Math. Struct. in Comp. Science (2018), *vol.* 28, *pp.* 1770–1785. © Cambridge University Press 2017
doi:10.1017/S0960129517000238 First published online 30 October 2017

Essential and density topologies on s_2-continuous posets

CHONGXIA LU and QINGGUO LI

College of Mathematics and Econometrics, Hunan University, Changsha, P.R. China
Emails: lcx19882016@aliyun.com, liqingguoli@aliyun.com

Received 8 December 2016; revised 18 July 2017

Recently, Rusu and Ciobanu established that for a continuous domain L, a subset B of L is a basis if and only if B is dense with respect to the d-topology, called the density topology, on L. In situations where directed completeness fails, Erné has proposed in 1991 an alternative definition of continuity called s_2-continuity which remedied the lack of stability of continuity under the classical Dedekind–MacNeille completion. In this paper, we show how the 'Rusu–Ciobanu' type of characterization can be formulated and established over the class of s_2-continuous posets with appropriate modifications. Although we obtain more properties of essential topologies and density topologies on s_2-continuous posets, respectively.

1. Introduction

Dana Scott introduced domain theory in the late 60s, it served as models for the denotational semantics of programming languages. Order structure and topological structure are two important mathematical structures in domain theory, it is a combination of general topology and order theory which are of important function. So some intrinsic topologies play crucial roles in the study of domains as well as general posets.

A basis B of a domain L is a dense subset in L endowed with the Scott topology but a Scott-dense subset need not be basis for L, even when L is a domain. A question naturally arises, namely whether there is a topology on domain L such that its bases are precisely dense subsets in this topology. In 2016, Rusu and Ciobanu provided an answer to this question Rusu and Ciobanu (2016). They introduced an intermediate topology over a dcpo that they called the e-topology, which consists of what they called e-open sets containing the essential parts of their elements. The smallest common refinement of the Scott topology and the e-topology on a dcpo L is called the density topology. The density topology is Hausdorff and finer than the Lawson topology. They proved that in a domain L, a subset is a basis if and only if it is a dense set in the density topology. Naturally, we would like to know whether this result holds in a continuous poset. In the poset, the directed completeness may fails. So Erné proposed in 1991 an alternative definition of continuity called s_2-continuity which remedied the lack of stability of continuity under the classical Dedekind–MacNeille completion (Erné 1991). It is easy to know that the s_2-continuous poset is always the continuous poset and the two continuities are coincident on dcpos.

In this paper, we show how the 'Rusu–Ciobanu' type of characterization can be formulated and established over the class of s_2-continuous posets with appropriate

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

modifications. We define an intermediate topology over a poset that we call the essential topology (for short, ε-topology). The smallest common refinement of σ_2-topology and the ε-topology on a poset L is called the density topology (for short, δ-topology). We prove that a poset with essential topology is a c-space, and a poset with density topology is a pospace and totally order-disconnected space. We also give some necessary conditions and sufficient conditions for a poset L with ε-topology $\varepsilon(L)$ to be compact or a Lindelöf space, respectively. Finally, we obtain the result that for two posets L and M, $\varepsilon(L) \times \varepsilon(M)$ and $\delta(L) \times \delta(M)$ are strictly coarser than $\varepsilon(L \times M)$ and $\delta(L \times M)$ in general, respectively, and a subset B of a s_2-continuous poset L is a basis iff B is δ-dense in L.

2. Preliminaries

We refer to Abramsky and Jung (1994) and Gierz et al. (2003) for the standard definitions and notations of order theory and domain theory, and to Engelking (1977) and Goubault-Larrecq (2013) for topology.

For a non-empty set X, we denote by $\mathscr{P}(X)$, $\mathscr{P}_{\mathrm{con}}(X)$ and $\mathscr{P}_{\mathrm{fin}}(X)$ the sets of all subsets, all non-empty countable subsets and all non-empty finite subsets of X, respectively. Let $(L, \leqslant)$ be a poset. For all $x \in L$, $A \subseteq L$, let $\uparrow x = \{y \in L \mid x \leqslant y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. A subset A of X is an upper set if $\uparrow A = A$, and a lower set if $\downarrow A = A$. A^u and A^l denote the sets of all upper and lower bounds of A defined by

$$A^u = \{x \in L \mid a \leqslant x \text{ for all } a \in A\} \quad \text{and} \quad A^l = \{x \in L \mid x \leqslant a \text{ for all } a \in A\},$$

respectively. Let $A^\delta = (A^u)^l$ be the cut of A. Furthermore, A^u is an upper set and A^l is a lower set.

The σ_2-topology and λ_2-topology on L are denoted $\sigma_2(L)$ and $\lambda_2(L)$, respectively (Zhang and Xu 2015). The interior of a subset A of L with respect to σ_2-topology and λ_2-topology on L are denoted $\mathrm{int}_{\sigma_2}(A)$ and $\mathrm{int}_{\lambda_2}(A)$, respectively.

Definition 2.1 (Zhang and Xu 2015). Let L be a poset. For any two elements x and y in L, we write $x \ll_2 y$, if for each directed subset $D \subseteq L$, $y \in D^\delta$ implies $x \leqslant d$ for some $d \in D$. If $x \ll_2 x$, then we called that x is a compact element of L. The set of compact elements of L is denoted by $K(L)$.

For $x \in L$, we denote by $\Uparrow x$ the set $\{y \in L \mid x \ll_2 y\}$ and by $\Downarrow x$ the set $\{y \in L \mid y \ll_2 x\}$. Then the set $\Downarrow A$ is equal to $\bigcup_{x \in A} \Downarrow x$, and $\Uparrow A$ is equal to $\bigcup_{x \in A} \Uparrow x$.

Proposition 2.1. Let L be a poset.

i. $x \ll_2 y$ implies $x \leqslant y$.
ii. $u \leqslant x \ll_2 y \leqslant v$ implies $u \ll_2 v$.
iii. If a least element $0 \in L$ exists, then $0 \ll_2 x$ for all $x \in L$.

Definition 2.2 (Zhang and Xu 2015). Let L be a poset.

i. L is called *s_2-continuous* if for all $x \in L$, the set $\Downarrow x$ is directed and $x = \sup \Downarrow x$.
ii. L is called *s_2-algebraic* if for all $x \in L$, the set $K(L) \cap \Downarrow x$ is directed and $x = \sup(K(L) \cap \Downarrow x)$.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Remark 2.1. For a dcpo, the relation $\ll_2$ coincides with the original definition of way-below, $\ll$; and hence the coincidence of the concepts of s_2-continuity and continuity.

Let L be a poset and $A, B \in \mathscr{P}(L)$. The set B is called a *cofinal subset* (Gierz et al. 2003) of A if $B \subseteq A$ and for any $x \in A$ there exists $y \in B$ such that $x \leqslant y$. Then for any cofinal subset B of A it holds that

i. B is directed iff A is directed;
ii. $A^u = B^u$;
iii. $\sup B$ exists iff $\sup A$ exists, and if these conditions are satisfied, then $\sup B = \sup A$.

Definition 2.3 (Zhang and Xu 2015). A subset B of a poset L is a *basis* for L if for every element x of L, the set $B \cap \downarrow x$ contains a directed subset D with $x = \sup D$.

Remark 2.2. Let L be a poset.

i. L is s_2-continuous iff it has a basis.
ii. L is s_2-algebraic iff the set $K(L)$ is a basis for L.
iii. If B is a basis of L, then $K(L) \subseteq B$.

Definition 2.4 (Goubault-Larrecq 2013). A topological space (X, τ) is called *c-space* if for every $x \in X$ and every open neighbourhood U of x, there exists a point $y \in U$ and an open subset $V \in \tau$ such that $x \in V \subseteq \mathrm{sat}\{y\} \subseteq U$, where $\mathrm{sat}\{y\} = \bigcap\{U \in \tau \mid y \in U\}$.

Definition 2.5 (Engelking 1977). A topological space (X, τ) is a *Lindelöf space* if for every open cover of X has a countable subcover.

Definition 2.6 (Gierz et al. 2003). Let X be a topological space. A partial order $\leqslant$ is said to be *closed* or have a *closed graph* if the relation $\leqslant$ is a closed subset of $X \times X$ in the product topology. In that case, $(X, \leqslant)$ is called a *pospace*.

Definition 2.7 (Davey and Priestley 2002). Let $(X, \leqslant)$ be a poset with a topology. It is called *totally order disconnected* if for any $x, y \in L$ with $x \not\leqslant y$, there exists a clopen lower set U such that $y \in U$ and $x \notin U$. We call a compact totally order-disconnected space a *Priestly space*.

3. ε-topology on s_2-continuous posets

In this section, we will introduce a kind of topology on a poset L:

Definition 3.1. A subset A of a poset L is called ε-open if $\downarrow A \subseteq A$; A is called ε-closed if $L \backslash A$ is ε-open.

Proposition 3.1. Let L be a poset and $A \subseteq L$. Then the following are equivalent:

i. A is ε-closed.
ii. $\Uparrow A \subseteq A$.

Proof. Suppose that A is ε-closed and assume that $\Uparrow A \not\subseteq A$. Then there exists $x \in \Uparrow A$ such that $x \notin A$, thus there is $y \in A$ such that $x \in \Uparrow y \backslash A$. Therefore, $y \in \downarrow x \subseteq \downarrow (L \backslash A) \subseteq L \backslash A$, this is a contradiction. For the converse, we can prove similarly. □

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Proposition 3.2. Let L be a poset and $A \in \mathscr{P}(L)$.

i. If $A = \uparrow A$, then A is ε-closed.
ii. If $A = \downarrow A$, then A is ε-open.

Proof.

i. If $A = \uparrow A$, then $\Uparrow A = \bigcup_{x \in A} \Uparrow x \subseteq \bigcup_{x \in A} \uparrow x = \uparrow A = A$. Thus, A is ε-closed.
ii. Similar to the proof of (i), we can prove that A is ε-open whenever $A = \downarrow A$. □

Proposition 3.3. The family $\varepsilon(L)$ of all ε-open sets on a poset L is a topology, called the essential topology (ε-topology).

Proof.

i. It is obvious that $\varnothing, L \in \varepsilon(L)$ by the definition of the operator $\Downarrow$.
ii. For any $\varnothing \neq \{A_i\}_{i \in I} \subseteq \varepsilon(L)$, we have $\Downarrow(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I}(\bigcup_{x \in A_i} \Downarrow x) = \bigcap_{i \in I} \Downarrow A_i \subseteq \bigcap_{i \in I} A_i$. Thus, $\bigcap_{i \in I} A_i \in \varepsilon(L)$.
iii. For any $\varnothing \neq \{A_i\}_{i \in I} \subseteq \varepsilon(L)$, we have $\Downarrow(\bigcup_{i \in I} A_i) = \bigcup_{i \in I}(\bigcup_{x \in A_i} \Downarrow x) = \bigcup_{i \in I} \Downarrow A_i \subseteq \bigcup_{i \in I} A_i$. Thus, $\bigcup_{i \in I} A_i \in \varepsilon(L)$.
Hence, $\varepsilon(L)$ is a topology. □

Remark 3.1. Let L be a poset. Then

i. $\varepsilon(L)$ is an Alexandrov topology since the intersection of any family of open sets is open.
ii. $(\varepsilon(L), \subseteq)$ is an algebraic lattice since $(L, \varepsilon(L))$ is a topped algebraic $\bigcap$-structure (Davey and Priestley 2002).

From now on, all the topological properties involving the ε-topology will be denoted by using the prefix 'ε.' A set is called ε-clopen if it is ε-open and ε-closed simultaneously. We denote by $\Gamma_\varepsilon(L)$ the family of all ε-closed sets on a poset L, and remark that $\sigma_2(L) \subseteq \Gamma_\varepsilon(L)$.

For a poset L, we denote by $\mathrm{int}_\varepsilon(A), \mathrm{cl}_\varepsilon(A)$ and $\mathrm{der}_\varepsilon(A)$, the interior, closure and derivative of a subset A of L in the ε-topology, respectively. For a poset L, every $x \in L$ has its smallest ε-open neighbourhood $U_x = \{x\} \cup \Downarrow x$ and its ε-closure $\mathrm{cl}_\varepsilon(\{x\}) = \{x\} \cup \Uparrow x$. Also, the family $\{\{x\} \cup \Downarrow x \mid x \in L\}$ is a basis for $\varepsilon(L)$ and $(L, \varepsilon(L))$ is T_0. If $\sqsubseteq_\varepsilon$ is the specialization order induced by the ε-topology (i.e., $x \sqsubseteq_\varepsilon y$ iff $x \in \mathrm{cl}_\varepsilon(\{y\})$), then we have $x \sqsubseteq_\varepsilon y$ iff $x = y$ or $x \in \Uparrow y$.

Definition 3.2. An arbitrary non-empty subset A of a poset P is ε-irreducible if $A \subseteq B \cup C$ for ε-closed subsets B and C implies $A \subseteq B$ or $A \subseteq C$. The family of all ε-irreducible subsets and the family of all ε-irreducible ε-closed subsets are denoted by $\mathrm{Irr}(\mathscr{P}(L))$ and $\mathrm{Irr}(\Gamma_\varepsilon(L))$, respectively.

Generally, the ε-topology is neither sober nor T_1. The next example illustrates that a poset with the ε-topology is not sober.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Example 3.1. Consider the poset $L = (0, 1)$ with the usual order of real numbers. For any $x \in L$, $\Uparrow x = \uparrow x \backslash \{x\} \in \Gamma_\varepsilon(L)$. Assume that there exists $x \in L$ such that $\Uparrow x \notin \mathrm{Irr}(\Gamma_\varepsilon(L))$. Then there are $y, z \in \Uparrow x$ and $B, C \in \Gamma_\varepsilon(L)$ such that $y \in B \backslash C$, $z \in C \backslash B$ and $\Uparrow x \subseteq B \cup C$. Since L is a chain, we have that $y < z$ or $z < y$. If $y < z$, then $y \ll_2 z$, i.e., $z \in \Uparrow y \subseteq \Uparrow B \subseteq B$. Similarly, we can get that $y \in \Uparrow z \subseteq \Uparrow C \subseteq C$ in another case. This is a contradiction. Thus, $\Uparrow x \in \mathrm{Irr}(\Gamma_\varepsilon(L))$ for all $x \in L$. But $\Uparrow x \notin \mathscr{E} = \{\mathrm{cl}_\varepsilon(\{y\}) \mid y \in L\}$ for all $x \in L$. Hence, $(L, \varepsilon(L))$ is not sober.

The following theorem gives a necessary and sufficient condition for the sobriety of ε-topology.

Theorem 3.1. Let L be a poset. The following statements are equivalent:

i. $(L, \varepsilon(L))$ is sober.
ii. For each $A \in \mathrm{Irr}(\Gamma_\varepsilon(L))$, every filtered subset $D \subseteq A$ has a lower bound in A.

Proof. Suppose that $A \in \mathrm{Irr}(\Gamma_\varepsilon(L))$ in which for every filtered subset $D \subseteq A$ has a lower bound in A. For any $x \in A$, by the Hausdorff Maximality Principle, there exists a maximal chain $C \subseteq A$ such that $x \in C$. By supposition C has a smallest element $x_0 \in C$ and $x_0 \notin \mathrm{cl}_\varepsilon\{y\}$ for all $y \in A \backslash \{x_0\}$. Since

$$A = \bigcup\nolimits_{x \in A} \mathrm{cl}_\varepsilon\{x\} = \mathrm{cl}_\varepsilon\{x_0\} \cup \left(\bigcup\nolimits_{y \in A \backslash \{x_0\}} \mathrm{cl}_\varepsilon\{y\} \right),$$

and $\bigcup_{y \in A \backslash \{x_0\}} \mathrm{cl}_\varepsilon\{y\} \in \Gamma_\varepsilon(L)$, we have that $A = \mathrm{cl}_\varepsilon\{x_0\}$. Because $(L, \varepsilon(L))$ is T_0, we have that x_0 is unique, and x_0 is a smallest element of A. Hence, $(L, \varepsilon(L))$ is sober.

Conversely, suppose $(L, \varepsilon(L))$ is sober and $A \in \mathrm{Irr}(\Gamma_\varepsilon(L))$. Then there exists unique $x_0 \in L$ such that $A = \mathrm{cl}_\varepsilon\{x_0\}$. Thus, $x_0 \in A = \{x_0\} \cup \Uparrow x_0 \subseteq \uparrow x_0$, which implies that x_0 is a smallest element of A. Therefore, x_0 is a lower bound of the filtered subset D of A. □

Remark 3.2. Let L be a poset. Then $(L, \varepsilon(L))$ is sober iff A has a greatest element with respect to specialization order for all $A \in \mathrm{Irr}(\Gamma_\varepsilon(L))$.

Proposition 3.4. Let L be a s_2-continuous poset. Then $(L, \varepsilon(L))$ is T_1 iff $x \leqslant y$ implies $x = y$ for all $x, y \in L$.

Proof. $(L, \varepsilon(L))$ is T_1 iff for every $x \in L$, $\mathrm{cl}_\varepsilon\{x\} = \{x\} \cup \Uparrow x = \{x\}$, iff for every $x \in L$, $\Uparrow x \subseteq \{x\}$. Suppose that $x \leqslant y$ implies $x = y$ for all $x, y \in L$. Then for each $x \in L$, $\Uparrow x \subseteq \uparrow x = \{x\}$. Conversely, suppose that $(L, \varepsilon(L))$ is T_1, and let $x \in L$. For $y \in \uparrow x$, $\downarrow\!\!\downarrow y \neq \varnothing$ since L is s_2-continuous. Thus, there exists $z \in \downarrow\!\!\downarrow y$, i.e., $y \in \Uparrow z \subseteq \{z\}$. Therefore, $y = z$, i.e., $y \in \Uparrow y \subseteq \Uparrow x \subseteq \{x\}$. □

Proposition 3.5. Let L be a poset. Then $(L, \varepsilon(L))$ is a c-space. Therefore, $(\varepsilon(L), \subseteq)$ is completely distributive.

Proof. For every $x \in L$, x has a smallest ε-open neighbourhood $\{x\} \cup \downarrow\!\!\downarrow x$. Thus, for any ε-open neighbourhood U of x, we have $x \in \{x\} \cup \downarrow\!\!\downarrow x \subseteq U$ and $\mathrm{sat}\{x\} = \{x\} \cup \downarrow\!\!\downarrow x$. Hence, $(L, \varepsilon(L))$ is a c-space.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

It was proved in Banaschewski (1977), Erné (1991) and Hoffmann (1981) that a T_0 space X is a c-space if and only if the lattice $(\mathcal{O}(X), \subseteq)$ is completely distributive. Thus, $(\varepsilon(L), \subseteq)$ is completely distributive. □

For a poset L, we denote by $\mathrm{Min}(L)$ the set of all minimal elements of L.

Lemma 3.1. A poset L is σ_2-compact iff L is finitely generated, i.e., $\mathrm{Min}(L) \in \mathscr{P}_{\mathrm{fin}}(L)$ and every element is above some minimal element of L.

Proof. Suppose that L is finitely generated. Since every finite subset is σ_2-compact, we have that $\mathrm{Min}(L)$ is σ_2-compact. Because the saturation of every σ_2-compact subset is also σ_2-compact and the saturation of $\mathrm{Min}(L)$ is L, we know that L is σ_2-compact.

Conversely, by the Hausdorff Maximality Principle, for any $x \in L$, there exists a maximality chain $C \subseteq L$ such that $x \in C$. Since L is σ_2-compact, we have that $\bigcap_{c\in C}{\downarrow}c \neq \varnothing$ and $\bigcap_{c\in C}{\downarrow}c$ is a single point set. Thus, $\mathrm{Min}(L) \neq \varnothing$ and every element is above some minimal element of L. Since the family $\{\mathrm{Min}(L)\backslash F \mid F \in \mathscr{P}_{\mathrm{fin}}(\mathrm{Min}(L))\}$ is filtered of σ_2-closed sets with an empty intersection, there is $F \in \mathscr{P}_{\mathrm{fin}}(\mathrm{Min}(L))$ such that $\mathrm{Min}(L)\backslash F = \varnothing$, i.e., $\mathrm{Min}(L) = F$ by the σ_2-compactness of L. □

Proposition 3.6. Let L be a poset. If L^{op} is not finitely generated, then $(L, \varepsilon(L))$ is not compact.

Proof. Assume that $(L, \varepsilon(L))$ is compact. By Proposition 3.2(2), we have that $\sigma_2(L^{\mathrm{op}}) \subseteq \varepsilon(L)$. Thus, $(L, \sigma_2(L^{\mathrm{op}}))$ is compact. Therefore, by Lemma 3.1, we know that L^{op} is finitely generated, this is a contradiction. □

The next example illustrates that the dual of a poset L is finitely generated but $(L, \varepsilon(L))$ is not actually compact.

Example 3.2. Consider the poset $L = \{(m, n) \mid m \in \{0, 1, 2, 3\} \text{ and } n \in \mathbb{N}\} \cup \{\omega_0, \omega_1\}$ with ordering defined by $(m_1, n_1) \leqslant (m_2, n_2)$ iff $m_1 = m_2$ and $n_1 \leqslant n_2$ for all $m_1, m_2 \in \{0, 1, 2, 3\}$, $n_1, n_2 \in \mathbb{N}$ and $(m, n) \leqslant \omega_0$ for all $m \in \{0, 1, 2\}$, $n \in \mathbb{N}$, as well as $(m, n) \leqslant \omega_1$ for all $m \in \{1, 2, 3\}$, $n \in \mathbb{N}$ (see Figure 1). It is easy to prove that

i. $\Downarrow(0, n) = {\downarrow}(0, n)$, for all $n \in \mathbb{N}$;
ii. $\Downarrow(3, n) = {\downarrow}(3, n)$, for all $n \in \mathbb{N}$;
iii. $\Downarrow(1, n) = \Downarrow(2, n) = \varnothing$, for all $n \in \mathbb{N}$;
(iv) $\Downarrow\omega_0 = \bigcup_{n\in\mathbb{N}}{\downarrow}(0, n)$ and $\Downarrow\omega_1 = \bigcup_{n\in\mathbb{N}}{\downarrow}(3, n)$.
Hence, $(L, \varepsilon(L))$ is not compact.

Proposition 3.7. Let L be a poset. If L^{op} is finitely generated and $\mathrm{Min}(L^{\mathrm{op}}) \subseteq K(L)$, then $(L, \varepsilon(L))$ is compact.

Proof. For $x \in \mathrm{Min}(L^{\mathrm{op}})$, $\{x\} \cup \Downarrow x = {\downarrow}x$ and ${\downarrow}x$ is also compact with respect to ε-topology, because $\{x\} \cup \Downarrow x$ is a compact saturated subset of $(L, \varepsilon(L))$. Since $L = \bigcup\{{\downarrow}x \mid x \in \mathrm{Min}(L^{\mathrm{op}})\}$ and $\mathrm{Min}(L^{\mathrm{op}}) \in \mathscr{P}_{\mathrm{fin}}(L)$, we have that $(L, \varepsilon(L))$ is compact. □

The next example illustrates that the dual of a poset L is finitely generated and $\mathrm{Min}(L^{\mathrm{op}}) \nsubseteq K(L)$ but $(L, \varepsilon(L))$ is compact.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

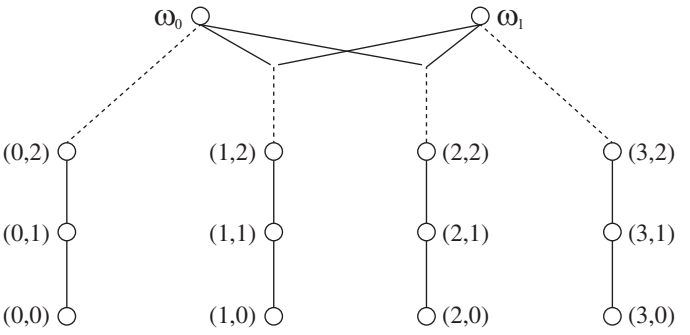


Fig. 1. The figure in Example 3.2.

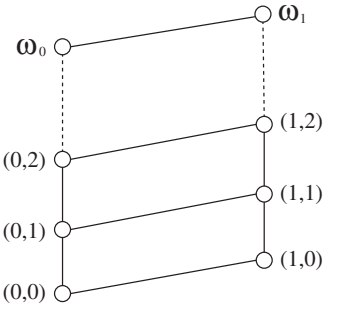


Fig. 2. The figure in Example 3.3.

Example 3.3. Consider the poset $L = \{(m,n) \mid m \in \{0,1\} \text{ and } n \in \mathbb{N}\} \cup \{\omega_0, \omega_1\}$ with ordering defined by $(m_1,n_1) \leqslant (m_2,n_2)$ iff $m_1 \leqslant m_2$ and $n_1 \leqslant n_2$ for all $m_1, m_2 \in \{0,1\}$, $n_1, n_2 \in \mathbb{N}$, and $(0,n) \leqslant \omega_0 \leqslant \omega_1$ for all $n \in \mathbb{N}$, as well as $(1,n) \leqslant \omega_1$ for all $n \in \mathbb{N}$ (see Figure 2). It is easy to prove that $\downarrow\!\!\downarrow \omega_1 = L \backslash \{\omega_0, \omega_1\}$. Hence, $(L, \varepsilon(L))$ is compact.

Proposition 3.8. Let L be a poset and $M = \text{Min}(L^{\text{op}}) \cup \bigcup\{\downarrow\!\!\downarrow x \mid x \in \text{Min}(L^{\text{op}})\}$. If L^{op} is finitely generated, then $L \backslash M$ is compact with respect to ε-topology if and only if $(L, \varepsilon(L))$ is compact.

Proof. Suppose that $L \backslash M$ is compact with respect to ε-topology. Since $\{x\} \cup \downarrow\!\!\downarrow x$ is a compact saturated subset of $(L, \varepsilon(L))$ for all $x \in L$ and $L = L \backslash M \cup M = L \backslash M \cup \text{Min}(L^{\text{op}}) \cup \bigcup\{\downarrow\!\!\downarrow x \mid x \in \text{Min}(L^{\text{op}})\} = L \backslash M \cup \bigcup\{\{x\} \cup \downarrow\!\!\downarrow x \mid x \in \text{Min}(L^{\text{op}})\}$, we have that $(L, \varepsilon(L))$ is compact.

Conversely, since $x \cup \downarrow\!\!\downarrow x \in \varepsilon(L)$ for all $x \in L$, we have that $M = \text{Min}(L^{\text{op}}) \cup \bigcup\{\downarrow\!\!\downarrow x \mid x \in \text{Min}(L^{\text{op}})\} \in \varepsilon(L)$. Thus, $L \backslash M$ is ε-closed. Because $(L, \varepsilon(L))$ is compact, we know that $L \backslash M$ is compact with respect to ε-topology. □

Corollary 3.1. Let L be a poset and $M = \text{Min}(L^{\text{op}}) \cup \bigcup\{\downarrow\!\!\downarrow x \mid x \in \text{Min}(L^{\text{op}})\}$. If L^{op} is finitely generated and $L \backslash M \in \mathscr{P}_{\text{fin}}(L)$, then $(L, \varepsilon(L))$ is compact.

Proof. Since every finite subset is always compact with respect to arbitrary topology on L, this result holds by Proposition 3.8. □

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Lemma 3.2. Let L be a poset with $L = \uparrow\mathrm{Min}(L)$. Then $(L, \sigma_2(L))$ is a Lindelöf space iff L is countably generated, i.e., $\mathrm{Min}(L) \in \mathscr{P}_{\mathrm{con}}(L)$ and every element is above some minimal element of L.

Proof. Suppose that L is countably generated. For any σ_2-open cover $\{U_i\}_{i\in I} \subseteq \sigma_2(L)$ of L, there exists a countable subset $J \subseteq I$ such that $\mathrm{Min}(L) \subseteq \bigcup_{i\in J} U_i$, which implies $L = \uparrow\mathrm{Min}(L) \subseteq \bigcup_{i\in J} U_i$. Thus, $(L, \sigma_2(L))$ is a Lindelöf space.

Conversely, since $\mathrm{Min}(L) \neq \varnothing$ and every element is above some minimal element of L, and $\{\mathrm{Min}(L)\backslash\{x\} \mid x \in \mathrm{Min}(L)\}$ is the family of σ_2-closed sets with an empty intersection, there is a $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathrm{Min}(L)$ such that $\mathrm{Min}(L)\backslash\{x_i\}_{i\in\mathbb{N}} = \varnothing$, i.e., $\mathrm{Min}(L) = \{x_i\}_{i\in\mathbb{N}}$ because $(L, \sigma_2(L))$ is a Lindelöf space. □

Corollary 3.2. Let L be a poset and L^{op} a dcpo. Then $(L, \sigma_2(L))$ is a Lindelöf space iff L is countably generated.

Proof. By the Zorn's Lemma, we know that $\mathrm{Min}(L) \neq \varnothing$ and every element is above some minimal element of L. Thus, $L = \uparrow\mathrm{Min}(L)$. □

Proposition 3.9. Let L be a poset with $L = \downarrow\mathrm{Min}(L^{\mathrm{op}})$. If L^{op} is not countably generated, then $(L, \varepsilon(L))$ is not a Lindelöf space.

Proof. Assume that $(L, \varepsilon(L))$ is a Lindelöf space. Since $\sigma_2(L^{\mathrm{op}}) \subseteq \varepsilon(L)$, we have that $(L, \sigma_2(L^{\mathrm{op}}))$ is a Lindelöf space. Thus, by Lemma 3.2, we know that L^{op} is countably generated, this is a contradiction. □

The next example illustrates that the dual of a poset L is countably generated but $(L, \varepsilon(L))$ is not a Lindelöf space.

Example 3.4. Consider the poset $L = \{(m, n) \mid m \in \{0, 1\} \text{ and } n \in [0, 1)\} \cup \{\omega\}$ with ordering defined by $(m_1, n_1) \leqslant (m_2, n_2)$ iff $m_1 = m_2$ and $n_1 \leqslant n_2$ for all $m_1, m_2 \in \{0, 1\}$, $n_1, n_2 \in [0, 1)$, and $(m, n) \leqslant \omega$ for all $m \in \{0, 1\}$ and $n \in [0, 1)$. It is easy to prove that $\downarrow\!\!\!\downarrow\omega = \downarrow\!\!\!\downarrow(m, n) = \varnothing$ for all $m \in \{0, 1\}$ and $n \in [0, 1)$. Thus, $\{x\} \in \varepsilon(L)$ for all $x \in L$. Hence, $(L, \varepsilon(L))$ is not a Lindelöf space since L is uncountable set.

Proposition 3.10. Let L be a poset. If L^{op} is countably generated and $\mathrm{Min}(L^{\mathrm{op}}) \subseteq K(L)$, then $(L, \varepsilon(L))$ is a Lindelöf space.

Proof. For $x \in \mathrm{Min}(L^{\mathrm{op}})$, $\{x\} \cup \downarrow\!\!\!\downarrow x = \downarrow x$. Thus, $\downarrow x$ is a smallest ε-open neighbourhood of x for all $x \in \mathrm{Min}(L^{\mathrm{op}})$. Since $L = \bigcup\{\downarrow x \mid x \in \mathrm{Min}(L^{\mathrm{op}})\}$ and $\mathrm{Min}(L^{\mathrm{op}}) \in \mathscr{P}_{\mathrm{con}}(L)$, we have that $(L, \varepsilon(L))$ is a Lindelöf space. □

Example 3.3 illustrates that the dual of a poset L is countably generated and $\mathrm{Min}(L^{\mathrm{op}}) \notin K(L)$ but $(L, \varepsilon(L))$ is a Lindelöf space.

Proposition 3.11. Let L be a poset and $M = \mathrm{Min}(L^{\mathrm{op}}) \cup \bigcup\{\downarrow\!\!\!\downarrow x \mid x \in \mathrm{Min}(L^{\mathrm{op}})\}$. If L^{op} is countably generated, then $L\backslash M$ with subtopology of ε-topology is a Lindelöf space if and only if $(L, \varepsilon(L))$ is a Lindelöf space.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Proof. Suppose that $L\backslash M$ with subtopology of ε-topology is a Lindelöf space. For any ε-open cover $\mathscr{U}$ of L, there exist countable subfamilies $\mathscr{U}_1$ and $\mathscr{U}_2$ of $\mathscr{U}$ such that $L\backslash M \subseteq \bigcup \mathscr{U}_1$ and $\mathrm{Min}(L^{\mathrm{op}}) \subseteq \bigcup \mathscr{U}_2$, i.e., $M \subseteq \bigcup \mathscr{U}_2$, since $\{x\} \cup \downarrow x$ is a smallest ε-open neighbourhood of x for all $x \in L$. Thus, $L = (\bigcup \mathscr{U}_1) \cup (\bigcup \mathscr{U}_2)$. Therefore, $(L, \varepsilon(L))$ is a Lindelöf space.

Conversely, since $x \cup \downarrow x \in \varepsilon(L)$ for all $x \in L$, we have that $M = \mathrm{Min}(L^{\mathrm{op}}) \cup \bigcup\{\downarrow x \mid x \in \mathrm{Min}(L^{\mathrm{op}})\} \in \varepsilon(L)$. Thus, $L\backslash M$ is ε-closed. Because $(L, \varepsilon(L))$ is a Lindelöf space, we know that $L\backslash M$ with subtopology of ε-topology is a Lindelöf space. □

Theorem 3.2. Let L and M be two posets and $f : L \to M$ injective. Then f preserves $\ll_2$ if and only if $f : (L, \varepsilon(L)) \to (M, \varepsilon(M))$ is continuous and $f(k) \in K(M)$ for all $k \in K(L)$.

Proof. Suppose that f preserves $\ll_2$. Then it is obvious that $f(k) \in K(M)$ for all $k \in K(L)$. To prove that f is continuous with respect to ε-topology, we only need to show that $f^{-1}(\{x\} \cup \downarrow x) \in \varepsilon(L)$ for all $x \in M$ since $\{\{x\} \cup \downarrow x \mid x \in M\}$ is a basis of $(M, \varepsilon(M))$. In fact, for every $y \in f^{-1}(\{x\} \cup \downarrow x)$ and each $u \in \downarrow y$, we have that $f(y) \subseteq \{x\} \cup \downarrow x$ and $f(u) \in \downarrow f(y)$ because f preserves $\ll_2$. Thus, $f(u) \in \downarrow f(y) \subseteq \downarrow x \subseteq \{x\} \cup \downarrow x$, i.e., $u \in f^{-1}(\{x\} \cup \downarrow x)$. Therefore, $y \in \{y\} \cup \downarrow y \subseteq f^{-1}(\{x\} \cup \downarrow x)$. Hence, $f^{-1}(\{x\} \cup \downarrow x) \in \varepsilon(L)$ since $\{\{y\} \cup \downarrow y \mid y \in L\}$ is a basis of $(L, \varepsilon(L))$.

Conversely, for any $x, y \in L$ with $y \ll_2 x$, $x \in \{x\} \cup \downarrow x \subseteq f^{-1}(\{f(x)\} \cup \downarrow f(x))$ because f is continuous with respect to ε-topology. Thus, $y \in f^{-1}(\{f(x)\} \cup \downarrow f(x))$. Therefore, $f(y) \in \{f(x)\} \cup \downarrow f(x)$.

i. If $y = x$, then $x \in K(L)$ and $f(y) = f(x)$. Thus, $f(x) \in K(M)$. Therefore, $f(y) \ll_2 f(x)$.
ii. If $y \neq x$, then $f(y) \neq f(x)$ since f is injective. Thus, $f(y) \ll_2 f(x)$. Therefore, f preserves $\ll_2$. □

Theorem 3.3. If L and M are posets, then the product topology of the topologies $\varepsilon(L)$ and $\varepsilon(M)$ is a subfamily of $\varepsilon(L \times M)$, i.e., $\varepsilon(L) \times \varepsilon(M) \subseteq \varepsilon(L \times M)$.

Proof. For L and M, we know that $\{\{(x, y)\} \cup \downarrow(x, y) \mid (x, y) \in L \times M\}$ and $\{(\{x\} \cup \downarrow x) \times (\{y\} \cup \downarrow y) \mid (x, y) \in L \times M\}$ are bases of $\varepsilon(L \times M)$ and $\varepsilon(L) \times \varepsilon(M)$, respectively. For every $(x, y) \in L \times M$ and each $(u, v) \in (\{x\} \cup \downarrow x) \times (\{y\} \cup \downarrow y)$, we have $\downarrow u \subseteq \downarrow x \subseteq \{x\} \cup \downarrow x$ and $\downarrow v \subseteq \downarrow y \subseteq \{y\} \cup \downarrow y$, which implies that $\downarrow(u, v) = \downarrow u \times \downarrow v \subseteq \{(\{x\} \cup \downarrow x) \times (\{y\} \cup \downarrow y)$. Thus, $(\{x\} \cup \downarrow x) \times (\{y\} \cup \downarrow y) \in \varepsilon(L \times M)$ for all $(x, y) \in L \times M$. Hence, $\varepsilon(L) \times \varepsilon(M) \subseteq \varepsilon(L \times M)$. □

The next example illustrates that for posets L and M, $\varepsilon(L) \times \varepsilon(M)$ is strictly coarser than $\varepsilon(L \times M)$ in general.

Example 3.5. Consider posets $L = \{0, 1\}$ and $M = [0, 1]$ with the usual order of real numbers, respectively. For every $x \in M$, $\{(1, x)\} \cup \downarrow(1, x) = \downarrow(1, x)\backslash\{(0, x)\}$ and $(\{1\} \cup \downarrow 1) \times (\{x\} \cup \downarrow x) = L \times \downarrow x = \downarrow(1, x)$, which implies that the interior with respect to $\varepsilon(L) \times \varepsilon(M)$ of $\{(1, x)\} \cup \downarrow(1, x)$ does not contain $(1, x)$, i.e., $\{(1, x)\} \cup \downarrow(1, x) \notin \varepsilon(L) \times \varepsilon(M)$. Thus, $\varepsilon(L) \times \varepsilon(M)$ is strictly coarser than $\varepsilon(L \times M)$.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Theorem 3.4. Let L and M be posets. If L and M satisfy one of the following two conditions:

1. $K(L) = L$ and $K(M) = M$.
2. $\downarrow\!\!\!\!\downarrow x = \varnothing$ and $\downarrow\!\!\!\!\downarrow y = \varnothing$ for all $(x, y) \in L \times M$,
   then $\varepsilon(L) \times \varepsilon(M) = \varepsilon(L \times M)$.

Proof. Suppose that $K(L) = L$ and $K(M) = M$. For every $(x, y) \in L \times M$, $\{(x, y)\} \cup \downarrow\!\!\!\!\downarrow(x, y) = \downarrow(x, y) = \downarrow x \times \downarrow y = (\{x\} \cup \downarrow\!\!\!\!\downarrow x) \times (\{y\} \cup \downarrow\!\!\!\!\downarrow y)$. Thus, $\varepsilon(L) \times \varepsilon(M) = \varepsilon(L \times M)$.

Suppose that $\downarrow\!\!\!\!\downarrow x = \varnothing$ and $\downarrow\!\!\!\!\downarrow y = \varnothing$ for all $(x, y) \in L \times M$. For every $(x, y) \in L \times M$, $\{(x, y)\} \cup \downarrow\!\!\!\!\downarrow(x, y) = \{(x, y)\} = \{x\} \times \{y\} = (\{x\} \cup \downarrow\!\!\!\!\downarrow x) \times (\{y\} \cup \downarrow\!\!\!\!\downarrow y)$. Thus, $\varepsilon(L) \times \varepsilon(M) = \varepsilon(L \times M)$. □

Next, we consider some properties of the operator $\Uparrow$.

Proposition 3.12. Let L be a poset. The operator $\Uparrow$ has the following properties:

1. $\Uparrow\varnothing = \varnothing$.
2. $\Uparrow(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \Uparrow A_i$, for every family of subsets $\{A_i\}_{i\in I} \subseteq \mathscr{P}(L)$.
3. $A \subseteq B$ implies $\Uparrow A \subseteq \Uparrow B$.
4. $\Uparrow A \backslash \Uparrow B \subseteq \Uparrow(A \backslash B)$ for all $A, B \in \mathscr{P}(L)$.
5. $\Uparrow(\Uparrow A) \subseteq \Uparrow A$ for all $A \in \mathscr{P}(L)$.

Proof. This follows immediately from the definition of the operator $\Uparrow$. □

Proposition 3.13. Let L be a s_2-continuous poset. Then $\Uparrow(\Uparrow A) = \Uparrow A$ for all $A \in \mathscr{P}(L)$. Furthermore, $\Uparrow U = U$ for all $U \in \sigma_2(L)$.

Proof. By Proposition 3.12(5), we have $\Uparrow(\Uparrow A) \subseteq \Uparrow A$. Now we only prove $\Uparrow(\Uparrow A) \supseteq \Uparrow A$. For $x \in \Uparrow A$, there exists $y \in A$ such that $x \in \Uparrow y$. Since L is a s_2-continuous poset, there is a $z \in \downarrow\!\!\!\!\downarrow x$ such that $z \in \Uparrow y$, i.e., $x \in \Uparrow(\Uparrow A)$.

For any $U \in \sigma_2(L)$, we have $\Uparrow U \subseteq U$ by Proposition 3.2(ii). For each $x \in U$, there exists $u \in U$ such that $x \in \Uparrow u \subseteq \Uparrow U$ since L is s_2-continuous. Thus, $\Uparrow U = U$. □

The arguments of Rusu and Ciobanu (2016) can now be reproduced to obtain an analogous result for posets or s_2-continuous posets, which are stated below without proof.

Proposition 3.14. Let L be a poset. For every $A \in \mathscr{P}(L)$, we have

1. $\mathrm{cl}_\varepsilon(A) = A \cup \Uparrow A$.
2. $\mathrm{int}_\varepsilon(A) = A \backslash \Uparrow(L \backslash A)$.
3. $\Uparrow A = \mathrm{der}_\varepsilon(A) \cup (A \cap K(L))$.
4. $\Uparrow \mathrm{cl}_\varepsilon(A) = \mathrm{cl}_\varepsilon(\Uparrow A) = \Uparrow A$.

Proposition 3.15. Let L be a s_2-continuous poset. The following assertions are equivalent:

1. For $x, y, z \in L$ with $x \in \downarrow\!\!\!\!\downarrow y \cap \downarrow\!\!\!\!\downarrow z$ implies $\Uparrow x \cap \downarrow\!\!\!\!\downarrow y \cap \downarrow\!\!\!\!\downarrow z \neq \varnothing$.
2. $A \in \mathrm{Irr}(\Gamma_\varepsilon(L))$ implies $\Uparrow A \in \mathrm{Irr}(\Gamma_\varepsilon(L))$.
3. $\Uparrow x \in \mathrm{Irr}(\Gamma_\varepsilon(L))$ for all $x \in L$.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

For a poset L, the property for $x, y, z \in L$ with $x \in \downarrow y \cap \downarrow z$ implies $\uparrow x \cap \downarrow y \cap \downarrow z \neq \varnothing$ will be denoted by R^*.

Theorem 3.5. Let L be a s_2-continuous poset which satisfies R^*. If $(L, \varepsilon(L))$ is sober, then L is s_2-algebraic.

Corollary 3.3. Let L be a s_2-continuous poset which satisfies R^*. If for every filtered subset $D \subseteq A$ has a lower bound in A for all $A \in \mathrm{Irr}(\Gamma_\varepsilon(L))$, then L is s_2-algebraic.

Proof. This follows immediately from Theorems 3.1 and 3.5. □

Let L be a poset. The condition $\uparrow L = L$ is important for what follows.

i. If L has a smallest element, then $\uparrow L = L$.
ii. If L is s_2-continuous, then $\uparrow L = L$.
iii. $\uparrow L = L$ iff $\downarrow x \neq \varnothing$ for all $x \in L$. Note that we cannot characterize s_2-continuous poset by $\uparrow L = L$ from (i).

Proposition 3.16. Let L be a poset with $\uparrow L = L$, and $A \in \mathscr{P}(L)$.

1. $\mathrm{int}_\varepsilon(A) \subseteq \uparrow A \subseteq \mathrm{cl}_\varepsilon(A)$.
2. $\mathrm{int}_\varepsilon(A \backslash \uparrow A) = \varnothing$.
3. If $A \in \varepsilon(L)$, then $A \subseteq \uparrow A$.
4. If $A \in \varepsilon(L)$, then $\uparrow A \in \sigma_2(L)$.

Remark 3.3. If L is a s_2-continuous poset, then $\sigma_2(L) = \{A \subseteq L \mid \uparrow A = A\} = \{\uparrow A \mid A \in \mathscr{P}(L)\}$. Therefore, $\mathrm{cl}_\varepsilon(A) = \uparrow A \in \sigma_2(L)$ for every $A \in \varepsilon(L)$, and so every ε-clopen set is σ_2-open.

The strong links between essential topology and σ_2-topology on a poset L are emphasized also by the fact that for every $A \in \mathscr{P}(L)$, there exists $U \in \varepsilon(L)$ such that $\mathrm{int}_{\sigma_2}(A) \subseteq U \subseteq \mathrm{cl}_{\sigma_2}(A)$. Moreover, if L is a s_2-continuous poset, then for any $A \in \mathscr{P}(L)$, there exists $V \in \sigma_2(L)$ such that $\mathrm{int}_\varepsilon(A) \subseteq V \subseteq \mathrm{cl}_\varepsilon(A)$.

If B is a basis in L, then $B \cap \downarrow x \neq \varnothing$ for all $x \in L$. Since $\{\{x\} \cup \downarrow x \mid x \in L\}$ is a basis for the ε-topology, it follows that B is ε-dense in L, i.e., $\mathrm{cl}_\varepsilon(B) = L$. The following result provides a characterization for ε-dense sets.

Proposition 3.17. Let L be a poset with $\uparrow L = L$ and $B \subseteq L$. Then $\mathrm{cl}_\varepsilon(B) = L$ iff $\uparrow B = L$.

Theorem 3.6. Let L be a s_2-continuous poset and $B \in \mathscr{P}(L)$. The following statements are equivalent:

1. B is a basis of L.
2. $\uparrow(\uparrow x \cap B) = \uparrow x$ for all $x \in L$.
3. $\uparrow(C \cap B) = \uparrow C$ for all $C \in \Gamma_\varepsilon(L)$.
4. $\uparrow(\uparrow A \cap B) = \uparrow A$ for all $A \in \mathscr{P}(L)$.
5. $\mathrm{cl}_\varepsilon(\uparrow A \cap B) = \uparrow A$ for all $A \in \mathscr{P}(L)$.
6. For each $U \in \sigma_2(L)$ and $V \in \varepsilon(L)$ with $U \cap V \neq \varnothing$, we have $U \cap V \cap B \neq \varnothing$.

Remark 3.4. By Remark 3.3 and Theorem 3.6, we know that a subset B is a basis of a s_2-continuous poset L iff B is ε-dense in every $U \in \sigma_2(L)$.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Theorem 3.7. A s_2-continuous poset L is algebraic iff $\mathrm{cl}_\varepsilon(A \cap K(L)) = \uparrow\!\!\uparrow A$ for all $A \in \Gamma_\varepsilon(L)$.

Proposition 3.18. Let L be a poset, $A \in \Gamma_\varepsilon(L)$ and $B \in \mathscr{P}(L)$. Then $\mathrm{cl}_\varepsilon(A \cap B) = A$ iff $\uparrow\!\!\uparrow(A \cap B) = \uparrow\!\!\uparrow A$ and $A \setminus \uparrow\!\!\uparrow A \subseteq B$.

Remark 3.5. By Theorem 3.6 and Proposition 3.18, we know that a subset B of a s_2-continuous poset L is ε-dense in every $A \in \Gamma_\varepsilon(L)$ implies that B is a basis.

4. δ-topology on s_2-continuous posets

In this section, we will introduce another kind of topology on a poset L and now proceed to the essential definition:

Definition 4.1. Let L be a poset. Then the common refinement $\sigma_2(L) \vee \varepsilon(L)$ of the upper topology and the ε-topology is called the *density topology* (for short, δ-topology) and is denoted by $\delta(L)$.

We note that the δ-topology on L has a base $\{U \cap V \mid U \in \sigma_2(L), V \in \varepsilon(L)\}$. Since $\{\{x\} \cup \downarrow\!\!\downarrow x \mid x \in L\}$ is a base for ε-topology $\varepsilon(L)$, we have that the family $\{(\{x\} \cup \downarrow\!\!\downarrow x) \cap U \mid x \in L, U \in \sigma_2(L)\}$ is a base for δ-topology $\delta(L)$. Therefore, $\sigma_2(L) \subseteq \lambda_2(L) \subseteq \delta(L)$.

From now on, all the topological properties involving the δ-topology will be denoted by using the prefix 'δ.' A set is called δ-clopen if it is δ-open and δ-closed simultaneously. We denote by $\Gamma_\delta(L)$ the family of all δ-closed sets on a poset L, and remark that $\sigma_2(L) \subseteq \Gamma_\varepsilon(L) \subseteq \Gamma_\delta(L)$. For a poset L, we denote by $\mathrm{int}_\delta(A)$ and $\mathrm{cl}_\delta(A)$, the interior and closure of a subset A of L in the δ-topology, respectively.

Proposition 4.1. Let L be a poset. Then U is a δ-clopen subset in $(L, \delta(L))$ for all $U \in \sigma_2(L)$, i.e., $\sigma_2(L) \subseteq \delta(L) \cap \Gamma_\delta(L)$. If L is s_2-continuous, then $x \in K(L)$ iff $\{x\} \in \delta(L)$.

Proof. We can prove similarly by the proof of Proposition 10 and Theorem 6(1) of Rusu and Ciobanu (2016). □

Corollary 4.1. Let L be a s_2-continuous poset.

1. If $\varnothing \neq K(L) \notin \mathscr{P}_{\mathrm{fin}}(L)$, then $(L, \delta(L))$ is not compact.
2. If $\varnothing \neq K(L) \notin \mathscr{P}_{\mathrm{con}}(L)$, then $(L, \delta(L))$ is not a Lindelöf space.

Proof. This follows immediately from Proposition 4.1. □

Proposition 4.2. Let L be a s_2-continuous poset. Then L is s_2-algebraic and $(L, \delta(L))$ is compact iff L is finite.

Proof. Suppose that L is finite. It is obvious that L is s_2-algebraic and $(L, \delta(L))$ is compact. For the reverse, by Corollary 4.1, we have $K(L) \in \mathscr{P}_{\mathrm{fin}}(L)$. Since L is s_2-algebraic, we know that $x = \sup(K(L) \cap \downarrow\!\!\downarrow x)$ and $K(L) \cap \downarrow\!\!\downarrow x$ is directed for all $x \in L$. Thus, $L = K(L)$, i.e., L is finite. □

Proposition 4.3. Let L be a poset. If L^{op} is not finitely generated, then $(L, \delta(L))$ is not compact.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Proof. Since $\varepsilon(L) \subseteq \delta(L)$ and $(L, \varepsilon(L))$ is not compact by Proposition 3.6, we have that $(L, \delta(L))$ is not compact. □

Proposition 4.4. Let L be a poset. If L^{op} is not countably generated, then $(L, \delta(L))$ is not a Lindelöf space.

Proof. Since $\varepsilon(L) \subseteq \delta(L)$ and $(L, \varepsilon(L))$ is not a Lindelöf space by Proposition 3.9, we have that $(L, \delta(L))$ is not a Lindelöf space. □

If L is a s_2-continuous poset, then $(L, \lambda_2(L))$ is a pospace. Thus, $(L, \delta(L))$ is also a pospace. We also have the following result.

Proposition 4.5. Let L be a poset. Then $(L, \delta(L))$ is a pospace, and hence is Hausdorff.

Proof. Suppose $x, y \in L$ with $x \not\leqslant y$. Then $x \in L\backslash{\downarrow}y \in \sigma_2(L) \subseteq \delta(L)$ and $y \in \{y\} \cup {\downarrow\!\!\!\downarrow}y \in \varepsilon(L) \subseteq \delta(L)$. Since $\{y\} \cup {\downarrow\!\!\!\downarrow}y \subseteq {\downarrow}y$, we have $(L\backslash{\downarrow}y) \times (\{y\} \cup {\downarrow\!\!\!\downarrow}y) \subseteq (L \times L)\backslash \leqslant$. Thus, $(L \times L)\backslash \leqslant$ is open in $(L, \delta(L)) \times (L, \delta(L))$, and hence $\leqslant$ is closed implies that $(L, \delta(L))$ is a pospace. Since every pospace is Hausdorff, we have that $(L, \delta(L))$ is Hausdorff. □

Proposition 4.6. Let L be a poset. Then $(L, \delta(L))$ is totally order-disconnected.

Proof. Suppose $x, y \in L$ with $x \not\leqslant y$. Then $y \in {\downarrow}y$ and $x \notin {\downarrow}y$. Since $L\backslash{\downarrow}y \in \sigma_2(L) \subseteq \delta(L) \cap \Gamma_\delta(L)$ by Proposition 4.1, we have ${\downarrow}y \in \delta(L) \cap \Gamma_\delta(L)$. Thus, the space $(L, \delta(L))$ is totally order-disconnected. □

Corollary 4.2. If a poset L is finite, then $(L, \delta(L))$ is a Priestly space.

Proof. Since every finite topology space is compact, we have that $(L, \delta(L))$ is a Priestly space by Proposition 4.6. □

Proposition 4.7. Let L be a poset, $x \in L$ and $U \in \varepsilon(L)$ with ${\downarrow}x \subseteq U$. Then there exists $V \in \delta(L)$ such that $x \in V \subseteq \mathrm{cl}_\delta(V) \subseteq U$.

Proof. Since $x \in \{x\} \cup {\downarrow\!\!\!\downarrow}x \subseteq {\downarrow}x \subseteq U$, $\{x\} \cup {\downarrow\!\!\!\downarrow}x \in \varepsilon(L) \subseteq \delta(L)$ and $L\backslash{\downarrow}x \in \sigma_2(L) \subseteq \delta(L) \cap \Gamma_\delta(L)$, we have that $x \in \{x\} \cup {\downarrow\!\!\!\downarrow}x \subseteq \mathrm{cl}_\delta(\{x\} \cup {\downarrow\!\!\!\downarrow}x) \subseteq U$. □

We consider the order between subsets G, H of a poset L by $G \leqslant H$ if $H \subseteq {\uparrow}G$. This implies that a family $\mathscr{F}$ of subsets is *directed* if the corresponding family $\{{\uparrow}F \mid F \in \mathscr{F}\}$ is a filter base.

Let L be a poset. The way-below relation $\ll_2$ on $\mathscr{P}(L)\backslash\{\varnothing\}$ is defined by $G \ll_2 H$ for all $G, H \subseteq L$, if for all directed subsets $D \subseteq L$ with ${\uparrow}H \cap D^\delta \neq \varnothing$, which implies that ${\uparrow}G \cap D \neq \varnothing$. For a poset L, we write $G \ll_2 x$ for $G \ll_2 \{x\}$ and $y \ll_2 H$ for $\{y\} \ll_2 H$. Note that $y \ll_2 x$ is unambiguously defined. A poset L is called s_2-quasicontinuous if for all $x \in L$, the family

$$Q_{\mathrm{fin}}(x) = \{F \subseteq L \mid F \in \mathscr{P}_{\mathrm{fin}}(L) \text{ and } F \ll_2 x\}$$

is a directed family and whenever $x \not\leqslant y$, there exists $F \in Q_{\mathrm{fin}}(x)$ with $y \notin {\uparrow}F$, i.e., ${\uparrow}x = \bigcap\{{\uparrow}F \mid F \in Q_{\mathrm{fin}}(x)\}$.

Remark 4.1. Let L be a poset. Then

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

1. $U \ll_2 U$ for all $U \in \sigma_2(L)$.
2. $G \ll_2 H$ if and only if $G \ll_2 x$ for all $x \in H$.

Lemma 4.1. (Zhang and Xu 2015) Let $\mathscr{F}$ be a directed family of non-empty finite subsets in a s_2-quasicontinuous poset L. For every $x \in L$ and $H \subseteq L$, if $H \ll_2 x$ and $\bigcap\{\uparrow F \mid F \in \mathscr{F}\} \subseteq \uparrow x$, then there exists a finite subset $F \in \mathscr{F}$ such that $H \ll_2 F$.

Theorem 4.1. Let L be a s_2-quasicontinuous poset, $x \in L$ and $U \in \lambda_2(L)$ with $x \in U$. Then there exists $V \in \delta(L)$ such that $x \in V \subseteq \mathrm{cl}_\delta(V) \subseteq U$.

Proof. Suppose that $x \in L$ and every $U \in \lambda_2(L)$ with $x \in U$. Then there exists $U_0 \in \sigma_2(L)$ and a finite subset $F \subseteq L$ such that $x \in U_0 \backslash \uparrow F \subseteq U$. For U_0, there is a finite subset $F_0 \in \mathscr{P}_{\mathrm{fin}}(U_0)$ such that $x \in \mathrm{int}_{\sigma_2}(\uparrow F_0)$, i.e., $x \in \uparrow F_0 \backslash \uparrow F \subseteq U$ by Lemma 4.1. Thus, $L \backslash U \subseteq (L \backslash \uparrow F_0) \cup \uparrow F$. For each $y \in F$, we have $y \not\leq x$, and there exists a finite subset $E_y \in \mathscr{P}_{\mathrm{fin}}(L)$ such that $y \in \mathrm{int}_{\sigma_2}(\uparrow E_y)$, $x \in L \backslash \uparrow E_y$ since L is s_2-quasicontinuous. Therefore, $x \in V = \mathrm{int}_{\sigma_2}(\uparrow F_0) \cap (\bigcap_{y \in F} L \backslash \uparrow E_y) \in \lambda_2(L) \subseteq \delta(L)$, $L \backslash U \subseteq W = (L \backslash \uparrow F_0) \cup (\bigcup_{y \in F} \lambda_2(L) \subseteq \delta(L)$ and $V \cap W = \varnothing$. Hence, $x \in V \subseteq L \backslash W \subseteq U$. Since $L \backslash W \in \Gamma_\delta(L)$, we have that $\mathrm{cl}_\delta(V) \subseteq L \backslash W$, i.e., $x \in V \subseteq \mathrm{cl}_\delta(V) \subseteq U$. □

Theorem 4.2. Let L be a s_2-quasicontinuous poset, $x \in L$ and $U \in \varepsilon(L)$ with $x \in U$. If $\downarrow x \backslash U$ is compact with respect to lower topology $\omega(L)$, then there exists $V \in \delta(L)$ such that $x \in V \subseteq \mathrm{cl}_\delta(V) \subseteq U$.

Proof. Suppose that $x \in L$ and $U \in \varepsilon(L)$ with $x \in U$ and $K = \downarrow x \backslash U$ is compact with respect to lower topology $\omega(L)$. For any $y \in K$, we have $x \not\leq y$, thus there exists a finite subset $F_y \in \mathscr{P}_{\mathrm{fin}}(L)$ such that $x \in \mathrm{int}_{\sigma_2}(\uparrow F_y) \in \sigma_2(L) \subseteq \delta(L)$, $y \in L \backslash \uparrow F_y \in \omega(L) \subseteq \varepsilon(L) \subseteq \delta(L)$, since L is s_2-quasicontinuous. Therefore, $K \subseteq \bigcup_{y \in K}(L \backslash \uparrow F_y)$, i.e., there exists a finite subset $E \subseteq K$ such that $K \subseteq \bigcup_{y \in E}(L \backslash \uparrow F_y)$. Let $V = (\downarrow x) \cap (\bigcap_{y \in E}(\mathrm{int}_{\sigma_2}(\uparrow F_y)))$ and $W = (L \backslash \downarrow x) \cup (\bigcup_{y \in E}(L \backslash \uparrow F_y))$. Then $x \in V \in \delta(L)$, $L \backslash U \subseteq W \in \delta(L)$ and $V \cap W = \varnothing$. Thus, $x \in V \subseteq L \backslash W \subseteq U$. □

Corollary 4.3. Let L be a s_2-quasicontinuous poset, $x \in L$ and $U \in \varepsilon(L)$ with $x \in U$. If $\downarrow x \backslash U \in \mathscr{P}_{\mathrm{fin}}(L)$, then there exists $V \in \delta(L)$ such that $x \in V \subseteq \mathrm{cl}_\delta(V) \subseteq U$.

Proof. Since every finite subset is always compact with respect to arbitrary topology on L, we have that this result holds by Theorem 4.2. □

Corollary 4.4. Let L be a s_2-continuous poset, $x \in L$ and $U \in \varepsilon(L)$ with $x \in U$. If $\downarrow x \backslash U$ is compact with respect to lower topology $\omega(L)$, then there exists $V \in \delta(L)$ such that $x \in V \subseteq \mathrm{cl}_\delta(V) \subseteq U$.

Proof. Since every s_2-continuous poset is always s_2-quasicontinuous, we have that this result holds by Theorem 4.2. □

Theorem 4.3. If L and M are posets, then the product topology of the topologies $\delta(L)$ and $\delta(M)$ is a subfamily of $\delta(L \times M)$, i.e., $\delta(L) \times \delta(M) \subseteq \delta(L \times M)$.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

Proof. By Exercise 4.5.19 in Goubault-Larrecq (2013), we know that $\sigma_2(L) \times \sigma_2(M) \subseteq \sigma_2(L \times M)$. By Theorem 3.3, we have $\varepsilon(L) \times \varepsilon(M) \subseteq \varepsilon(L \times M)$. Thus, $\delta(L) \times \delta(M) \subseteq \delta(L \times M)$. □

Example 3.5 and Johnstone space illustrate that for posets L and M, $\delta(L) \times \delta(M)$ is strictly coarser than $\delta(L \times M)$ in general.

Theorem 4.4. Let L and M be posets. If $K(L) = L$ and $K(M) = M$, then $\varepsilon(L) \times \varepsilon(M) = \varepsilon(L \times M)$.

Proof. Suppose that $K(L) = L$ and $K(M) = M$. Then by Theorem II-4.13 in Gierz et al. (2003), we obtain that $\sigma_2(L) \times \sigma_2(M) = \sigma_2(L \times M)$ since L and M are algebraic, and by Theorem 3.4, we know that $\varepsilon(L) \times \varepsilon(M) = \varepsilon(L \times M)$. Thus, $\delta(L) \times \delta(M) = \delta(L \times M)$. □

The following Theorem is an important result for δ-topology.

Theorem 4.5. Let L be a s_2-continuous poset and $B \in \mathscr{P}(L)$. Then B is a basis of L iff B is δ-dense in L.

Proof. This follows immediately from Theorem 3.6. □

A topological space is called *separable space* if it has a countable dense subsets (Goubault-Larrecq 2013).

Corollary 4.5. An s_2-continuous poset L has a countable basis iff $(L, \delta(L))$ is a separable space.

Proof. This follows immediately from Theorem 4.5. □

We denote by $\mathscr{B}_\delta(L)$ the family of all δ-dense subsets of a poset L. We note that $\mathscr{B}_\delta(L)$ is an upper set with respect to inclusion order, and a s_2-continuous poset is algebraic iff $K(L) \in \mathscr{B}_\delta(L)$. The following theorem gives some basic properties of $\mathscr{B}_\delta(L)$. For s_2-continuous posets, we have the following similar results (see Rusu and Ciobanu (2016),Theorem 6).

Theorem 4.6. Let L be a s_2-continuous poset. Then $\mathscr{B}_\delta(L)$ has the following properties:

1. $K(L) \subseteq \mathrm{int}_\delta(B)$ for all $B \in \mathscr{B}_\delta(L)$.
2. If $B \in \mathscr{B}_\delta(L)$ and $x \notin K(L)$, then $x \in \mathrm{cl}_\delta(B \backslash \{x\})$.
3. $K(L) = \bigcap \mathscr{B}_\delta(L)$.

Proof. Similarly, we can prove it by the proofs of Theorem 6(2), (4), (6) of Rusu and Ciobanu (2016). □

For a poset L, a subset A of L is δ-nowhere dense set wherever $\mathrm{int}_\delta(\mathrm{cl}_\delta(A)) = \varnothing$. We denote by $\mathscr{N}_\delta(L)$ the family of all δ-nowhere dense subsets of a poset L. Now we can get the following result (see Rusu and Ciobanu (2016), Theorem 7).

Theorem 4.7. Let L be a s_2-continuous poset. The following statements are equivalent:

1. L is s_2-algebraic.
2. $\bigcap \mathscr{B}_\delta(L) \in \mathscr{B}_\delta(L)$.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press

3. $(\mathscr{B}_\delta(L), \supseteq)$ is a dcpo.
4. $\bigcap(\mathscr{B}_\delta(L) \cap \delta(L)) \in \mathscr{B}_\delta(L)$.
5. For any $\varnothing \neq \{A_i\}_{i\in I} \subseteq \mathscr{N}_\delta(L)$, $\bigcup_{i\in I} A_i \neq \varnothing$ implies $\bigcup_{i\in I} A_i \notin \delta(L)$.

Proof. Likewise, we can prove it by the proof of Theorem 7 of Rusu and Ciobanu (2016). □

A topological space is called *Baire space* if for any countable intersection of dense open subsets is again dense (Gierz et al. 2003).

Corollary 4.6. If a poset L is s_2-algebraic, then $(L, \delta(L))$ is a Baire space.

Acknowledgements

This work is supported by National Natural Science Foundation of China (No.11371130, 11611130169) and the Research Fund for the Doctoral Program of Higher Education of China (No.20120161110017).

References

Abramsky, S. and Jung, A. (1994). Domain theory. In: Abramsky, S., Gabbay, D.M., and Maibaum, T. S. E. (eds.) *Semantic Structures*, *Handbook of Logic in Computer Science*, vol. 3, Clarendon Press, Oxford, 1–168.

Banaschewski, B. (1977). Essential extensions of T_0-spaces. *General Topology and its Applications* **7** (3) 233–246.

Davey, B.A. and Priestley, H.A. (2002). *Introduction to Lattices and Order,* 2nd ed., Cambridge University Press, Cambridge.

Engelking, R. (1977). *General Topology*, Polish Scientific Publishers, Warszawa.

Erné, M. (1991). The ABC of order and topology. In: H. Herlich and H.-E. Porst (eds.) *Category Theory at Work*, Heldermann Verlag, Berlin, 57–83.

Erné, M. (1991). The Dedekind-MacNeille completion as a reflector. *Order* **8** (2) 159–173.

Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M. and Scott, D.S. (2003). *Continuous Lattices and Domains*, Encyclopedia of Mathematics and Its Applications, vol. 93, Cambridge University Press, Cambridge.

Goubault-Larrecq, J. (2013). *Non-Hausdorff Topology and Domain Theory*, New Mathematical Monographs, vol. 22, Cambridge University Press, Cambridge.

Hoffmann, R.-E. (1981). Continuous posets, prime spectra of completely distrbutive lattices and Hausdorff compactifications. In: B. Banaschewski and R.-E. Hoffmann (eds.) *Continuous Lattices*, Lecture Notes in Mathematics, vol. 871, Springer-Verlag, Berlin, 159–208.

Rusu, D. and Ciobanu, G. (2016). Essential and density topologies of continuous domains. *Annals of Pure and Applied Logic* **167** (9) 726–736.

Zhang, W.F. and Xu, X.Q. (2015). s_2-quasicontinuous posets. *Theoretical Computer Science* **574** 78–85.

https://doi.org/10.1017/S0960129517000238 Published online by Cambridge University Press