{\gamma(t-1) + \min\{v_p(c(Q_j)) - \gamma, 0\}/g_j} \\ &\leqslant g^{t-1} (\nu+1)^{t-2} p^{\gamma(t-1-1/g) + v_p(c(Q))/g}, \end{aligned}$$

since $\min_{0 \leqslant j \leqslant g} v_p(c(Q_j)) = v_p(c(Q))$. This completes the proof of (4·2). □

Let us now state the analogue in our context of [13, lemma 6]. This is a straightforward consequence of the fundamental lemma of the combinatorial sieve. It coincides with [3, lemma 3·3] when $t = 2$ and with [13, lemma 6] when $t = 1$. As in [13, lemma 6], we let α be an arbitrary constant in]0,1] and define

$$\alpha_1 := \tfrac{3}{25}\alpha, \quad \alpha_2 := \varepsilon_1/6g. \tag{4·3}$$

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LEMMA 4·3. *Let $t \geqslant 1$, $0 < \alpha < 1$, and let $Q, R_h \in \mathbb{Z}[X_1, \dots, X_t]$ be polynomials satisfying the hypotheses of Theorem 3·1. Under the conditions*

$$\boldsymbol{a} \in \mathbb{N}^{*r},\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^{*t},\ w := \min_j x_j,\ \wp\boldsymbol{a} \leqslant w^{\alpha_1},\ z \leqslant w^{\alpha_2},\ x_j^{\alpha} \leqslant y_j \leqslant x_j,$$

we have

$$1 \ll \wp\boldsymbol{y} \frac{\varrho_{\boldsymbol{R}}^{\#}(\boldsymbol{a})}{\mathcal{K}(\boldsymbol{a})^t} \prod_{\substack{g < p \leqslant z \\ p \nmid \wp\boldsymbol{a}}} \Big(1 - \frac{\varrho_Q^{+}(p)}{p^r}\Big). \tag{4·4}$$
$$\sum_{\substack{x_j < n_j \leqslant x_j + y_j\ (1 \leqslant j \leqslant t) \\ a_h \| R_h(\boldsymbol{n})\ (1 \leqslant h \leqslant r) \\ (p \mid Q(\boldsymbol{n}),\ g < p \leqslant z) \Rightarrow p \mid \wp\boldsymbol{a} \\ (R_h(\boldsymbol{n})/a_h, \wp\boldsymbol{a}) = 1\ (1 \leqslant h \leqslant r)}}$$

Note that, since the product in (3·3) only covers primes $p > g$, it avoids any fixed prime divisor of the R_h. It is also worthwhile to keep in mind that

$$\frac{\varrho_{\boldsymbol{R}}^{\#}(\boldsymbol{a})}{\mathcal{K}(\boldsymbol{a})^t} = \operatorname{dens}\Big\{\boldsymbol{n} \in \mathbb{Z}^t : a_h \parallel R_h(\boldsymbol{n})\ (1 \leqslant h \leqslant r),\ (p \mid \wp\boldsymbol{a},\ p \nmid a_h) \Rightarrow (p \nmid R_h(\boldsymbol{n}))\Big\}. \tag{4·5}$$

Proof. In the above statement, the parameters α_1 and α_2 defined in (4·3) coincide respectively with ε_1 and ε_3 from [**13**, lemma 6]. Lemma 4·3 follows by applying the combinatorial sieve to the sequence

$$\mathcal{A} := \left\{ Q(n_1, \dots, n_t) : \begin{array}{l} x_j < n_j \leqslant x_j + y_j\ (1 \leqslant j \leqslant t),\ a_h \| R_h(\boldsymbol{n})\ (1 \leqslant h \leqslant r), \\ (p \mid \wp\boldsymbol{a}/a_h,\ p \nmid a_h) \Rightarrow (p \nmid R_h(n)) \end{array} \right\}.$$

Thus, we can select $A_1 = g + 1$ in the hypotheses of the sieve—see [**10**, lemma 2·2], condition (Ω_1)— since the upper bound (4·1) implies that

$$\frac{\varrho_Q^{+}(p)}{p^t} \leqslant \frac{g}{p} \leqslant 1 - \frac{1}{g+1} \quad \big(p \geqslant g + 1,\ p \nmid \wp\boldsymbol{a}\big).$$

Further details are omitted. □

Substituting Lemma 4·2 to [**13**, lemma 6] in the proof of [**13**, theorem 5] readily furnishes estimate (3·3). Note that Lemma 4·2 is crucial in order to get effective bounds in terms of $\|Q\|$.

5. *Proof of Theorem 1·1*

5·1. Statements

It follows from (1·6) that

$$V_2(x; \boldsymbol{c},\ \boldsymbol{\ell}) = \Big| \Big\{ (\boldsymbol{m}, \boldsymbol{n}) \in (\mathbb{N}^*)^{t+1} \times (\mathbb{N}^*)^{t+1} : \sum_{0 \leqslant j \leqslant t} c_j m_j^{\ell_j} = \sum_{0 \leqslant j \leqslant t} c_j n_j^{\ell_j} \leqslant x \Big\} \Big|,$$

and so $V_2(x) = V_2(x; \boldsymbol{c},\ \boldsymbol{\ell}) = V_2^{\neq}(x) + V_2^{=}(x)$, where the subsums $V_2^{\neq}(x)$ and $V_2^{=}(x)$ respectively correspond to the extra conditions $m_0 \neq n_0$ and $m_0 = n_0$. We obtain the following estimates.

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PROPOSITION 5·1. *Let $t \geqslant 1$, $\delta > 0$, $\boldsymbol{c}$, $\boldsymbol{\ell} \in (\mathbb{N}^*)^{t+1}$, satisfy $\ell_0 \geqslant 2$ and $\delta = \sum_{1 \leqslant j \leqslant t} 1/\ell_j$. Then*

$$V_2^{\neq}(x) \ll x^{2\delta} \mathfrak{S}(x) \ll x^{2\delta} (\log_2 x)^{5/2} \qquad (x \geqslant 3). \tag{5·1}$$

The second bound follows from (1·1). The exponent $5/2$ hence measures, in the current state of knowledge, the deviation of this general estimate from optimality. In [17, Section 8], Koukoulopoulos and Tao indicate the exponent $= 2^{4L} + 15/4$ for $L = \max \ell_j$, $\ell_0 = 2$, $\delta = 1/2$. The condition $m_0 \neq n_0$ has been omitted in [17]. Since it is necessary for a general result, the argument given in [17] only yields the bound [17, (1·5)] in the historical case $\boldsymbol{\ell} = (2, 4, 4)$ considered by Hooley. The exponent of $\log_2 x$ stated in [17] is then 65539.75.

Our approach to Theorem 1·1 is based on an inductive argument described in the following statement.

PROPOSITION 5·2. *Let $t \geqslant 2$, $\boldsymbol{c} = \{c_j\}_{j=0}^t \in (\mathbb{N}^*)^{t+1}$, $\boldsymbol{\ell} = \{\ell_j\}_{j=0}^t \in (\mathbb{N}^*)^{t+1}$ with $\ell_0 \geqslant 2$. Put $\check{\boldsymbol{c}} := (c_1, \dots, c_t)$, $\check{\boldsymbol{\ell}} = (\ell_1, \dots, \ell_t)$. We have*

$$V_2^{=}(x; \boldsymbol{c}, \boldsymbol{\ell}) \leqslant \Big(\frac{x}{c_0}\Big)^{1/\ell_0} V_2(x; \check{\boldsymbol{c}}, \check{\boldsymbol{\ell}}) \qquad (x \geqslant 3). \tag{5·2}$$

Moreover, if $t \in \{1, 2\}$ and $\ell_{t-1} = \ell_t \geqslant 3$, or if $t = 3$ and $\ell_{t-2} = \ell_{t-1} = \ell_t \geqslant 26$, then

$$V_2(x; \check{\boldsymbol{c}}, \check{\boldsymbol{\ell}}) \ll x^{t/\ell_t}. \tag{5·3}$$

Proof. The bound (5·2) is clear, and so is (5·3) for $t = 1$. The remaining bounds (5·3) are due to Salberger: for $t = 2$, see [23, cor. 0.7] ; for $t = 3$, $\ell_t \geqslant 33$, see [5] ; for $t = 3$, $\ell_t \geqslant 26$, see [22, cor. 4·6].

As mentioned in the introduction, we have been informed by Salberger that he intends to write up the proof that (5·2) holds for $t = 3$, $\ell_t \geqslant 16$, and hopefully $\ell_t \geqslant 12$.

5·2. A lemma from the literature

Define $P^+(n)$ as the largest prime factor of an integer n, with the convention that $P^+(1) := 1$. The following statement is established in [13, lemmas 2 and 5].

LEMMA 5·3. *Let $r, g \geqslant 1$, $A \geqslant 1$, $B \geqslant 1$, $\varepsilon > 0$ and let $\{\sigma_j\}_{j=1}^r$, $\{\vartheta_j\}_{j=1}^r$ be two families of multiplicative arithmetical functions. Assume that, uniformly for prime powers p^ν, we have $\sigma_j(p^\nu) \ll 1$, $\vartheta_j(p^\nu) = 1 + O(1/p)$. Let $\widehat{F} \in \mathcal{M}_r(A^g, B, g\varepsilon)$, and define*

$$H(\boldsymbol{s}) := \widehat{F}(\boldsymbol{s}) \prod_{1 \leqslant j \leqslant r} \sigma_j(s_j) \frac{\varrho_{\boldsymbol{R}}^{\#}(\boldsymbol{s})}{\mathcal{K}(\boldsymbol{s})^t} \qquad (\boldsymbol{s} \in \mathbb{N}^{*r}).$$

Then, for $x \geqslant 3$, we have

$$\sum_{\substack{\boldsymbol{s} \in \mathbb{N}^{*r} \\ \wp \boldsymbol{s} \leqslant x}} H(\boldsymbol{s}) \prod_{1 \leqslant j \leqslant r} \vartheta_j(s_j) \ll \sum_{\substack{\boldsymbol{s} \in \mathbb{N}^{*r} \\ \wp \boldsymbol{s} \leqslant x}} H(\boldsymbol{s}), \tag{5·4}$$

$$\sum_{\substack{\boldsymbol{s} \in \mathbb{N}^{*r} \\ P^+(\wp \boldsymbol{s}) \leqslant x}} H(\boldsymbol{s}) \ll \sum_{\substack{\boldsymbol{s} \in \mathbb{N}^{*r} \\ \wp \boldsymbol{s} \leqslant x}} H(\boldsymbol{s}). \tag{5·5}$$

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5·3. Proof of Proposition 5·1.

Consider

$$Q(X_1,\dots,X_t,Y_1,\dots,Y_t) := \sum_{1\leqslant j\leqslant t} c_j X_j^{\ell_j} - \sum_{1\leqslant j\leqslant t} c_j Y_j^{\ell_j}. \tag{5·6}$$

The following statement gathers the necessary estimates regarding function ϱ_Q defined in (2·4).

LEMMA 5·4. *Let $t\geqslant 1$, $\boldsymbol{c},\ \boldsymbol{\ell}\in\mathbb{N}^{*t}$ be such that*

$$L := \max_{1\leqslant j\leqslant t} \ell_j, \quad \gcd(c_1,\dots,c_t)=1.$$

For all prime numbers p and all integers $\nu\geqslant 1$ and Q defined by (5·6), we have

$$\varrho_Q(p)=p^{2t-1}+O(p^t), \qquad \varrho_Q(p^\nu)\leqslant Lp^{2t\nu-1}, \qquad \varrho_Q(p^\nu)\ll \nu p^{2(t-\delta)\nu}$$

with $\delta := \sum_{j=1}^t 1/\ell_j$.

Proof. The second estimate follows from Lemma 4·1. The first formula is included in the first estimate of [**21**, lemma 3·4], itself resting on an estimate of Korobov [**16**, chapter 1, theorem 5]. The third bound is a consequence of the second estimate in [**21**, lemma 3·4], resting on Korobov's [**16**, chapter. 1, theorem 6], observing that the proof given in [**21**] remains valid as it stands when $\delta<1/2$. □

We are now ready to complete the proof of Proposition 5·1. Put $L:=\max \ell_j$. Proposition 5·1 will follow from Theorem 3·1, applied with $k=1$, $Q_1=Q$, $F=\Delta$. Up to dividing through by $\gcd(c_1,\dots,c_t)$, the c_j may be assumed coprime. The polynomial Q is then irreducible in $\mathbb{Z}[X_1,\dots,X_t,Y_1,\dots,Y_t]$, so that $r=1$. Indeed, we may write $Q=c_1X_1^{\ell_1}+Q_1$, where $Q_1\in\mathbb{Z}[X_2,\dots X_t,Y_1,\dots,Y_t]$ has no square factor in $\mathbb{Q}[X_2,\dots,X_t,Y_1,\dots,Y_t]$. This enables applying Eisenstein's criterion associated to a non constant irreducible factor of Q_1 in $\mathbb{Z}[X_2,\dots,X_t,Y_1,\dots,Y_t]$.

Taking the above into account, Lemma 5·4 furnishes

$$\prod_{L<p\leqslant x}\Big(1-\frac{\varrho_Q^+(p)}{p^{2t}}\Big)\asymp \log x \qquad (x\geqslant 2L).$$

Moreover, the quantity $E_Q(v)$ defined in (3·4) satisfies

$$E_Q(v)\asymp \sum_{n\leqslant v}\frac{\Delta(n)}{n} \qquad (v\geqslant 1). \tag{5·7}$$

This follows from Lemmas 5·3 and 5·4. Indeed, since $k=r=1$, we have

$$E_Q(v)=\sum_{n\leqslant v}\frac{\Delta(n)\varrho_Q^{\#}(n)}{\mathcal{K}(n)^{2t}}\cdot \tag{5·8}$$

By Lemma 5·4, there exist two multiplicative functions ϑ_1 and ϑ_2 such that:

(i) $\vartheta_j(p^\nu)=1+O(1/p)$ $(j\in\{1,2\},\ \nu\geqslant 1)$,

(ii) $\varrho^{\#}(n)/\mathcal{K}(n)^{2t}=\varrho^{+}(n)\vartheta_1(n)/n^{2t}=\vartheta_2(n)/n$ $(n\geqslant 1)$.

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A first application of (5·4) provides the upper bound in (5·7). To establish the corresponding lower bound, we note that, by condition (i), we have $\vartheta_2(p) \neq 0$ for all but only a bounded number of primes p. Let $\vartheta_3(p^\nu) = \vartheta_2(p^\nu) + \mathbf{1}_{\{\vartheta_2(p^\nu)=0\}}(p^\nu)$. Applying (5·5) to replace condition $n \leqslant v$ by $p|n \Rightarrow p \leqslant v$, we hence see that substituting ϑ_3 to ϑ_2 in (5·8) does not alter the order of magnitude of the sum. A second application of (5·4) with $H(n) := \Delta(n)\vartheta_3(n)$ and $\vartheta = 1/\vartheta_3$ then furnishes the lower bound in (5·7).

We may now apply Theorem 3·1 in order to derive 5·1. Put

$$(\boldsymbol{m}, \boldsymbol{n}) = (m_1, \dots, m_t, n_1, \dots, n_t).$$

By symmetry, we may assume $n_0 > m_0$. Let us then split the variation range of $m_0 + n_0$ into dyadic intervals $]N,2N]$ with $N = 2^\nu \leqslant 2x^{1/\ell_0}$. Up to neglecting a contribution $\ll x^{2\delta}$, we may assume $N > x^{1/\ell_0 - 1/2\ell_0\ell_k}$. Indeed, if $N \leqslant x^{1/\ell_0-1/2\ell_0\ell_k}$, then for all fixed $(m_1, \dots, m_k)$, $(n_1, \dots, n_{k-1})$, the power $n_k^{\ell_k}$ lies in a fixed interval of length $\ll N^{\ell_0} \ll x^{1-1/2\ell_k}$. The number of admissible n_k is hence $\ll x^{1/\ell_k - 1/2\ell_k^2}$. However, for each fixed n_k, the number of admissible pairs (m_0, n_0) is $\ll x^\varepsilon$ for any $\varepsilon > 0$. This plainly implies that the contribution of those $N \leqslant x^{1/\ell_0-1/2\ell_0\ell_k}$ is $\ll x^{2\delta-1/2\ell_k^2+\varepsilon} \ll x^{2\delta}$ provided that $\varepsilon \leqslant 1/2\ell_k^2$.

If $(m_0, \boldsymbol{m}, n_0, \boldsymbol{n})$ is counted in $V_2^{\neq}(x)$, we have $Q(\boldsymbol{m}, \boldsymbol{n}) = c_0(n_0^{\ell_0} - m_0^{\ell_0})$, hence the divisor $n_0^{\ell_0-1} + m_0 n_0^{\ell_0-1} + \ldots + m_0^{\ell_0-1}$ of $Q(\boldsymbol{m}, \boldsymbol{n})$ is of size N^{ℓ_0-1}. As a consequence, for each $2t$-tuple $(\boldsymbol{m}, \boldsymbol{n})$, the number of admissible pairs (m_0, n_0) does not exceed $\Delta(Q(\boldsymbol{m}, \boldsymbol{n}))$. Now let us split the variation ranges of the $m_j^{\ell_j}$ and $n_h^{\ell_h}$ into intervals of size N^{ℓ_0}, say $m_j^{\ell_j} \in]r_j N^{\ell_0}, (r_j+1)N^{\ell_0}]$ and $n_h^{\ell_h} \in]s_h N^{\ell_0}, (s_h+1)N^{\ell_0}]$ with $r_j, s_h \ll x/N^{\ell_0}$. For each r_j, the integer m_j is hence restricted to an interval of length $\ll N^{\ell_0/\ell_j}/(r_j+1)^{(1-1/\ell_j)}$.

For each pair (r_j, s_j), Theorem 3·1 furnishes a contribution

$$\ll \prod_{1\leqslant j\leqslant t} \frac{N^{2\ell_0/\ell_j}}{(r_j+1)^{1-1/\ell_j}(1+s_j)^{1-1/\ell_j}} \frac{E_Q(x)}{\log x}. \tag{5·9}$$

Here, the hypothesis $N > x^{1/\ell_0-1/2\ell_0\ell_k}$ has been used in a crucial way since it implies that m_j lies in a range of size $\gg x^{1/\ell_j-1/2\ell_k} \gg x^{1/2\ell_k}$. This enables us to check hypotheses (3·2) of Theorem 3·1, which require that conditions (3·2) be satisfied by

$$x_j = (r_j N^{\ell_0})^{1/\ell_j}, \quad x_{j+k} = (s_j N^{\ell_0})^{1/\ell_j}, \quad y_j = \{(r_j+1)N^{\ell_0}\}^{1/\ell_j} - (r_j N^{\ell_0})^{1/\ell_j},$$
$$y_{j+k} = ((s_j+1)N^{\ell_0})^{1/\ell_j} - (s_j N^{\ell_0})^{1/\ell_j},$$

for $1 \leqslant j \leqslant k$. Thus, the choice $\alpha = 1/2\ell_k$, $\beta = 1/\ell_k - 1/2\ell_k^2$, is admissible in (3·2).

Now observe that relations $\left|m_j^{\ell_j} - n_j^{\ell_j} - (r_j - s_j)N^{\ell_0}\right| \leqslant N^{\ell_0}$ and

$$c_0|n_0^{\ell_0} - m_0^{\ell_0}| = \Big|\sum_{1\leqslant j\leqslant t} c_j(m_j^{\ell_j} - n_j^{\ell_j})\Big| \leqslant 2^{\ell_0} c_0 N^{\ell_0}$$

imply $\left|\sum_{1\leqslant j\leqslant t} c_j(r_j - s_j)\right| \leqslant 2^{\ell_0}c_0 + \sum_{1\leqslant j\leqslant t} c_j$. Therefore, when all r_j and all s_h are fixed except s_1, the number of admissible values for s_1 is bounded. Summing (5·9) over r_j and s_h, taking the condition $\sum_{1\leqslant j\leqslant t} 1/\ell_j = \delta$ into account then yields a contribution

$$\ll N^{\ell_0/\ell_1} x^{2\delta - 1/\ell_1} \frac{E_Q(x)}{\log x}.$$

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It remains to sum over ν such that $N^{\ell_0} = 2^{\ell_0\nu} \leqslant 2^{\ell_0}x$ to get, in view of (5·7),

$$V_2^{\neq}(x; \boldsymbol{c},\ \boldsymbol{\ell}) \ll x^{2\delta}\frac{E_Q(x)}{\log x} \ll x^{2\delta}\mathfrak{S}(x).$$

Note that the manipulation resting on the condition $m_0 + n_0 \in]N, 2N]$ saved a factor $\log_2 x$.

This completes the proof of Proposition 5·1.

5·4. Proof of Theorem 1·1.

Since $V_1(x) \asymp x$ (see [**21**, lemma 3·1] for further precisions) and taking (1·5) into account the required estimate (1·3) immediately results from a suitable bound for $V_2(x, \boldsymbol{c}, \boldsymbol{\ell})$. In view of (5·1), we write $\boldsymbol{c} = (c_1, \dots, c_t)$, $\boldsymbol{\ell} = (\ell_1, \dots, \ell_t)$.

For $1 \leqslant k \leqslant t$, put $\delta_k = \sum_{t-k\leqslant j\leqslant t} 1/\ell_j$, $\boldsymbol{c}_k = (c_{t-k}, \dots, c_t)$, $\boldsymbol{\ell}_k = (\ell_{t-k}, \dots, \ell_t)$. The parameter s being defined in the statement of Theorem 1·1 and $\boldsymbol{\ell}$ satisfying conditions (i), (ii), (iii) of this statement, we shall show by induction on $k \geqslant \max(1, s-1)$ that

$$V_2(x; \boldsymbol{c}_k, \boldsymbol{\ell}_k) \ll x^{\delta_k}\mathfrak{S}(x). \tag{5·10}$$

Note that hypothesis (iii) may be rewritten as $\delta_k \leqslant 1/\ell_{t-k-1}$ for $s-1 \leqslant k \leqslant t-1$.

For $k = s = 1$, $t \geqslant 2$, the stated bound (5·3) follows from [**21**, lemma 3·11]. The case $k = 1$, $s = 2$, $t \geqslant 2$, follows from (5·3). For $k = 2$, $t \geqslant 3$, $s = 3$, we only need to bound the number of solutions $(\boldsymbol{x}, \boldsymbol{y})$ such that $x_j, y_j \leqslant x^{1/\ell}$ and

$$x_1^\ell + x_2^\ell + x_3^\ell = y_1^\ell + y_2^\ell + y_3^\ell$$

with $\ell = \ell_t$. The required estimate then follows from Proposition 5·2. The induction is thus initialised. In these cases, the factor $\mathfrak{S}(x)$ is actually superfluous. Let $k \geqslant s-1$ and assume (5·14) holds for k. We want to prove that it persists for $k+1$. By Propositions 5·2 and 5·1, we have

$$V_2(x; \boldsymbol{c}_{k+1},\ \boldsymbol{\ell}_{k+1}) \ll x^{2\delta_k}\mathfrak{S}(x) + x^{1/\ell_{t-k-1}}V_2(x; \boldsymbol{c}_k, \boldsymbol{\ell}_k) \ll (x^{2\delta_k} + x^{1/\ell_{t-k-1}+\delta_k})\mathfrak{S}(x).$$

This provides the desired bound since $\delta_k \leqslant 1/\ell_{t-k-1}$, and $1/\ell_{t-k-1} + \delta_k = \delta_{k+1}$.

The induction step is thus established, and so we get the result for $k = t+1$. This completes the proof of Theorem 1·1 since $\delta_{t+1} = 1$.

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