Math. Proc. Camb. Phil. Soc. (1985), **97**, 525
Printed in Great Britain

A tabular method for correcting skewness

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(*Received* 30 *March* 1984)

Abstract. We propose a smooth correction for skewness in an asymptotically normal statistic. Unlike the case of approximation by Edgeworth expansion, our correction applies uniformly in all values of the level. The correction is based on a 'comparison statistic', and, in the case of the Studentized mean, it enables removal of all effects of skewness up to terms of order $n^{-\frac{3}{2}}$.

1. *Introduction*

Our purpose in this paper is to describe an approach for obtaining *completely uniform* corrections for skewness in normal approximations. The corrections are not obtained from an Edgeworth expansion but, rather, from approximation by a penultimate limit distribution. Their advantage over Edgeworth expansion technology is that they are available *uniformly* in the level α. Edgeworth corrections usually only hold uniformly on compact intervals, and this can cause problems when working with the tail of a distribution. Pfanzagl [**13**] has made an excellent survey of Edgeworth expansion techniques. A disadvantage of our approach is that it requires tabulation of the distribution of a 'comparison statistic'. However, the table need only be compiled once, and it may be used for an extremely wide range of parent distributions.

The 'tabular method' may be regarded as a generalization of methodology behind chi square approximations to the distributions of sums of independent random variables; see [**10**]. A table of the distribution of a 'comparison statistic' plays the role of chi square tables.

To introduce our methodology and compare it with Edgeworth expansions, let us consider the case of a 'Studentized' statistic T_n which has an asymptotic standard normal distribution, and for which a single parameter can be identified as the main cause of skewness. Suppose T_n admits an expansion of the form

$$P(T_n \leqslant x) = \Phi(x) + n^{-\frac{1}{2}}\lambda p_1(x)\,\phi(x) + n^{-1}x p_2(x)\,\phi(x) + o(n^{-1}) \tag{1·1}$$

uniformly in x, where Φ and ϕ are the standard normal distribution and density functions, respectively. Here p_j denotes an even polynomial of degree $2j$, $j = 1$ and 2, and λ stands for the parameter describing skewness. In the argument below we shall assume that p_1 does not depend on any unknown parameters, although p_2 may depend on such parameters. If the term of order $n^{-\frac{1}{2}}$ is a function of more than one unknown parameter, the argument can be modified by using tables of a multiparameter distribution. In the case of Student's t-statistic we may take λ to equal the standardized third moment.

The Edgeworth expansion approach usually runs as follows. We may deduce from (1·1) that

$$P\{T_n \leqslant x - n^{-\frac{1}{2}}\lambda p_1(x)\} = \Phi(x) + O(n^{-1})$$

https://doi.org/10.1017/S0305004100063118 Published online by Cambridge University Press

uniformly on compact intervals. Only very seldom (for example, when $\lambda = 0$) is the result valid uniformly in *all* x. Since λ is unknown, we construct an estimate $\hat{\lambda}$ from the distribution on which T_n was based. Under suitable regularity conditions on $\hat{\lambda}$, the expansion remains valid if λ is replaced by $\hat{\lambda}$; see for example [9]. Thus

$$P\{T_n \leqslant x - n^{-\frac{1}{2}}\hat{\lambda}p_1(x)\} = \Phi(x) + O(n^{-1})$$

uniformly in x on *compact* intervals. Since this expansion usually fails as either $x \to +\infty$ or $x \to -\infty$, there is a need for an alternative, 'smoother' approach. We introduce this next.

Let T_n^0 be a 'comparison statistic' whose distribution is known. We require that T_n^0 admit an Edgeworth expansion like (1·1), and that the first term in the expansion be proportional to the first term in (1·1). Thus,

$$P(T_n^0 \leqslant x) = \Phi(x) + n^{-\frac{1}{2}}ap_1(x)\,\phi(x) + n^{-1}xp_3(x)\,\phi(x) + o(n^{-1}), \tag{1·2}$$

where a is a known constant. We shall take $a > 0$. Suppose the upper and lower critical points of T_n^0, say $z_n^+(\alpha)$ and $-z_n^-(\alpha)$, are tabulated for selected values of α and a wide range of values of n. The T_n^0 tables would closely resemble chi square tables. The critical points are defined by the formulae

$$P\{T_n^0 \leqslant z_n^+(\alpha)\} = P\{T_n^0 > -z_n^-(\alpha)\} = 1 - \alpha \quad (0 < \alpha < 1).$$

Since T_n^0 is asymptotically normal then $z_n^+(\alpha)$ and $z_n^-(\alpha)$ both converge to $z = z(\alpha)$, the solution of the equation $\Phi(z) = 1 - \alpha$, as $n \to \infty$.

Given an n-sample from which we calculate the statistic T_n, let us also compute an estimate $\hat{\lambda}$ of the skewness λ. Define the random integer N to equal $[na^2/\hat{\lambda}^2]$, which may be taken as either the integer part of $na^2/\hat{\lambda}^2$ or the integer nearest to $na^2/\hat{\lambda}^2$. Let $z_N(\alpha)$ equal $z_N^+(\alpha)$ if $\hat{\lambda} > 0$, $z(\alpha)$ if $\hat{\lambda} = 0$, and $z_N^-(\alpha)$ otherwise. Then it is intuitively clear that

$$P\{T_n \leqslant z_N(\alpha)\} = (1 - \alpha) + n^{-1}z(\alpha)\,p_4(z(\alpha))\,\phi(z(\alpha)) + o(n^{-1}) \tag{1·3}$$

uniformly in $0 < \alpha < 1$, as $n \to \infty$, where p_4 is another polynomial. Thus, by referring to tables of the comparison statistic T_n^0, rather than to standard normal tables, we have eliminated the first-order effect of skewness. The new approximation is to order n^{-1}, whereas the old was only to order $n^{-\frac{1}{2}}$. Most importantly, the new approximation is valid uniformly in all $\alpha \in (0, 1)$, whereas the former was uniformly available only for $\epsilon < \alpha < 1 - \epsilon$, $\epsilon > 0$.

This is the basic principle behind our correction. In practice it will usually be desirable to adjust the distribution of T_n^0 so as to further reduce the error. The form of the polynomial p_4 in (1·3) is likely to depend upon three different factors: (i) a second-order effect due to skewness, possibly described by terms in λ^2; (ii) an effect due to other unknown parameters, which might be termed a generalized kurtosis; and (iii) a residual effect, after the influences (i) and (ii) have been removed. The kurtosis could be corrected by tabulating a version of T_n^0 which depended on kurtosis as well as on n, but this would entail a considerable increase in the bulk of the tables. (Compare chi square tables with tables of Fisher's F.) It therefore seems desirable to correct for kurtosis in a different manner, which we shall discuss in Section 3. However, by slightly adjusting the distribution of T_n^0 it may be possible to remove all of the skewness term from p_4. Thus p_4 can be assumed to be influenced only by the factors (ii) and (iii)

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above. A secondary, deterministic correction, $\zeta_n(\alpha)$, should remove the effect of type (iii), and then we may write

$$P\{T_n \leqslant z_N(\alpha) + \zeta_n(\alpha)\} = (1-\alpha) + n^{-1} z(\alpha)\, p_5(z(\alpha))\, \phi(z(\alpha)) + o(n^{-1}) \tag{1·4}$$

uniformly in $0 < \alpha < 1$, where p_5 depends only on the kurtosis. In particular, if the kurtosis were zero then the right-hand side of (1·4) would equal $1-\alpha+o(n^{-1})$ uniformly in $0 < \alpha < 1$, irrespective of the size of the skewness.

Under suitable regularity conditions, the remainder of $o(n^{-1})$ in (1·4) may be sharpened to $O(n^{-\frac{3}{2}})$. Thus, our correction has removed all the effects of skewness up to terms of order $n^{-\frac{3}{2}}$. We call the correction 'smooth' because it is available *uniformly* in α. A correction of a very different type was suggested in [**9**], but has the disadvantage that it is uniform only in $\alpha \in (\epsilon, 1-\epsilon)$, for $\epsilon > 0$. In important special cases like Student's t-statistic, this uniformity cannot be extended to $0 < \alpha < 1$. The smooth correction may be expected to work better in the tails than an unsmooth correction.

The theory of first-order corrections, leading to results like (1·3), is discussed in Section 2. In that work we carry the expansion only to terms of small order in $n^{-\frac{1}{2}}$, since the form of the polynomial p_4 depends intimately on the structure of the statistic under discussion. Furthermore, the nature of the estimator $\hat{\lambda}$ depends closely on the situation being investigated, and so it is not really possible to present a rigorous, general description of the effect of removing skewness terms from p_4. For these reasons we shall confine our investigation of second-order corrections to the case of Student's t-statistic, in which the term 'kurtosis' has an unambiguous and well-accepted meaning. This topic is presented in Section 3. The proofs for Section 3 are deferred until Section 4.

Approximation by asymptotic expansion is studied in much greater depth in [**1**, **2**, **3**, **5**, **6**, **7**, **12**, **13**]. Our techniques rely heavily on multivariate normal approximations developed by Bhattacharya and Ghosh [**2**] and Bhattacharya and Rao [**3**]. The influence of non-normality on Student's t-statistic has been very extensively documented, and the literature has been reviewed by Bowman, Beauchamp and Shenton [**4**] and Cressie [**8**].

2. *First-order correction*

Under suitable regularity conditions on the statistic T_n^0, the expansion (1·2) may be inverted to yield

$$\left.\begin{aligned} z_n^+(\alpha) &= z(\alpha) - n^{-\frac{1}{2}} a p_1(z(\alpha)) + O(n^{-\xi}) \\ \text{and} \qquad z_n^-(\alpha) &= z(\alpha) + n^{-\frac{1}{2}} a p_1(z(\alpha)) + O(n^{-\xi}) \end{aligned}\right\} \tag{2·1}$$

uniformly in $\gamma n^{-\beta} \leqslant \alpha \leqslant 1 - \gamma n^{-\beta}$, for some $\beta > \frac{1}{2}$ and $\xi > \frac{1}{2}$ and all $\gamma > 0$. The inversion may be accomplished using techniques from the proof of Lemma 1 in Section 4. Since we have a wide latitude for the choice of T_n^0, the nature of the regularity condition is relatively unimportant. Therefore we shall regard (2·1) as the assumption, rather than (1·2). We shall further assume that, for some $\epsilon > 0$, the estimator $\hat{\lambda}$ satisfies

$$P(|\hat{\lambda} - \lambda| > n^{-\epsilon}) = o(n^{-\frac{1}{2}}) \tag{2·2}$$

as $n \to \infty$. The first-order correction is described by the following theorem, which uses the notation of Section 1.

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THEOREM 1. *Suppose T_n admits the expansion*

$$P(T_n \leqslant x) = \Phi(x) + n^{-\frac{1}{2}} \lambda p_1(x)\, \phi(x) + o(n^{-\frac{1}{2}}) \tag{2·3}$$

uniformly in x, as $n \to \infty$. Then, under conditions (2·1) *and* (2·2),

$$P\{T_n \leqslant z_N(\alpha)\} = (1-\alpha) + o(n^{-\frac{1}{2}}) \tag{2·4}$$

uniformly in $0 < \alpha < 1$, as $n \to \infty$.

Proof. It will be shown during the proof of Theorem 2 that, for large n,

$$|z(\alpha)| < (2\beta \log n)^{\frac{1}{2}}$$

uniformly in $n^{-\beta} \leqslant \alpha \leqslant 1 - n^{-\beta}$. If $\lambda \neq 0$ then, on the set $E = \{|\hat{\lambda} - \lambda| \leqslant n^{-\epsilon}\}$, we have $N \geqslant na^2/2\lambda^2$ for large n. On the other hand, if $\lambda = 0$ then $N \geqslant n^{1+2\epsilon}a^2/2$ on E. Hence there exists $C_1 > 0$ such that

$$\sup_{n^{-\beta} \leqslant \alpha \leqslant 1-n^{-\beta}} |z_N^{\pm}(\alpha) - \{z(\alpha) \mp N^{-\frac{1}{2}} a p_1(z(\alpha))\}| \leqslant C_1 n^{-\xi}$$

on E, for large n. A similar argument shows that $|N^{-\frac{1}{2}} - n^{-\frac{1}{2}} a^{-1}|\lambda|| \leqslant C_2 n^{-\frac{1}{2}-\epsilon}$ on E if $0 < \epsilon < \frac{1}{2}$, and therefore

$$\sup_{n^{-\beta} \leqslant \alpha \leqslant 1-n^{-\beta}} |z_N(\alpha) - z(\alpha) + n^{-\frac{1}{2}} \lambda p_1(z(\alpha))| \leqslant C_3 n^{-\frac{1}{2}-\eta}$$

on E, for some $\eta > 0$. Consequently,

$$\begin{aligned} P\{T_n \leqslant z_N(\alpha)\} &\leqslant P\{T_n \leqslant z(\alpha) - n^{-\frac{1}{2}} \lambda p_1(z(\alpha)) + C_3 n^{-\frac{1}{2}-\eta}\} + P(\bar{E}) \\ &= \Phi(z(\alpha)) + o(n^{-\frac{1}{2}}) = (1-\alpha) + o(n^{-\frac{1}{2}}) \end{aligned}$$

uniformly in $n^{-\beta} \leqslant \alpha \leqslant 1 - n^{-\beta}$. (Set $x = z - n^{-\frac{1}{2}} \lambda p_1(z) + C_3 n^{-\frac{1}{2}-\eta}$ in (2·3).) The reverse inequality may be obtained in the same way, and so (2·4) is true uniformly in $n^{-\beta} \leqslant \alpha \leqslant 1 - n^{-\beta}$. To handle the case $\alpha < \alpha_n \equiv n^{-\beta}$, observe that $z(\alpha_n) \sim (2\beta \log n)^{\frac{1}{2}}$ and, since $\beta > \frac{1}{2}$, $1 - \Phi\{z(\alpha_n)\} \sim \{z(\alpha_n)\}^{-1} \phi\{z(\alpha_n)\} = o(n^{-\frac{1}{2}})$. Furthermore,

$$\sup_{\alpha \leqslant \alpha_n} P\{T_n > z_N(\alpha)\} \leqslant P\{T_n > \inf_{m \geqslant r} z_m(\alpha_n)\} + P(N < r), \tag{2·5}$$

where we take $r = r(n) = na^2/2\lambda^2$ if $\lambda \neq 0$, and $r = n^{1+2\epsilon}a^2/2$ if $\lambda = 0$. It follows from (2·1) that, for any β' lying between $\frac{1}{2}$ and β,

$$\inf_{m \geqslant r} z_m(\alpha_n) \geqslant z(\alpha_n) + o(1) > (2\beta' \log n)^{\frac{1}{2}},$$

provided n is large. The expansion (2·3) implies that $P\{T_n > (2\beta' \log n)^{\frac{1}{2}}\} = o(n^{-\frac{1}{2}})$, and it follows from (2·2) that $P(N < r) = o(n^{-\frac{1}{2}})$. Therefore the right-hand side of (2·5) equals $o(n^{-\frac{1}{2}})$ uniformly in $\alpha \leqslant n^{-\beta}$. This proves (2·4) uniformly in $\alpha \leqslant n^{-\beta}$, and the case $\alpha > 1 - n^{-\beta}$ may be treated similarly.

3. *Second-order correction*

Here we consider only the case of Student's t-statistic, and provide a method of eliminating both the first- and second-order effects of skewness. Let $Y, Y_1, Y_2, \ldots$ be independent and identically distributed random variables with mean μ and variance σ^2, both assumed unknown. Inference about μ is usually based on the statistic

$$T_n = \left\{n^{-\frac{1}{2}} \sum_1^n (Y_j - \mu)\right\} \Big/ \left\{(n-1)^{-1} \sum_1^n (Y_j - \bar{Y})^2\right\}^{\frac{1}{2}} = \left(n^{-\frac{1}{2}} \sum_1^n X_j\right) \Big/ \left\{(n-1)^{-1} \sum_1^n (X_j - \bar{X})^2\right\}^{\frac{1}{2}},$$

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where $X = (Y-\mu)/\sigma$ and $X_j = (Y_j-\mu)/\sigma$, $j \geqslant 1$, are independent and identically distributed random variables with zero mean and unit variance, and where

$$\overline{X} = n^{-1}\sum_1^n X_j \quad \text{and} \quad \overline{Y} = n^{-1}\sum_1^n Y_j.$$

We shall take as our 'comparison statistic' any random variable T_n^0 which admits the expansion

$$P(T_n^0 \leqslant x) = \Phi(x) + n^{-\frac{1}{2}}\tfrac{1}{6}a(1+2x^2)\,\phi(x) + n^{-1}x\,\tfrac{1}{36}a^2(15+14x^2-2x^4)\,\phi(x) + n^{-\frac{3}{2}}p_1(x)\,\phi(x) + n^{-2}p_2(x)\,\phi(x) + O(n^{-\frac{5}{2}}) \quad (3{\cdot}1)$$

uniformly in x, where a is a known positive constant and where p_1 and p_2 are polynomials. The precise form of p_1 and p_2 is unimportant. We shall also assume a short 'local' expansion:

$$(d/dx)\{P(T_n^0 \leqslant x) - \Phi(x) - n^{-\frac{1}{2}}\tfrac{1}{6}a(1+2x^2)\,\phi(x) - n^{-1}x\,\tfrac{1}{36}a^2(15+14x^2-2x^4)\,\phi(x)\} = O(n^{-\frac{3}{2}}) \quad (3{\cdot}2)$$

uniformly in x. Define $\mu_3 = E(X^3)$ and $\mu_4 = E(X^4)$, and let

$$\hat{\mu}_3 = \left\{n^{-1}\sum_1^n (Y_j-\overline{Y})^3\right\} \Big/ \left\{(n-1)^{-1}\sum_1^n (Y_j-\overline{Y})^2\right\}^{\frac{3}{2}} = \left\{n^{-1}\sum_1^n (X_j-\overline{X})^3\right\} \Big/ \left\{(n-1)^{-1}\sum_1^n (X_j-\overline{X})^2\right\}^{\frac{3}{2}}$$

be an estimator of μ_3. Alternatively, we may define $\hat{\mu}_3$ using the biased estimate of variance in the denominator, without affecting the asymptotics. Let $z_n^+(\alpha)$ and $z_n^-(\alpha)$ be the critical points of T_n^0, set $N = [na^2/\hat{\mu}_3^2]$, and define $z_N(\alpha)$ to equal $z_N^+(\alpha)$ if $\hat{\mu}_3 > 0$, $z(\alpha)$ if $\hat{\mu}_3 = 0$, and $z_N^-(\alpha)$ if $\hat{\mu}_3 < 0$. Write $t_n(\alpha)$ and $z = z(\alpha)$ for the upper $(1-\alpha)$-level critical points of Student's t-distribution on $(n-1)$ degrees of freedom, and the standard normal distribution, respectively.

THEOREM 2. *Suppose the comparison statistic T_n^0 admits the expansions* (3·1) *and* (3·2), *that the distribution of the random variable X has a nontrivial absolutely continuous component, and that $E(|X|^{10+\eta}) < \infty$ for some $\eta > 0$. Then*

$$P\{T_n \leqslant z_N(\alpha) + t_n(\alpha) - z(\alpha)\} = (1-\alpha) - n^{-1}\tfrac{1}{12}(\mu_4-3)\,z(3z^2+5)\,\phi(z) + o(n^{-1}) \quad (3{\cdot}3)$$

uniformly in $0 < \alpha < 1$, as $n \to \infty$, where $z = z(\alpha)$.

The expansion (3·3) shows that the correction term $z_N(\alpha) - z(\alpha)$ has eliminated both the first- and second-order effects of skewness, leaving only a second-order effect due to kurtosis. Note that

$$\sup_{-\infty < z < \infty} |\tfrac{1}{12}z(3z^2+5)\,\phi(z)| = 0{\cdot}1905 < \tfrac{1}{5},$$

and so

$$\sup_{0<\alpha<1} |P\{T_n \leqslant z_N(\alpha) + t_n(\alpha) - z(\alpha)\} - (1-\alpha)| \leqslant |\kappa|/5n + o(n^{-1}),$$

where $\kappa = \mu_4 - 3$ denotes the kurtosis. The influence of kurtosis to order n^{-1} can be removed by making a secondary approximation, which we now describe.

Let $\hat{\mu}_4 = \{n^{-1}\Sigma_1^n(X_i-\overline{X})^4\}/\{(n-1)^{-1}\Sigma_1^n(X_i-\overline{X})^2\}^2$ be an estimate of μ_4.

THEOREM 3. *Assume the conditions of Theorem 2. Then for each $\epsilon \in (0, \frac{1}{2})$,*

$$P\{T_n \leqslant z_N(\alpha) + t_n(\alpha) - z(\alpha) + n^{-1}\tfrac{1}{12}(\hat{\mu}_4-3)z(3z^2+5)\} = (1-\alpha) + o(n^{-1})$$

uniformly in $\epsilon \leqslant \alpha \leqslant 1-\epsilon$, as $n \to \infty$, where $z = z(\alpha)$.

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Theorem 3 may be derived in the manner of Theorem 3 of Hall [9], and so is not proved here.

There exists an infinity of possible comparison statistics T_n^0 which satisfy the expansions (3·1) and (3·2). Suppose $Z, Z_1, Z_2, \ldots$ are independent with a common absolutely continuous distribution having all moments finite, zero mean and unit variance. Define $\lambda_j = E(Z^j)$ for $j \geqslant 3$, and let

$$n^{-\frac{1}{2}} T_n^0 = \left(n^{-1}\sum_1^n Z_j\right) \Big/ \left\{n^{-1}\sum_1^n (Z_i - \bar{Z})^2\right\}^{\frac{1}{2}} + \xi_1\left(n^{-1}\sum_1^n Z_i\right)^3 + \xi_2\left(n^{-1}\sum_1^n Z_i\right)\left(n^{-1}\sum_1^n Z_i^2 - 1\right)^2,$$

where ξ_1 and ξ_2 are fixed constants (not depending on n). Note that $n^{-\frac{1}{2}} T_n^0 = H(Z^{(1)}, Z^{(2)})$ where $Z^{(1)} = n^{-1}\Sigma_1^n Z_i$, $Z^{(2)} = n^{-1}\Sigma_1^n (Z_i^2 - 1)$ and

$$H(z^{(1)}, z^{(2)}) = z^{(1)} / \{1 + z^{(2)} - (z^{(1)})^2\}^{\frac{1}{2}} + \xi_1 (z^{(1)})^3 + \xi_2 z^{(1)} (z^{(2)})^2.$$

Therefore theorem 2 of Bhattacharya and Ghosh [2] may be applied to T_n^0, and from that result we may deduce after some tedious algebra that

$$\begin{aligned} P(T_n^0 \leqslant x) = {} & \Phi(x) + n^{-\frac{1}{2}} \tfrac{1}{6}\lambda_3 (1 + 2x^2)\,\phi(x) \\ & - n^{-1} x\{\tfrac{1}{18}\lambda_3^2 (x^4 + 2x^2 - 3) - \tfrac{1}{12}\lambda_4 (x^2 - 3) + \tfrac{1}{2}x^2 \\ & + 3\xi_1 + \xi_2 (2\mu_3^2 + \mu_4 - 1) + (\xi_1 + \xi_2 \mu_3^2)(x^2 - 3)\}\,\phi(x) \\ & + n^{-\frac{3}{2}} p_1(x)\,\phi(x) + n^{-2} p_2(x)\,\phi(x) + O(n^{-\frac{5}{2}}) \end{aligned} \tag{3·4}$$

uniformly in x, where p_1 and p_2 are polynomials. The identification of this expansion is outlined at the end of Section 4. On comparing (3·4) with (3·1), and taking $a = \lambda_3$, we see that it is necessary to have

$$\tfrac{1}{18}\lambda_3^2 (x^3 + 2x^2 - 3) - \tfrac{1}{12}\lambda_4 (x^2 - 3) + \tfrac{1}{2}x^2 + 3\xi_1 + \xi_2 (2\lambda_3^2 + \lambda_4 - 1) + (\xi_1 + \xi_2 \lambda_3^2)(x^2 - 3) \equiv \tfrac{1}{36}\lambda_3^2 (2x^4 - 14x^2 - 15),$$

which entails

$$\xi_1 = (7\lambda_4 + 4\lambda_3^2 \lambda_4 - \lambda_4^2 - 12\lambda_3^2 - 9\lambda_3^4 - 6)/12(\lambda_3^2 + 1 - \lambda_4)$$

and

$$\xi_2 = (\lambda_3^2 + \lambda_4)/4(\lambda_3^2 + 1 - \lambda_4).$$

In particular, if the Z_i are standardized exponential variables then $\lambda_3^2 = 4$, $\lambda_4 = 9$, $\xi_1 = \frac{3}{2}$ and $\xi_2 = -\frac{13}{16}$. The method of proof of Bhattacharya and Ghosh's [2] theorem 2 can be used to derive a short local expansion of the type (3·2).

4. *Proofs for Section* 3

Proof of Theorem 2. Throughout this proof the symbol C denotes a positive generic constant, while $I(E)$ stands for the indicator function of an event E. The symbol z, if unqualified, always means $z(\alpha)$. We set $d = 10 + \eta$. In order to clarify the reasoning behind our argument we define $a_0 = 5a^2/12$, $a_1 = 7a^2/18$ and $a_2 = -a^2/18$. Thus, the expansions (3·1) and (3·2) may be written as

$$\begin{aligned} P(T_n^0 \leqslant x) = {} & \Phi(x) + n^{-\frac{1}{2}} \tfrac{1}{6} a (1 + 2x^2)\,\phi(x) + n^{-1} x (a_0 + a_1 x^2 + a_2 x^4)\,\phi(x) \\ & + n^{-\frac{3}{2}} p_1(x)\,\phi(x) + n^{-2} p_2(x)\,\phi(x) + O(n^{-\frac{5}{2}}) \end{aligned} \tag{4·1}$$

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and

$$(d/dx)\{P(T_n^0 \leqslant x) - \Phi(x) - n^{-\frac{1}{2}}\tfrac{1}{6}a(1+2x^2)\,\phi(x) - n^{-1}x(a_0+a_1x^2+a_2x^4)\,\phi(x)\} = O(n^{-\frac{3}{2}}) \quad (4{\cdot}2)$$

uniformly in x.

The early part of our proof consists of a sequence of six preparatory lemmas. Lemma 1 provides expansions of $z_n^+(\alpha)$ and $z_n^-(\alpha)$. These results are used to derive Lemma 2, which describes an expansion of $z_N(\alpha)$. Lemma 3 is a simple algebraic inequality. Lemma 4 shows that statistics based on the X's can be approximated by statistics based on a simpler Y-sample. Lemma 5 provides an expansion of $T_n - z_N(\alpha)$. Finally, Lemma 6 presents the classical Edgeworth expansion for the Studentized mean. Thereafter the lemmas are combined to derive the required expansion of the distribution function of $T_n - z_N(\alpha)$.

LEMMA 1. *Let $z = z(\alpha)$ be the solution of the equation $1 - \Phi(z) = \alpha$, and set*

$$y_n^+ = y_n^+(\alpha) = z - n^{-\frac{1}{2}}\tfrac{1}{6}a(1+2z^2) + n^{-1}z(b_0+b_1z^2) + n^{-\frac{3}{2}}q_1(z) + n^{-2}q_2(z)$$

and

$$y_n^- = y_n^-(\alpha) = z + n^{-\frac{1}{2}}\tfrac{1}{6}a(1+2z^2) + n^{-1}z(b_0+b_1z^2) + n^{-\frac{3}{2}}q_3(z) + n^{-2}q_4(z),$$

where $q_1, \ldots, q_4$ are polynomials, $b_0 = 7a^2/72 - a_0$ and $b_1 = a^2/6 - a_1$. We may choose $q_1, \ldots, q_2$ such that, for all $\gamma > 0$ and $0 < \delta < \frac{1}{2}$, $|z_n^\pm(\alpha) - y_n^\pm(\alpha)| = O(n^{\delta-\frac{3}{2}})$ uniformly in $\gamma n^{-1-\delta} \leqslant \alpha \leqslant 1 - \gamma n^{-1-\delta}$, as $n \to \infty$.

Proof. Replacing x by $y_n^+ = z + y_n'$, say, in the expansion (4·1), we find that

$$P(T_n^0 \leqslant y_n^+) = \Phi(z) + n^{-1}z\{(b_0 - 7a^2/72 + a_0) + (b_1 - a^2/6 + a_1)z^2 + (a^2/18 + a_2)z^4\}\,\phi(z) + n^{-\frac{3}{2}}\{q_1(z) + q_5(z)\}\,\phi(z) + n^{-2}\{q_2(z) + q_6(z)\}\,\phi(z) + O(n^{-\frac{5}{2}}) \quad (4{\cdot}3)$$

uniformly in values of α for which $|z(\alpha)| \leqslant (2\beta\log n)^{\frac{1}{2}}$, where $\beta = 1 + \delta$ and q_5 and q_6 are polynomials. Neither q_5 nor q_6 depends on q_2, and only q_6 depends on q_1. Therefore if we define $q_1 = -q_5$ and then $q_2 = -q_6$, we may deduce from (4·3) that

$$P\{T_n^0 \leqslant y_n^+(\alpha)\} = P\{T_n^0 \leqslant z_n^+(\alpha)\} + O(n^{-\frac{5}{2}}) \quad (4{\cdot}4)$$

uniformly in values of α for which $|z(\alpha)| \leqslant (2\beta\log n)^{\frac{1}{2}}$.

Set $\alpha_n = \gamma n^{-1-\delta} = \gamma n^{-\beta}$, and note that

$$z(\alpha_n) = (2\beta\log n)^{\frac{1}{2}} - (\log\log n)/(8\beta\log n)^{\frac{1}{2}} + O(1/\log n)$$

as $n \to \infty$. Therefore by Lemma 1, $\max\{y_n(\alpha), z_n^+(\alpha)\} < (2\beta\log n)^{\frac{1}{2}}$ uniformly in $\alpha \geqslant \alpha_n$ for all large n and, similarly, $\min\{y_n(\alpha), z_n^+(\alpha)\} > -(2\beta\log n)^{\frac{1}{2}}$ uniformly in $\alpha \leqslant 1 - \alpha_n$. Let f_n denote the density of T_n^0. In view of (4·2),

$$\inf_{|u| < (2\beta\log n)^{\frac{1}{2}}} f_n(u) \sim (2\pi)^{-\frac{1}{2}}n^{-\frac{1}{2}-\delta}, \quad (4{\cdot}5)$$

provided $0 < \delta < \frac{1}{2}$. By inverting a Taylor expansion we may deduce from (4·4) and (4·5) that

$$\begin{aligned}\sup_{\alpha_n \leqslant \alpha \leqslant 1-\alpha_n} |z_n^+(\alpha) - y_n^+(\alpha)| &\leqslant [\sup_{\alpha_n \leqslant \alpha \leqslant 1-\alpha_n} |P\{T_n^0 \leqslant z_n^+(\alpha)\} - P\{T_n^0 \leqslant y_n^+(\alpha)\}|] \\ &\quad \times \{\inf_{|u| < (2\beta\log n)^{\frac{1}{2}}} f_n(u)\}^{-1} \\ &= O(n^{\delta-\frac{3}{2}}),\end{aligned}$$

which proves Lemma 1 for z_n^+. The case of z_n^- may be handled similarly.

The next step is to expand $z_N(\alpha)$.

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LEMMA 2. *We have*

$$z_N(\alpha) = z - n^{-\frac{1}{2}}\tfrac{1}{6}\mu_3(1+2z^2) + n^{-1}\mu_3^2 a^{-2} z(b_0 + b_1 z^2)$$
$$- n^{-\frac{3}{2}}\tfrac{1}{6}(1+2z^2)\sum_1^n \{(X_j'^3 - \mu_3') - 3X_j - (3\mu_3/2)(X_j'^2 - \lambda\mu_2')\} + \Delta_n^{(1)}, \quad (4{\cdot}6)$$

where, on the set

$$E^{(1)} \equiv \left\{\left|\sum_1^n X_j\right| \leqslant n^{\frac{1}{2}+1/d}\right\} \cap \left\{\left|\sum_1^n (X_j^2 - 1)\right| \leqslant n^{\frac{1}{2}+2/d}\right\} \cap \left\{\left|\sum_1^n (X_j^3 - \mu_3)\right| \leqslant n^{\frac{1}{2}+3/d}\right\}$$
$$\cap \{|X_j| \leqslant n^{\frac{1}{2}} \text{ for } 1 \leqslant j \leqslant n\},$$

we have

$$|\Delta_n^{(1)}| \leqslant Cn^{-\frac{3}{2}+5/d}\log n \quad (4{\cdot}7)$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$.

Proof. Note that

$$\hat{\mu}_3 = (1-n^{-1})^{\frac{3}{2}}\left\{\mu_3 + n^{-1}\sum_1^n (X_j^3 - \mu_3) - 3\bar{X} - 3\bar{X}n^{-1}\sum_1^n (X_j^2 - 1) + 2\bar{X}^2\right\}$$
$$\times \left\{1 + n^{-1}\sum_1^n (X_j^2 - 1) - \bar{X}^2\right\}^{-\frac{3}{2}}$$
$$= \mu_3 + n^{-1}\sum_1^n (X_j^3 - \mu_3) - 3\bar{X} - (3\mu_3/2)n^{-1}\sum_1^n (X_j^2 - 1) + \Delta_{n1}, \quad (4{\cdot}8)$$

where $|\Delta_{n1}| \leqslant Cn^{-1+5/d}$ on the set

$$E_1 = \left\{\left|\sum_1^n X_j\right| \leqslant n^{\frac{1}{2}+1/d}\right\} \cap \left\{\left|\sum_1^n (X_j^2 - 1)\right| \leqslant n^{\frac{1}{2}+2/d}\right\} \cap \left\{\left|\sum_1^n (X_j^3 - \mu_3)\right| \leqslant n^{\frac{1}{2}+3/d}\right\}.$$

Furthermore, $|\hat{\mu}_3 - \mu_3| \leqslant Cn^{-\frac{1}{2}+3/d}$ on E_1, and so, with $N^* = na^2/\hat{\mu}_3^2$, we have $N^* \geqslant Cn$ on E_1. Therefore

$$y_{N^*}^+(\alpha) = z - n^{-\frac{1}{2}}\tfrac{1}{6}|\hat{\mu}_3|(1+2z^2) + n^{-1}\hat{\mu}_3^2 a^{-2} z(b_0 + b_1 z^2) + \Delta_{n2},$$

where $|\Delta_{n2}| \leqslant Cn^{-\frac{3}{2}}(\log n)^{r/2}$ uniformly in values of α for which $|z| \leqslant (2\beta\log n)^{\frac{1}{2}}$, for an integer r. But $|N^{-\frac{1}{2}k} - (N^*)^{-\frac{1}{2}k}| \leqslant C_k n^{-\frac{1}{2}k-1}$ on E_1, for each integer $k \geqslant 0$, and thus

$$y_N^+(\alpha) = z - n^{-\frac{1}{2}}\tfrac{1}{6}|\hat{\mu}_3|(1+2z^2) + n^{-1}\hat{\mu}_3^2 a^{-2} z(b_0 + b_1 z^2) + \Delta_{n3},$$

where $|\Delta_{n3}| \leqslant Cn^{\delta-\frac{3}{2}}$ uniformly in values of α for which $|z| \leqslant (2\beta\log n)^{\frac{1}{2}}$. Similarly,

$$y_N^-(\alpha) = z + n^{-\frac{1}{2}}\tfrac{1}{6}|\hat{\mu}_3|(1+2z^2) + n^{-1}\hat{\mu}_3^2 a^{-2} z(b_0 + b_1 z^2) + \Delta_{n4},$$

and combining these estimates with Lemma 1 we find that

$$z_N(\alpha) = z - n^{-\frac{1}{2}}\tfrac{1}{6}\hat{\mu}_3(1+2z^2) + n^{-1}\hat{\mu}_3^2 a^{-2} z(b_0 + b_1 z^2) + \Delta_{n5},$$

where, on the set E_1, $|\Delta_{n5}| \leqslant Cn^{\delta-\frac{3}{2}}$ uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. We may now deduce from (4·8) that

$$z_N(\alpha) = z - n^{-\frac{1}{2}}\tfrac{1}{6}\mu_3(1+2z^2) + n^{-1}\mu_3^2 a^{-2} z(b_0 + b_1 z^2)$$
$$- n^{-\frac{3}{2}}\tfrac{1}{6}(1+2z^2)\sum_1^n \{(X_j^3 - \mu_3) - 3X_j - (3\mu_3/2)(X_j^2 - 1)\} + \Delta_{n6},$$

where $|\Delta_{n6}| \leqslant Cn^{-\frac{3}{2}+5/d}\log n$ uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$, on the set E_1.

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Set $X_j' = X_j I(|X_j| \leqslant n^{\frac{1}{2}})$ and $\mu_r' = E(X_j')^r$. On the set $E_2 = \{|X_j| \leqslant n^{\frac{1}{2}} \text{ for } 1 \leqslant j \leqslant n\}$, we have

$$n^{-\frac{3}{2}}\tfrac{1}{6}(1+2z^2)\Big|\sum_1^n [\{(X_j^3-\mu_3)-3X_j-(3\mu_3/2)(X_j^2-1)\} - \{(X_j'^3-\mu_3')-3X_j-(3\mu_3/2)(X_j'^2-\mu_2')\}]\Big|$$
$$\leqslant C_1 n^{-\frac{1}{2}}(\log n)[E\{|X|^3 I(|X|>n^{\frac{1}{2}})\}+E\{X^2 I(|X|>n^{\frac{1}{2}})\}]$$
$$\leqslant C_2 n^{-\frac{3}{2}}$$

uniformly in $\alpha_n \leqslant z \leqslant 1-\alpha_n$, using Chebyshev's inequality. Lemma 2 is immediate. Note that the set $E^{(1)}$ defined there is actually $E_1 \cap E_2$. ∎

We now develop an expansion for T_n.

LEMMA 3. *There exists a positive constant C such that*

$$\left|(v+x-y^2)^{-\frac{1}{2}}-v^{-\frac{1}{2}}\left\{\sum_{j=0}^3\binom{-\frac{1}{2}}{j}(x/v)^j-(y^2/v)\sum_{j=1}^3 j\binom{-\frac{1}{2}}{j}(x/v)^{j-1}\right\}\right| \leqslant C(x^4+y^4)$$

uniformly in real numbers v, x and y satisfying $\frac{1}{2} \leqslant v \leqslant \frac{3}{2}$, $|x| \leqslant \frac{1}{8}$ and $y^2 \leqslant \frac{1}{8}$.

Proof. Since $(1+a)^{-\frac{1}{2}} = \Sigma_0^3\binom{-\frac{1}{2}}{j}a^j + r(a)$ where $|r(a)| \leqslant Ca^4$ uniformly in $|a| \leqslant \frac{1}{2}$, then

$$(v+x+y^2)^{-\frac{1}{2}} = v^{-\frac{1}{2}}\sum_0^3\binom{-\frac{1}{2}}{j}(x/v-y^2/v)^j + r,$$

where $|r| \leqslant C(x^4+y^4)$ uniformly in $|x/v| \leqslant \frac{1}{4}$ and $y^2/v \leqslant \frac{1}{4}$. Lemma 3 now follows from a binomial expansion of $(x/v-y^2/v)^j$. ∎

Let $Y_j = X_j + n^{-1}\frac{1}{6}(1+2z^2)Z_j$, where $Z_j = (X_j'^3-\mu_3')-3X_j-(3\mu_3/2)(X_j'^2-\mu_2')$, and define

$$\tau = \operatorname{var}(Y_j) = 1+n^{-1}\tfrac{1}{6}(1+2z^2)(2\mu_4-6-3\mu_3^2)+O(n^{-2}\log^2 n) \tag{4·9}$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. The next lemma shows how a statistic based on the X's approximates one based on the Y's.

LEMMA 4. *On the set*

$$E^{(2)} \equiv \left\{\left|\sum_1^n (X_j^2-1)\right| \leqslant n^{\frac{1}{2}+2/d}\right\} \cap \left\{\left|\sum_1^n (X_jZ_j-EX_jZ_j)\right| \leqslant n^{\frac{1}{2}+4/d}\right\}$$
$$\cap \left\{\left|\sum_1^n (Z_j^2-EZ_j^2)\right| \leqslant n^{\frac{1}{2}+6/d}\right\} \cap \left\{\left|\sum_1^n X_j\right| \leqslant n^{\frac{1}{2}+1/d}\right\} \cap \left\{\left|\sum_1^n Z_j\right| \leqslant n^{\frac{1}{2}+3/d}\right\}$$

we have

$$\left|\left(n^{-1}\sum_1^n X_j\right)^2\left\{n^{-1}\sum_1^n (X_j^2-1)\right\}^k - \tau^{-1}\left(n^{-1}\sum_1^n Y_j\right)^2\left\{\tau^{-1}n^{-1}\sum_1^n (Y_j^2-\tau)\right\}^k\right| \leqslant Cn^{-2+4/d}\log n \tag{4·10}$$

for integers $0 \leqslant k \leqslant 3$.

Proof. Since

$$\sum_1^n (Y_j^2-\tau) = \sum_1^n (X_j^2-1) + n^{-1}\tfrac{1}{3}(1+2z^2)\sum_1^n (X_jZ_j-EX_jZ_j)$$
$$+ n^{-2}\tfrac{1}{36}(1+2z^2)^2\sum_1^n (Z_j^2-EZ_j^2),$$

https://doi.org/10.1017/S0305004100063118 Published online by Cambridge University Press

then on the set

$$E_3 = \left\{\left|\sum_1^n (X_j^2-1)\right| \leqslant n^{\frac{1}{2}+2/d}\right\} \cap \left\{\left|\sum_1^n (X_jZ_j - EX_jZ_j)\right| \leqslant n^{\frac{1}{2}+4/d}\right\} \cap \left\{\left|\sum_1^n (Z_j^2 - EZ_j^2)\right| \leqslant n^{\frac{1}{2}+6/d}\right\},$$

we have

$$\left|n^{-1}\sum_1^n (Y_j^2-\tau) - n^{-1}\sum_1^n (X_j^2-1)\right| \leqslant Cn^{-3/2+4/d}\log n$$

and

$$\left|n^{-1}\sum_1^n (X_j^2-1)\right| |\tau^{-1}-1| \leqslant Cn^{-3/2+2/d}\log n,$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. Consequently,

$$\left|\tau^{-1}n^{-1}\sum_1^n (Y_j^2-\tau) - n^{-1}\sum_1^n (X_j^2-1)\right| \leqslant Cn^{-3/2+4/d}\log n$$

and

$$\left|\left\{\tau^{-1}n^{-1}\sum_1^n (Y_j^2-\tau)\right\}^k - \left\{n^{-1}\sum_1^n (X_j^2-1)\right\}^k\right| \leqslant Cn^{-3/2+4/d}\log n \tag{4·11}$$

on the set E_3, uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$ and for integers $0 \leqslant k \leqslant 3$.

On the set

$$E_4 = \left\{\left|\sum_1^n X_j\right| \leqslant n^{\frac{1}{2}+1/d}\right\} \cap \left\{\left|\sum_1^n Z_j\right| \leqslant n^{\frac{1}{2}+3/d}\right\},$$

we have

$$\left|n^{-1}\sum_1^n (X_j-Y_j)\right| \leqslant Cn^{-3/2+3/d}\log n \tag{4·12}$$

and

$$\begin{aligned}\left|\left(n^{-1}\sum_1^n X_j\right)^2 - \tau^{-1}\left(n^{-1}\sum_1^n Y_j\right)^2\right| &\\ \leqslant \tau^{-1}\left|n^{-1}\sum_1^n (X_j-Y_j)\right|\left(\left|n^{-1}\sum_1^n X_j\right| + \left|n^{-1}\sum_1^n Y_j\right|\right) + |\tau^{-1}-1|\left(n^{-1}\sum_1^n X_j\right)^2 & \quad (4·13)\\ \leqslant Cn^{-2+4/d}\log n, &\end{aligned}$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. Combining (4·11) and (4·13) we obtain Lemma 4. Note that $E^{(2)} = E_3 \cap E_4$. ∎

Combining Lemmas 2, 3 and 4 we obtain the following expansion of $T_n - z_N(\alpha)$.

LEMMA 5. *We have*

$$T_n - z_N(\alpha) = \tau^{\frac{1}{2}}\left(n^{-\frac{1}{2}}\sum_1^n Y_j\right) \Big/ \left\{(n-1)^{-1}\sum_1^n (Y_j-\overline{Y})^2\right\}^{\frac{1}{2}} - \{z - n^{-\frac{1}{2}}\tfrac{1}{6}\mu_3(1+2z^2) + n^{-1}\mu_3^2 a^{-2}z(b_0+b_1z^2)\} + \Delta_n^{(2)}, \tag{4·14}$$

where, on the set $E^{(3)} \equiv E^{(1)} \cap E^{(2)}$,

$$|\Delta_n^{(2)}| \leqslant Cn^{-3/2+5/d}\log n \quad \textit{uniformly in} \quad \alpha_n \leqslant \alpha \leqslant 1-\alpha_n. \tag{4·15}$$

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Proof. Let $v_1 = 1$, $v_2 = \tau$, $x_1 = n^{-1}\Sigma_1^n(X_j^2 - 1)$, $x_2 = \Sigma_1^n(Y_j^2 - \tau)$, $y_1 = n^{-1}\Sigma_1^n X_j$ and $y_2 = n^{-1}\Sigma_1^n Y_j$. Then $|x_i|^4 + |y_i|^4 \leqslant C_n^{-2+8/d}$ on $E_3 \cap E_4$ and, by (4·11) and (4·10),

$$|(x_1/v_1)^k - (x_2/v_2)^k| \leqslant Cn^{-3/2+4/d}\log n$$

and

$$|(y_1^2/v_1)(x_1/v_1)^k - (y_2^2/v_2)(x_2/v_2)^k| \leqslant Cn^{-2+4/d}\log n.$$

We may now deduce from Lemma 3 that, on $E_3 \cap E_4$,

$$|(v_1 + x_1 - y_1^2)^{-\frac{1}{2}} - v_2^{\frac{1}{2}}(v_2 + x_2 - y_2^2)^{-\frac{1}{2}}| \leqslant Cn^{-3/2+4/d}\log n \tag{4·16}$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1 - \alpha_n$.

Set $y_3 = y_1 - y_2$, and observe that

$$\begin{aligned}|y_1(v_1 + x_1 - y_1^2)^{-\frac{1}{2}} - y_3 - y_2 v_2^{\frac{1}{2}}(v_2 + x_2 - y_2^2)^{-\frac{1}{2}}| \\ \leqslant |y_2|\,|(v_1 + x_1 - y_1^2)^{-\frac{1}{2}} - v_2^{\frac{1}{2}}(v_2 + x_2 - y_2^2)^{-\frac{1}{2}}| + |y_3|\,|(v_1 + x_1 - y_1^2)^{-\frac{1}{2}} - 1|.\end{aligned} \tag{4·17}$$

It follows from (4·12) that, on E_4, $|y_2| \leqslant Cn^{-\frac{1}{2}+1/d}$ and $|y_3| \leqslant Cn^{-3/2+3/d}\log n$, and from Lemma 3 that, on $E_3 \cap E_4$, $|(v_1 + x_1 - y_1^2)^{-\frac{1}{2}} - 1| \leqslant C_1(|x_1| + y_1^2) \leqslant C_2 n^{-\frac{1}{2}+2/d}$, all inequalities holding uniformly in $\alpha_n \leqslant \alpha \leqslant 1 - \alpha_n$. From these estimates and (4·16) we see that the left-hand side of (4·17) is dominated by $Cn^{-2+5/d}\log n$ uniformly in $\alpha_n \leqslant \alpha \leqslant 1 - \alpha_n$. Remembering that

$$v_1 + x_1 - y_1^2 = n^{-1}\sum_1^n (X_j - \bar{X})^2 \quad \text{and} \quad v_2 + x_2 - y_2^2 = n^{-1}\sum_1^n (Y_j - \bar{Y})^2,$$

we may now deduce that

$$\begin{aligned}\left(n^{-\frac{1}{2}}\sum_1^n X_j\right)\Big/\left\{n^{-1}\sum_1^n (X_j - \bar{X})\right\}^{\frac{1}{2}} + n^{-3/2}\tfrac{1}{6}(1 + 2z^2)\sum_1^n Z_j \\ = \tau^{\frac{1}{2}}\left(n^{-\frac{1}{2}}\sum_1^n Y_j\right)\Big/\left\{n^{-1}\sum_1^n (Y_j - \bar{Y})^2\right\}^{\frac{1}{2}} + \Delta_{n7},\end{aligned}$$

where, on the set $E_3 \cap E_4$, $|\Delta_{n7}| \leqslant Cn^{-\frac{3}{2}+5/d}\log n$ uniformly in $\alpha_n \leqslant \alpha \leqslant 1 - \alpha_n$. This leads quickly to the result,

$$\begin{aligned}T_n + n^{-\frac{3}{2}}\tfrac{1}{6}(1 + 2z^2)\sum_1^n \{(X_j'^3 - \mu_3') - 3X_j - (3\mu_3/2)(X_j'^2 - \mu_2')\} \\ = \tau^{\frac{1}{2}}\left(n^{-\frac{1}{2}}\sum_1^n Y_j\right)\Big/\left\{(n-1)^{-1}\sum_1^n (Y_j - \bar{Y})^2\right\}^{\frac{1}{2}} + \Delta_{n8},\end{aligned}$$

where $|\Delta_{n8}| \leqslant Cn^{-\frac{3}{2}+5/d}\log n$ on the set $E^{(1)} \cap E^{(2)} = E_1 \cap E_2 \cap E_3 \cap E_4$. Combining this estimate with (4·6) and (4·7) we see $T_n - z_N(\alpha)$ has the expansion given in Lemma 5. ‖

The following asymptotic expansion for the studentized mean may be deduced from theorem 5 of Bhattacharya and Ghosh [2].

LEMMA 6. *If $U, U_1, U_2, \ldots$ are independent and identically distributed with zero mean, unit variance and finite eighth moment, if the joint distribution of (U, U^2) satifies Cramér's continuity condition,*

$$\limsup_{|s|+|t|\to\infty} |E\exp(isU + itU^2)| < 1,$$

https://doi.org/10.1017/S0305004100063118 Published online by Cambridge University Press

and if we define $\lambda_3 = E(U^3)$ *and* $\lambda_4 = E(U^4)$, *then*

$$P\left[\left(n^{-\frac{1}{2}}\sum_1^n U_j\right)\Big/\left\{(n-1)^{-1}\sum_1^n (U_j-\overline{U})^2\right\}^{\frac{1}{2}} \leqslant x\right] = \Phi(x) + n^{-\frac{1}{2}}\tfrac{1}{6}\lambda_3(1+2x^2)\,\phi(x)$$
$$+ n^{-1}x\{\tfrac{1}{2}(1-x^2) + \tfrac{1}{12}\lambda_4(x^2-3) + \tfrac{1}{18}\lambda_3^2(3-2x^2-x^4)\}\,\phi(x) + o(n^{-1}) \quad (4{\cdot}18)$$

uniformly in x.

We now begin the proof of Theorem 2 proper. The first step is to obtain an expansion of the distribution function of Student's t-statistic calculated from the Y-sample. This expansion is given at (4·20) below. We proceed by applying Lemma 6 to the case where $U_j = U_j(n,\alpha) = Y_j/\tau^{\frac{1}{2}}$, $1 \leqslant j \leqslant n$. These U_j's form a triangular array with the following properties. (i) For each fixed n and α, $U_1, \ldots, U_n$ are independent and identically distributed with zero mean and unit variance. (ii) $E\{|U_1(n,\alpha)|^9\} < \infty$ uniformly in n and $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$, as may be deduced by observing that $E(X^{10}) < \infty$ and

$$E|n^{-1}(1+2x^2)\,(X_1')^3|^9 \leqslant Cn^{-9}(\log n)^9\, n^{\frac{1}{2}(27-10)} = O(n^{-\frac{1}{2}}\log^9 n)$$

uniformly in $|z| \leqslant (2\beta\log n)^{\frac{1}{2}}$. (iii) The joint characteristic function $\psi(s,t|n,\alpha)$ of (U_1, U_1^2) satisfies

$$\limsup_{|s|+|t|\to\infty} \sup_{n\geqslant m,\, \alpha_n\leqslant\alpha\leqslant 1-\alpha_n} |\psi(s,t|n,\alpha)| < 1 \quad (4{\cdot}19)$$

for some $m \geqslant 1$. (We shall prove this result shortly.) The proof of theorem 2(b) of Bhattacharya and Ghosh [2] may now be reworked, giving rise to the expansion (4·18) uniformly in all x and in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$, as $n\to\infty$. In our case, $\lambda_3 = \mu_3 + o(n^{-\frac{1}{2}})$ and $\lambda_4 = \mu_4 + o(1)$ uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. Consequently,

$$P\left[\left(n^{-\frac{1}{2}}\sum_1^n Y_j\right)\Big/\left\{(n-1)^{-1}\sum_1^n (Y_j-\overline{Y})^2\right\}^{\frac{1}{2}} \leqslant x\right]$$
$$= \Phi(x) + n^{-\frac{1}{2}}\tfrac{1}{6}\mu_3(1+2x^2)\,\phi(x) + n^{-1}x\{\tfrac{1}{2}(1-x^2)$$
$$+ \tfrac{1}{12}\mu_4(x^2-3) + \tfrac{1}{18}\mu_3^2(3-2x^2-x^4)\}\,\phi(x) + o(n^{-1}). \quad (4{\cdot}20)$$

We now return to (4·14) and (4·15), and note that

$$t_n(\alpha) - z(\alpha) = (1/4n)\,z(1+z^2) + o(n^{-1})$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. Since $d > 10$ then, on the set $E_5 \equiv E^{(1)} \cap E^{(2)}$, the remainder $\Delta_n^{(2)}$ is essentially bounded by a deterministic sequence equal to $o(n^{-1})$ uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. Thus, it may be deduced from (4·20) that the error between $P\{T_n \leqslant z_N(\alpha) + t_n(\alpha) - z(\alpha)\}$ and the expansion

$$\Phi(x) + n^{-\frac{1}{2}}\tfrac{1}{6}\mu_3(1+2x^2)\,\phi(x) + n^{-1}x\{\tfrac{1}{2}(1-x^2) + \tfrac{1}{12}\mu_4(x^2-3)$$
$$+ \tfrac{1}{18}\mu_3^2(3-2x^2-x^4)\}\,\phi(x) + o(n^{-1}), \quad (4{\cdot}21)$$

in which

$$x = \{z - n^{-\frac{1}{2}}\tfrac{1}{6}\mu_3(1+2z^2) + n^{-1}\mu_3^2 a^{-2} z(b_0 + b_1 z^2) + n^{-1}\tfrac{1}{4}z(1+z^2)\}/\tau^{\frac{1}{2}}$$
$$= z - n^{-\frac{1}{2}}\tfrac{1}{6}\mu_3(1+2z^2) + n^{-1}z\{\mu_3^2 a^{-2}(b_0 + b_1 z^2) + \tfrac{1}{4}(1+z^2)$$
$$+ \tfrac{1}{12}(1+2z^2)\,(3\mu_3^2 + 6 - 2\mu_4)\} + o(n^{-1}) \quad (4{\cdot}22)$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$ (refer (4·9)), is dominated by $P(\tilde{E}_5) + o(n^{-1})$ uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. We shall prove shortly that

$$P(\tilde{E}_5) = o(n^{-1}). \quad (4{\cdot}23)$$

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When the formula (4·22) is substituted for x in (4·21), we obtain the expression

$$(1-\alpha)+n^{-1}z[\mu_3^2\{a^{-2}(b_0+b_1z^2)+\tfrac{1}{72}(23+16z^2)\}-\tfrac{1}{12}(\mu_4-3)(3z^2+5)]\,\phi(z)+o(n^{-1})$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$. Since $b_0 = -23a^2/72$ and $b_1 = -16a^2/72$ (see Lemma 1) then the coefficient of μ_3^2 vanishes, and so

$$P\{T_n \leqslant z_N(\alpha)+t_n(\alpha)-z(\alpha)\} = (1-\alpha)-n^{-1}\tfrac{1}{12}(\mu_4-3)\,z(3z^2+5)\,\phi(z)+o(n^{-1})$$

uniformly in $\alpha_n \leqslant \alpha \leqslant 1-\alpha_n$.

It remains to consider the cases $\alpha \leqslant \alpha_n$ and $\alpha \geqslant 1-\alpha_n$; we treat only the former. We showed following the definition of the set E_1 that there exists a constant C such that $N \geqslant Cn$ on E_1. Since $\inf z(\alpha) > \{(2+\delta)\log n\}^{\frac{1}{2}}$ for large n, and since $P(E_1) = o(n^{-1})$, then

$$\begin{aligned}\sup_{\alpha\leqslant\alpha_n} &|P\{T_n \leqslant z_N(\alpha)+t_n(\alpha)-z(\alpha)\}-(1-\alpha)+n^{-1}\tfrac{1}{12}(\mu_4-3)\,z(3z^2+5)\,\phi(z)| \\ &\leqslant \sup_{\alpha\leqslant\alpha_n} P\{T_n > z_N(\alpha)+t_n(\alpha)-z(\alpha)\}+o(n^{-1}) \\ &\leqslant \sup_{\alpha\leqslant\alpha_n} P[T_n > \inf_{m\geqslant Cn}\{z_m^+(\alpha), z_m^-(\alpha)\}+t_n(\alpha)-z(\alpha)]+o(n^{-1}). \quad (4\cdot24)\end{aligned}$$

We may deduce from Lemma 1 that

$$\inf_{m\geqslant Cn} z_m^{\pm}(\alpha_n) \sim z(\alpha_n) \sim (2\beta\log n)^{\frac{1}{2}},$$

and so, for large n,

$$\inf_{m\geqslant Cn}\{z_m^+(\alpha), z_m^-(\alpha)\}+t_n(\alpha)-z(\alpha) > \{(2+\delta)\log n\}^{\frac{1}{2}}.$$

Therefore the last-written probability in (4·24) equals $o(n^{-1})$ uniformly in $\alpha \leqslant \alpha_n$.

To complete the proof of Theorem 2 we must check the conditions (4·19) and (4·23). We turn first to (4·19). Since the distribution of X has a nontrivial absolutely continuous component, we may write $dF(x) = pf(x)\,dx+(1-p)\,dG(x)$, where F is the distribution function of X, G is another proper distribution function, f is a density on $(-\infty,\infty)$, and $0 < p < 1$. Let Z be a random variable with density f, and define the function

$$\xi(x) = x+n^{-1}\tfrac{1}{6}(1+2z^2)\{(x^3-\mu_3')-3x-(3\mu_3/2)(x^2-\mu_2')\}.$$

Then

$$\begin{aligned}|\psi(\tau^{\frac{1}{2}}s,\tau t|n,\alpha)| &\leqslant |E\exp\{is\xi(X)+ut\xi^2(X)\}|+2P(|X| > n^{\frac{1}{2}}) \\ &= \left|\int_{-\infty}^{\infty}\exp\{is\xi(x)+it\xi^2(x)\}\{pf(x)\,dx+(1-p)\,dG(x)\}\right|+2P(|X| > n^{\frac{1}{2}}) \\ &\leqslant p|E\exp\{is\xi(Z)+it\xi^2(Z)\}|+(1-p)+2P(|X| > n^{\frac{1}{2}}) \\ &= p\left|E\exp\left\{isZ+itZ^2+i\sum_{j=1}^{6}(c_js+d_jt)Z^j\right\}\right|+(1-p)+2P(|X| > n^{\frac{1}{2}}),\end{aligned}$$

where c_j and d_j are constants depending on n and α and which satisfy

$$\sup_{\alpha_n\leqslant\alpha\leqslant1-\alpha_n}\sum_{j=1}^{6}(|c_j|+|d_j|)\to 0 \quad \text{as} \quad n\to\infty.$$

Therefore (4·19) will follow from the following lemma.

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LEMMA 7. *If Z is an absolutely continuous random variable and k is a positive integer, then*

$$\lim_{\sum_1^k |t_j| \to \infty} \left| E \exp\left(i \sum_1^k t_j Z^j \right) \right| = 0.$$

Proof. Following the lines of most contemporary proofs of the Riemann–Lebesgue lemma (see for example the proof on page 303 of Indritz [**11**]), we see that it suffices to show that, on any compact interval $[a, b]$, and with ζ denoting either the sine or cosine function, we have

$$\lim_{\sum_1^k |t_j| \to \infty} \left| \int_a^b \zeta\left(\sum_1^k t_j x^j \right) dx \right| = 0.$$

If this result fails then it fails along a particular subsequence $t_j(\lambda)$, $1 \leqslant j \leqslant k$, as $\lambda \to \infty$. By passing to a subsubsequence if necessary, we see that we may assume that, for some l in $1 \leqslant l \leqslant k$, we have $\rho_j(\lambda) \equiv t_j(\lambda)/t_l(\lambda) \to \rho_{j0} \in [-1, 1]$ as $\lambda \to \infty$, for $1 \leqslant j \leqslant k$, and also that, for some $\epsilon > 0$,

$$\left| \int_a^b \zeta\left\{ \sum_1^k t_j(\lambda)\, x^j \right\} dx \right| > \epsilon \tag{4·25}$$

for all λ. We may divide the interval $(a, b]$ up into k intervals $I_j = (a_{j-1}, a_j]$, $1 \leqslant j \leqslant k$, such that $a = a_0 \leqslant a_1 \leqslant \ldots \leqslant a_k = b$ and the polynomial $p_\lambda(x) = \Sigma_{j=1}^k \rho_j(\lambda) x^j$ is strictly monotone on each I_j. Possibly some of the I_j's are empty. The endpoints a_j depend on λ but, since $p_\lambda(x) \to p_0(x) \equiv \Sigma_{j=1}^k \rho_{j0} x^j$ as $\lambda \to \infty$, we may choose the indices j such that each a_j converges to a number $a_{j0} \in [a, b]$ as $\lambda \to \infty$. The result (4·25) will be contradicted, and the proof of Lemma 4 completed, if we show that

$$\int_{a_{j-1}}^{a_j} \zeta\{t(\lambda)\, p_\lambda(x)\}\, dx \to 0$$

as $\lambda \to \infty$ for $1 \leqslant j \leqslant k$, where $t(\lambda) \equiv t_l(\lambda)$. Clearly this is equivalent to proving that

$$\int_{a_{j-1,0}}^{a_{j0}} \zeta\{t(\lambda)\, p_\lambda(x)\}\, dx \to 0.$$

Therefore we may assume that $a_{j0} > a_{j-1,0}$, and without loss of generality that p is increasing on $[a_{j-1,0}, a_{j0}]$, and show that for each small, positive ϵ

$$\int_{a_{j-1,0}+\epsilon}^{a_{j0}-\epsilon} \zeta\{t(\lambda)\, p_\lambda(x)\}\, dx \to 0.$$

Now, $(d/dx) p_\lambda(x)$ is bounded away from zero on $[a_{j-1,0}+\epsilon, a_{j0}-\epsilon]$ uniformly in large λ, and so the derivative of $q_\lambda(y) \equiv p_\lambda^{-1}(y)$ is bounded on

$$J_j \equiv [p_\lambda(a_{j-1,0}+\epsilon),\, p_\lambda(a_{j0}-\epsilon)]$$

uniformly in large λ. Hence

$$\begin{aligned} \int_{a_{j-1,0}+\epsilon}^{a_{j0}-\epsilon} \zeta\{t(\lambda) p_\lambda(x)\}\, dx &= \int_{J_j} \zeta\{t(\lambda) y\}\, q_\lambda'(y)\, dy \\ &= \int_{p_0(a_{j-1,0}+\epsilon)}^{p_0(a_{j0}-\epsilon)} \zeta\{t(\lambda) y\}\, (d/dy)\{p_0^{-1}(y)\}\, dy + o(1) = o(1) \end{aligned}$$

as $\lambda \to \infty$, using the Riemann–Lebesgue lemma. This proves Lemma 7. ∎

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Finally we prove (4·23). The set E_5 consists of the set $\{|X_j| \leqslant n^{\frac{1}{2}} \text{ for } 1 \leqslant j \leqslant n\}$ intersected with several sets of the form $\{|\Sigma_1^n V_j| \leqslant n^{\frac{1}{2}+k/d}\}$, where $k \geqslant 1$ and $V_1, \ldots, V_n$ are independent and identically distributed random variables with zero mean. The result (4·23) is proved by noting that

$$nP(|X| > n^{\frac{1}{2}}) \leqslant n^{-4} E(X^{10}),$$

and

$$P\left(\left|\sum_1^n V_j\right| > n^{\frac{1}{2}+k/d}\right) \leqslant n^{-1} E\left\{\left|n^{-\frac{1}{2}} \sum_1^n V_j\right|^{d/k} I\left(\left|n^{-\frac{1}{2}} \sum_1^n V_j\right| > n^{k/d}\right)\right\}. \tag{4·26}$$

It can be checked that in each individual case the variable $n^{-\frac{1}{2}} \Sigma_1^n V_j$ is asymptotically normal $N(0, v)$, where $0 < v < \infty$, and that (d/k)th moments converge. Therefore the right-hand side of (4·26) equals $o(n^{-1})$ as $n \to \infty$.

We close this section by outlining the proof of (3·4). Let

$$V_1 = n^{-\frac{1}{2}} \sum_1^n Z_j \quad \text{and} \quad V_2 = n^{-\frac{1}{2}} \sum_1^n (Z_j^2 - 1),$$

and observe that

$$\begin{aligned} T_n^0 &= V_1 + n^{-\frac{1}{2}}(-\tfrac{1}{2} V_1 V_2) + n^{-1}\{(\tfrac{1}{2} + \xi_1) V_1^3 + (\tfrac{3}{8} + \xi_2) V_1 V_2^2\} + O_p(n^{-\frac{3}{2}}) \\ &= W_n' + O_p(n^{-\frac{3}{2}}), \end{aligned}$$

say. The statistic W_n' equals that defined by (1·9) of Bhattacharya and Ghosh [2], and so, up to terms of order $n^{-\frac{3}{2}}$, T_n^0 has the same expansion as W_n'. It may be shown after some algebra that

$$\begin{aligned} E(W_n') &= -n^{-\frac{1}{2}} \tfrac{1}{2} \lambda_3 + O(n^{-\frac{3}{2}}), \\ E(W_n')^2 &= 1 + n^{-1}\{2\lambda_3^2(1 + 2\xi_2) + 3(1 + 2\xi_1) + 2\xi_2(\lambda_4 - 1)\} + O(n^{-\frac{3}{2}}), \\ E(W_n')^3 &= -n^{-\frac{1}{2}} \tfrac{7}{2} \lambda_3 + O(n^{-\frac{3}{2}}) \end{aligned}$$

and

$$E(W_n')^4 = 3 + n^{-1}\{60\xi_1 + 12\xi_2(\lambda_4 - 1 + 4\lambda_3^2) + 28\lambda_3^2 + 30 - 2\lambda_4\} + O(n^{-\frac{3}{2}}).$$

Therefore the cumulants of W_n' are

$$\begin{aligned} \kappa_1' &= -n^{-\frac{1}{2}} \tfrac{1}{2} \lambda_3 + O(n^{-\frac{3}{2}}), \\ \kappa_2' &= 1 + n^{-1}\{\lambda_3^2(\tfrac{7}{4} + 4\xi_2) + 3(1 + 2\xi_1) + 2\xi_2(\lambda_4 - 1)\} + O(n^{-\frac{3}{2}}), \\ \kappa_3' &= -n^{-\frac{1}{2}} 2\lambda_3 + O(n^{-\frac{3}{2}}) \end{aligned}$$

and

$$\kappa_4' = n^{-1}(24\xi_1 + 24\xi_2\lambda_3^2 + 12\lambda_3^2 + 12 - 2\lambda_4) + O(n^{-\frac{3}{2}}).$$

It is now relatively easy to show that W_n' has the expansion (3·4), up to terms of order $n^{-\frac{3}{2}}$.

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