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Is a typical bi-Perron algebraic unit a pseudo-Anosov dilatation?

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Abstract. In this note, we deduce a partial answer to the question in the title. In particular, we show that asymptotically almost all bi-Perron algebraic units whose characteristic polynomial has degree at most $2n$ do not correspond to dilatations of pseudo-Anosov maps on a closed orientable surface of genus n for $n \geq 10$. As an application of the argument, we also obtain a statement on the number of closed geodesics of the same length in the moduli space of area-one abelian differentials for low-genus cases.

Thurston's celebrated theorem [**21**] classified self-homeomorphisms of compact orientable surfaces up to isotopy. As long as a homeomorphism does not admit a finite power which is isotopic to the identity and does not admit a finite union of disjoint simple closed curves which are preserved up to isotopy, it is isotopic to a so-called *pseudo-Anosov map*. Associated to a pseudo-Anosov map is a positive number called its *dilatation*. The logarithm of the dilatation of a pseudo-Anosov map is the topological entropy of the map. See [**23**] for details.

We call the number λ a *pseudo-Anosov dilatation* if it is the dilatation constant for a pseudo-Anosov map on some compact orientable surface. We call an algebraic integer λ *bi-Perron* if all the Galois conjugates of λ are contained in an annulus of outer radius λ itself and inner radius $1/\lambda$. Fried [**8**] showed that every pseudo-Anosov dilatation is a bi-Perron algebraic unit, and he conjectured that this is also a sufficient condition: every bi-Perron algebraic unit is realized as the dilatation constant of some pseudo-Anosov map.

This turned out to be a difficult question and there are many related works. Thurston [**22**] gave an answer to an $\mathrm{Out}(F_n)$-version of this question. In [**22**], Thurston also gave a few examples of constructions of pseudo-Anosov maps for given bi-Perron

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algebraic units (here the numbers are given as the slopes of some post-critically finite piecewise linear self-homeomorphisms of the unit interval). These examples motivated the authors' previous work on constructing pseudo-Anosov maps in [**2**] for a bi-Perron algebraic unit given as the leading eigenvalue of a Perron–Frobenius matrix satisfying some additional properties. Both constructions in [**22**] and [**2**] are similar to that given in [**4**]. See also, for instance, [**1**, **13**–**17**, **19**, **20**] for various constructions of pseudo-Anosov maps.

In this short note, we formulate an easier version of Fried's question, and give a statistical answer to it. To state our result precisely, we introduce some notation.

Let $n \geq 2$ be fixed, and R be any positive real number. Define $\mathcal{B}_n(R)$ to be the set of bi-Perron algebraic units no larger than R whose characteristic polynomial has degree at most $2n$. Here, by the characteristic polynomial of a bi-Perron algebraic unit λ, we mean the monic palindromic integral polynomial whose leading root is λ and has the lowest degree among all such polynomials. And let $\mathcal{D}_n(R)$ be the set of dilatations no larger than R of pseudo-Anosov maps with orientable invariant foliations on a closed orientable surface S_n of genus n. Similarly, let $\mathcal{D}'_n(R)$ be the set of dilatations no larger than R of pseudo-Anosov maps, not necessarily with orientable invariant foliation, on a closed orientable surface with genus n.

We remark that $\mathcal{D}_n(R)$ is contained in $\mathcal{B}_n(R)$, and $\mathcal{D}'_n(R)$ is contained in $\mathcal{B}_{3n-3}(R)$. A pseudo-Anosov dilatation λ on a surface of genus n is a root of an integral palindromic polynomial of degree at most $2n$ if its invariant foliations are orientable. This is because λ is the leading eigenvalue of the induced symplectic action on the homology group of the surface, which is $\mathbb{Z}^{2n}$. If we do not require the invariant foliation to be orientable, the upper bound on degree is $6n - 6$: we can reduce this case to the case of orientable foliation by taking a double cover of the surface, and this bound follows from the fact that a quadratic differential on a surface of genus n has at most $2n$ zeros, which is due to Gauss and Bonnet together with the Riemann–Hurwitz formula (compare [**7**]).

Note that Fried's conjecture is equivalent to $\mathcal{B}_n(R)$ being contained in $\mathcal{D}_m(R)$ for some large enough m. But a priori, m could be arbitrarily large and we do not know how to prove or disprove the claim. Instead we show the following theorem.

THEOREM 1. *Let $\mathcal{B}_n(R)$, $\mathcal{D}'_n(R)$ and $\mathcal{D}_n(R)$ be as above. Then*

(1)
$$\lim_{R\to\infty} \frac{|\mathcal{D}_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0,$$
where $4m - 3 \leq n(n+1)/2$. In particular, $\lim_{R\to\infty} (|\mathcal{D}_n(R)|/|\mathcal{B}_n(R)|) = 0$, for all $n \geq 6$.

(2)
$$\lim_{R\to\infty} \frac{|\mathcal{D}'_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0,$$
where $6m - 6 \leq n(n+1)/2$. In particular, $\lim_{R\to\infty} (|\mathcal{D}'_n(R) \cap \mathcal{B}_n(R)|/|\mathcal{B}_n(R)|) = 0$, for all $n \geq 10$.

Here $|A|$ denotes the cardinality of A for a finite set A.

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Theorem 1 says that asymptotically almost all bi-Perron algebraic units whose characteristic polynomial has degree at most $2n$ do not correspond to dilatations of pseudo-Anosov maps on a surface of genus n for all $n \geq 10$, and for $n \geq 6$ if the invariant foliation is further assumed to be orientable. It would be interesting to see if the statement still holds for lower genera.

Let Γ_n be the set of all periodic orbits for the Teichmüller flow on the moduli space of area-one abelian differentials on S_n. Then there exists a surjective map $\Gamma_n \to \bigcup_{R>0} \mathcal{D}_n(R)$ defined by $\gamma \mapsto e^{\ell(\gamma)}$ where $\ell(\gamma)$ is the length of the orbit $\gamma \in \Gamma_n$. Let $\Gamma_n(R)$ be the preimage of $\mathcal{D}_n(R)$ of this map.

By $f \sim g$ we mean that there exists C such that $(1/C)f(x) \leq g(x) \leq Cf(x)$ when $x \gg 0$. By $f \lesssim g$ we mean $f = O(g)$ when $x \to \infty$. To the best of our knowledge, the following theorem (rephrased for our purposes) is independently due to Eskin, Mirzakhani and Rafi [**6**] and Hamenstädt [**11**].

THEOREM 2. [**6**, **11**] $|\Gamma_n(R)| \sim R^{4n-3}/((4n-3)\log R)$.

We give a brief explanation of the above statement. In [**6**] and [**11**], it was stated that when restricted to each connected component of the strata of the moduli space of area-one abelian differentials on S_n,

$$\begin{aligned} |\Gamma_n(R)| &= |\{\gamma \in \Gamma_n : e^{\ell(\gamma)} \leq R\}| \\ &= |\{\gamma \in \Gamma_n : \ell(\gamma) \leq \log R\}| \\ &\sim \frac{e^{(2n+\ell-1)\log R}}{(2n+\ell-1)\log R} = \frac{R^{2n+\ell-1}}{(2n+\ell-1)\log R}. \end{aligned}$$

Here ℓ is the maximum number of zeros of an area-one abelian differential on S_n, so it is $2n-2$. See [**5**, **6**, **10**, **18**, **24**] for the relevant background to this theorem.

As a result, we have $|\Gamma_n(R)| \sim R^{4n-3}/((4n-3)\log R)$ on each connected component of the strata. But for fixed n, there exist only a finite number of such components. Therefore we get $|\Gamma_n(R)| \sim R^{4n-3}/((4n-3)\log R)$ without restricting to the components. Note that since n is fixed, we can just say $|\Gamma_n(R)| \sim R^{4n-3}/\log R$.

As a direct corollary, we have the following result.

COROLLARY 3. $|\mathcal{D}_n(R)| \lesssim R^{4n-3}/\log R$.

In the exactly same way, we can obtain an analogue of Corollary 3 for $|\mathcal{D}'_n(R)|$ from the following theorem, which is due to Eskin and Mirzakhani [**5**].

THEOREM 4. [**5**, Theorem 1.1] $|\{$*Closed geodesics in the moduli space of genus n surface of length at most* $\log(R)\}| \sim R^{6n-6}/((6n-6)\log(R))$.

And as a direct corollary, just like Corollary 3, we have our next result.

COROLLARY 5. $|\mathcal{D}'_n(R)| \lesssim R^{6n-6}/\log R$.

We remark that this does not imply $|\mathcal{D}_n(R)| \sim R^{4n-3}/\log R$ or $|\mathcal{D}'_n(R)| \sim R^{6n-6}/\log R$, since each element in $\mathcal{D}_n(R)$ or $\mathcal{D}'_n(R)$ may correspond to a lot of different closed geodesics in the moduli space.

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Now we study how $|\mathcal{B}_n(R)|$ grows. Let $P_n(R)$ be the set of Perron polynomials of degree n with roots no larger than R ($x > 1$ is Perron if it is the root of a monic irreducible polynomial with integer coefficients, so that the other roots of the polynomial are (strictly) less than x in absolute value).

LEMMA 6. $|P_n(R)| \sim R^{n(n+1)/2}$.

Proof. $|P_n(R)| \lesssim R^{n(n+1)/2}$. Due to Vieta's formula (see [**9**], for instance) the absolute value of the coefficient of x^k of a monic polynomial of degree n with all roots no larger than R is $\lesssim R^{n-k}$. Hence, the total number of such polynomials must be $\lesssim \prod_k R^{n-k} = R^{n(n+1)/2}$.

$R^{n(n+1)/2} \lesssim |P_n(R)|$. Let a_k be the coefficient of x^k in a monic polynomial of degree n (so $a_n = 1$). By Rouché's theorem (see [**3**] for the statement and a proof), when

$$1 > \left|\frac{a_0}{R^n}\right| + \left|\frac{a_1}{R^{n-1}}\right| + \cdots + \left|\frac{a_{n-1}}{R}\right|, \tag{1}$$

$$\left(\frac{1}{2}\right)^{n-1}\left|\frac{a_{n-1}}{R}\right| > \left|\frac{a_0}{R^n}\right| + \cdots + \left|\frac{a_{n-2}}{R^2}\right|\left(\frac{1}{2}\right)^{n-2} + \left(\frac{1}{2}\right)^n, \tag{2}$$

and

$$\left(\frac{1}{3}\right)^{n-1}\left|\frac{a_{n-1}}{R}\right| > \left|\frac{a_0}{R^n}\right| + \cdots + \left|\frac{a_{n-2}}{R^2}\right|\left(\frac{1}{3}\right)^{n-2} + \left(\frac{1}{3}\right)^n, \tag{3}$$

one root λ of this polynomial has magnitude between $R/2$ and R while all other roots have magnitude smaller than $R/3$. Hence λ must be real. If $\lambda < 0$ one can multiply a_k by $(-1)^{n-k}$ to get a polynomial with a root $-\lambda$. Hence, half of those polynomials have a leading positive real root, so they are in $P_n(R)$.

Inequalities (1)–(3) are satisfied if and only if the point $(a_0/R^n, \ldots, a_{n-1}/R)$ lies in a non-empty open subset $U \subset [-1, 1]^n$. The number of such points as $R \to \infty$ converges to the volume of this open subset divided by the co-volume of the lattice $\mathbb{Z}/R^n \times \mathbb{Z}/R^{n-1} \times \cdots \times \mathbb{Z}/R$, and the co-volume of this lattice is $R^{-n(n+1)/2}$. □

LEMMA 7. $\lim_{R\to\infty} (|\{\textit{reducible elements in } P_n(R)\}|/|P_n(R)|) = 0$.

Proof. Because any reducible monic integer polynomial can be written as the product of two monic integer polynomials of lower degree, we have

$$|\{\text{reducible elements in } P_n(R)\}| \le \sum_k |b_k(R)||b_{n-k}(R)|$$

where $b_k(R)$ is the set of monic polynomials with roots bounded by R. The first part of the proof of the previous lemma implies that $|b_k(R)| \lesssim R^{k(k+1)/2}$, hence $|\{\text{reducible elements in } P_n(R)\}| \sim o(R^{n(n+1)/2})$. □

Let $\mathcal{P}_n(R)$ be the set of Perron numbers of degree n no larger than R. They have a one–one correspondence with irreducible elements in $P_n(R)$ which by Lemma 7 constitute almost all of $P_n(R)$ asymptotically. Hence by Lemma 6, $|\mathcal{P}_n(R)| \sim R^{n(n+1)/2}$.

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LEMMA 8. $|\mathcal{B}_n(R)| \sim R^{n(n+1)/2}$.

Proof. When $x > 1$, $1/x < 1$. Hence, $x \in \mathcal{B}_n(R)$ implies that $x + 1/x \in \mathcal{P}_n(R+1)$, hence $|\mathcal{B}_n(R)| \lesssim (R+1)^{n(n+1)/2} \sim R^{n(n+1)/2}$.

On the other hand, from the proof of Lemma 6 and the discussion above, the number of Perron numbers of degree n which lie between R and $R/2$ and have all conjugates smaller than $R/3$ is $\sim R^{n(n+1)/2}$. When R is sufficiently large, each such Perron number y corresponds to a bi-Perron number x by the relation $x + 1/x = y$. And in fact x is an algebraic unit (so it is in $\mathcal{B}_n(R)$). Note that both roots of the polynomial $x^2 - yx + 1$ are algebraic integers, since it is a monic polynomial with algebraic integer coefficients. Furthermore, the product of these roots is 1, so they must be algebraic units. Hence $|\mathcal{B}_n(R)| \gtrsim R^{n(n+1)/2}$. □

Now we are ready to prove our main theorem.

Proof of Theorem 1. By Corollary 3 and Lemma 8, we get

$$\frac{|\mathcal{D}_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} \lesssim \frac{R^{4m-3}}{\log R \cdot R^{n(n+1)/2}}.$$

The right-hand side goes to 0 as R goes to $+\infty$ as long as we have $4m - 3 \le n(n+1)/2$. When $m = n$, this inequality is satisfied if and only if $n \ge 6$ (recall that n is always assumed to be at least 2), which proves part (1). Part (2) follows from the same argument above but using Corollary 5 instead of Corollary 3. □

Recall that Γ_n is the set of all closed geodesics in the moduli space of area-one abelian differentials on the surface S_n, and $\Gamma_n(R)$ is the subset of Γ_n which consists of the closed geodesics of length no larger than $\log R$. For each $\gamma \in \Gamma_n$, let m_γ be the number $|\{\gamma' \in \Gamma_n : \ell(\gamma') = \ell(\gamma)\}|$.

We remark that if m_γ were uniformly bounded, one could have obtained Theorem 1(1) for $n \ge 2$ instead of $n \ge 6$ using [**12**, Theorem 1(1)]. But, at least in the low-genus cases, this is not true. As an application of our argument, we obtain the following theorem.

THEOREM 9. *Suppose that $n \le 5$. For any positive integer k, the set $\{\gamma \in \Gamma_n : m_\gamma \ge k\}$ is typical, that is,*

$$\lim_{R\to\infty} \frac{|\{\gamma \in \Gamma_n(R) : m_\gamma \ge k\}|}{|\Gamma_n(R)|} \to 1.$$

Proof. Suppose otherwise. Then, for some k, we have

$$\limsup \frac{|\{\gamma \in \Gamma_n(R) : m_\gamma < k\}|}{|\Gamma_n(R)|} > 0.$$

But this implies that

$$\limsup \frac{|\mathcal{D}_n(R)|}{|\Gamma_n(R)|} \ge \limsup \frac{(1/k)|\{\gamma \in \Gamma_n(R) : m_\gamma < k\}|}{|\Gamma_n(R)|} > 0.$$

On the other hand, we know that $\mathcal{D}_n(R) \subset \mathcal{B}_n(R)$. As a consequence, $\lim (|\mathcal{B}_n(R)|/|\Gamma_n(R)|) \ge \limsup (|\mathcal{D}_n(R)|/|\Gamma_n(R)|)$.

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By Corollary 3 and Lemma 8, we get

$$\frac{|\mathcal{B}_n(R)|}{|\Gamma_n(R)|} \sim \frac{R^{n(n+1)/2}\log R}{R^{4n-3}} \to 0, \quad \text{for } n \leq 5,$$

a contradiction. □

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