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Asymptotic Enumeration of Eulerian Circuits in the Complete Graph

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We determine the asymptotic behaviour of the number of Eulerian circuits in a complete graph of odd order. One corollary of our result is the following. If a maximum random walk, constrained to use each edge at most once, is taken on K_n, then the probability that all the edges are eventually used is asymptotic to $e^{3/4}n^{-1/2}$. Some similar results are obtained about Eulerian circuits and spanning trees in random regular tournaments. We also give exact values for up to 21 nodes.

1. Introduction

Let D be a directed graph with node set VD and edge set ED. A *walk of length m on D* is a sequence $v_0, v_1, \ldots, v_m$ of nodes of D such that $(v_{i-1}, v_i) \in ED$ for $1 \leqslant i \leqslant m$. This walk is a *circuit* if $v_m = v_0$ and $(v_{i-1}, v_i) \neq (v_{j-1}, v_j)$ for $1 \leqslant i < j \leqslant m$. An *Eulerian circuit* in D is a circuit of length $|ED|$. Two Eulerian circuits are called *equivalent* if one is a cyclic permutation of the other. It is clear that the size of such an equivalence class equals the common length of the walks in the class. Let $\mathrm{Eul}(D)$ denote the number of equivalence classes of Eulerian circuits in D.

If G is an undirected graph, we can define the concepts of *walk*, *circuit*, and *Eulerian circuit* and $\mathrm{Eul}(G)$ in the same way. The only clarification needed is that a walk $v_0, v_1, \ldots, v_m$ is only a circuit if $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$ for $1 \leqslant i < j \leqslant m$; that is, edges may be traversed in one direction only. Also note that an Eulerian circuit and its reverse are counted separately for $n > 1$ in an undirected graph.

A *tournament* is a digraph in which, for each pair of distinct nodes v and w, either (v, w) or (w, v) is an edge, but not both. A tournament is *regular* if the in-degree is equal

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to the out-degree at each node. Each Eulerian circuit in K_n induces a regular tournament by orienting the edges according to the directions in which they are traversed.

It is clear that $\mathrm{Eul}(K_n) = 0$ if n is even. In this paper we determine the asymptotic value of $\mathrm{Eul}(K_n)$ as $n \to \infty$ with n odd. In Section 2 we express $\mathrm{Eul}(K_n)$ in terms of an n-dimensional integral using Cauchy's formula. The value of the integral is estimated in Sections 3 and 4. In Section 5 we present and discuss the major results, and in the final section we present the exact values to 21 nodes and compare them against the asymptotic formula.

As the appearance of Euler's name suggests, this problem is one of the oldest in graph theory. In his 1736 paper on the famous Königsberg Bridges Problem, Euler [3] proved that $\mathrm{Eul}(K_n) = 0$ for even n and stated without proof a theorem implying that $\mathrm{Eul}(K_n) > 0$ for odd n. A proof of the latter result was not provided until a paper of Hierholzer in 1873 [4]. English translations of these two papers can be found in [2].

$\mathrm{Eul}(K_7)$ was calculated by Reiss [12] as the number of legal circular arrangements of the 21 dominoes (doubles excluded) over the set $\{1, \ldots, 7\}$. Reiss's paper was dated 1859, and published posthumously. He viewed the problem as that of counting circular sequences from the above set in which each element appears 3 times and each unordered pair appears as a consecutive pair exactly once. He then broke the enumeration down further by considering all the subsequences that could be formed from $\{1, 2, 3\}$, and then all the ways that subsequences from $\{4, 5, 6, 7\}$ could be interpolated. Lucas [9] (pp. 125–128) attributes to C.-A. Laisant the observation that the circular domino arrangements with doubles excluded could be thought of as Eulerian circuits in a complete graph. Taking this point of view, greatly simplified calculations of $\mathrm{Eul}(K_7)$ were obtained by l'abbé Jolivald in 1885, and by Tarry [16] a year later. Lucas also reported the value of $\mathrm{Eul}(K_9)$, calculated by Jolivald and Tarry independently, which apparently neither published.

Tarry's method of calculating $\mathrm{Eul}(K_n)$ was a recurrence based on deleting a vertex and pairing its incident edges in all possible ways. A modern description and extension of Tarry's method can be found in [7]. It could be used on the computer to find $\mathrm{Eul}(K_n)$ up to at least $n = 15$, but in Section 6 we will report on another method that is feasible for larger orders.

A 1969 paper of Sorokin [15] claims to determine $\mathrm{Eul}(K_n)$ exactly, but it is not correct. Firstly, the formula stated for the number of regular tournaments is wrong (as first noticed by Liskovets [8]), and secondly there is a false assumption that different regular tournaments have the same number of Eulerian circuits. Sorokin gives no proofs, so we have no way to analyse his paper further.

Another approach to the problem was provided by Shishov and Thuan [13], who showed how $\mathrm{Eul}(G)$, for any graph G, can be written as the solution of a triangular set of linear equations. Unfortunately, the number of equations is extremely large for complete graphs. It is an interesting open question to determine whether Shishov and Thuan's method can be used for reasonable exact or asymptotic counting.

In this paper, the only equivalences between Eulerian circuits that we will consider are cyclic permutations, as defined above. However, one could also consider equivalences defined by permutations of the labellings of the graph. Counts of equivalence classes of this type up to $n = 7$ are given in [18]. One would expect that, as $n \to \infty$, almost no

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Eulerian circuits admit symmetries and that our asymptotic value need only be divided by $(n-1)!$ or $n!$ (depending on whether the starting point is fixed), but this remains conjectural.

2. The result expressed as an integral

A *directed tree with root* v is a connected directed graph T such that $v \in VT$ has out-degree zero, and each other node has out-degree one. Thus, T is a tree that has each edge oriented towards v.

Let D be a directed graph with n nodes, and let $v \in VD$. A *directed spanning tree of* D *with root* v is a spanning subgraph of D which is a directed tree with root v.

The following famous theorem, sometimes called the BEST Theorem, is due to de Bruijn, van Aardenne-Ehrenfest, Smith and Tutte [1, 14].

Theorem 2.1. *Let D be a directed graph with nodes $v_1, v_2, \ldots, v_n$. Suppose that there are numbers $d_1, d_2, \ldots, d_n$ such that, for every node v_i, both the in-degree and the out-degree of v_i are equal to d_i. Let $t_i = t_i(D)$ be the number of directed spanning trees of D rooted at v_i. Then t_i is independent of i, and*

$$\mathrm{Eul}(D) = t_i \prod_{j=1}^{n} (d_j - 1)!\,.$$

Define $\mathscr{T}_n$ to be the set of directed rooted trees on nodes $v_1, v_2, \ldots, v_n$, with the root being v_n. Note that these are just unique orientations of the undirected trees and hence $|\mathscr{T}_n| = n^{n-2}$. For $T \in \mathscr{T}_n$, define $RT(T)$ to be the number of regular tournaments of n nodes that contain T.

From Theorem 2.1, we find that in the case of a regular tournament R of odd order n, $\mathrm{Eul}(R) = (\frac{1}{2}n - \frac{3}{2})!^n t_n(R)$. Denote by $\mathscr{R}_n$ the set of all regular tournaments of order n, and group the Eulerian circuits according to the induced tournaments. This gives

$$\mathrm{Eul}(K_n) = \sum_{R \in \mathscr{R}_n} \mathrm{Eul}(R) = (\tfrac{1}{2}n - \tfrac{3}{2})!^n \sum_{R \in \mathscr{R}_n} t_n(R).$$

Regrouping the terms of the final summation according to the directed subtrees rooted at v_n, we find that

$$\mathrm{Eul}(K_n) = \left(\tfrac{1}{2}n - \tfrac{3}{2}\right)!^n \sum_{T \in \mathscr{T}_n} RT(T).$$

For $n \geqslant 1$ and $x \geqslant 0$, define $U_n(x) = \{(x_1, x_2, \ldots, x_n) : |x_i| < x \text{ for all } i\}$. The value of $RT(T)$ is the constant term in

$$\prod_{1 \leqslant j < k \leqslant n} \left(x_j^{-1} x_k + x_j x_k^{-1}\right) \prod_{jk \in ET} \frac{x_j x_k^{-1}}{x_j^{-1} x_k + x_j x_k^{-1}},$$

which we can extract via Cauchy's theorem using the unit circle as a contour for each variable. Making the substitution $x_j = e^{i\theta_j}$ for each j, we find

$$\mathrm{Eul}(K_n) = \left(\tfrac{1}{2}n - \tfrac{3}{2}\right)!^n \pi^{-n} 2^{\binom{n}{2} - n + 1} S,$$

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where

$$S = \int_{U_n(\pi/2)} \prod_{1\leqslant j<k\leqslant n} \cos\Delta_{jk} \sum_{T\in\mathscr{T}_n} \prod_{jk\in ET} \left(1 + i\tan\Delta_{jk}\right) d\boldsymbol{\theta}, \tag{2.1}$$

having put $\Delta_{jk} = \theta_j - \theta_k$ and using the fact that the integrand is unchanged by the substitutions $\theta_j \mapsto \theta_j + \pi$ for odd n.

We approach the integral by first estimating it in the region that will turn out to be the asymptotically significant one. Then we bound the integral over the remaining regions and show that it is vanishingly small in comparison.

3. The dominant part of the integral

Define $V_0 = \{\boldsymbol{\theta} \in U_n(\pi/2) : |\Delta_{jn}| \leqslant n^{-1/2+\epsilon} \text{ for } j = 1,\ldots,n-1\}$, and let S_0 denote the contribution to S of $\boldsymbol{\theta} \in V_0$. Since the integrand is invariant under uniform translation of all the θ_js mod π, we can fix $\theta_n = 0$ and multiply by its range π. Thus we have

$$S_0 = \pi \int_{U_{n-1}(n^{-1/2+\epsilon})} \prod_{1\leqslant j<k\leqslant n} \cos\Delta_{jk} \sum_{T\in\mathscr{T}_n} \prod_{jk\in ET} \left(1 + i\tan\Delta_{jk}\right) d\boldsymbol{\theta}', \tag{3.1}$$

where $\boldsymbol{\theta}' = (\theta_1,\ldots,\theta_{n-1})$ and $\theta_n = 0$.

The sum over $\mathscr{T}_n$ in the integrand of (3.1) can be expressed as a determinant, as shown by Tutte [17].

Theorem 3.1. *Let w_{jk} $(1 \leqslant j,k \leqslant n,\ j \neq k)$ be arbitrary. Define the $n \times n$ matrix A by*

$$A_{jk} = \begin{cases} -w_{jk}, & \text{for } k \neq j, \\ \sum_{r\neq j} w_{jr}, & \text{for } k = j, \end{cases}$$

the sum being over $1 \leqslant r \leqslant n$ with $r \neq j$. For any r with $1 \leqslant r \leqslant n$, let M_r denote the principal minor of A formed by removing row r and column r. Then

$$\det(M_r) = \sum_T \prod_{jk\in ET} w_{jk},$$

where the sum is over all directed trees T with $VT = \{1,2,\ldots,n\}$ and root r.

Let I denote the identity matrix, and J the matrix with every entry 1. In each usage, the order will be clear from the context. The following lemma will be applied to estimate the determinant of a matrix close to the identity matrix.

Lemma 3.1. *Let $\|\cdot\|$ denote any matrix norm. Let X be an $n \times n$ matrix over the complex numbers such that $\|X\| < 1$. For fixed $m \geqslant 2$,*

$$\det(I + X) = \exp\left(\sum_{r=1}^{m-1} \frac{(-1)^{r+1}}{r} \operatorname{tr} X^r + E_m(X)\right),$$

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where tr *denotes the trace function, and*

$$|E_m(X)| \leqslant \frac{n}{m}\,\frac{\|X\|^m}{1-\|X\|}.$$

Proof. Since $\|X\| < 1$, $I+X$ is nonsingular. The trace and the determinant of $I+X$ are, respectively, the sum and the product of the eigenvalues of $I+X$, and so $\det(I+X) = \exp\bigl(\operatorname{tr}\log(I+X)\bigr)$. The Taylor expansion of $\log(I+X)$ converges for $\|X\| < 1$, and so

$$E_m(X) = \operatorname{tr}\sum_{r=m}^{\infty}\frac{(-1)^{r+1}}{r}X^r.$$

Furthermore, the spectral norm of X is bounded above by any matrix norm, so $|\operatorname{tr} X^r| \leqslant n\|X^r\| \leqslant n\|X\|^r$, and we can bound the tail of the series using a geometric progression. See [6] (Chapter 9) for the necessary matrix theory. □

Lemma 3.2. *For* $\boldsymbol{\theta} \in U_{n-1}(n^{-1/2+\epsilon})$,

$$\sum_{T\in\mathscr{T}_n}\prod_{jk\in ET}\bigl(1+i\tan\Delta_{jk}\bigr) = n^{n-2}\exp\left(\frac{1}{2n}\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^2 + O(n^{-1/2+3\epsilon})\right).$$

Proof. Define the $(n-1)\times(n-1)$ matrix B by

$$B_{jk} = \begin{cases} -i\tan\Delta_{jk}, & \text{for } k\neq j,\\ i\sum_{r=1}^{n}\tan\Delta_{jr}, & \text{for } k = j.\end{cases}$$

Then, by Theorem 3.1,

$$\begin{aligned}\sum_{T\in\mathscr{T}_n}\prod_{jk\in ET}\bigl(1+i\tan\Delta_{jk}\bigr) &= |nI-J+B|\\ &= |I+B(nI+J)^{-1}|\,|nI-J|\\ &= n^{n-2}|I+\Phi|,\end{aligned}$$

where $\Phi = \frac{1}{n}B(I+J)$. Here we have used the easily verified facts that $(nI-J)^{-1} = \frac{1}{n}(I+J)$ and $|nI-J| = n^{n-2}$.

The advantage of this transform is that Φ is small. In fact

$$\Phi_{jk} = \begin{cases} \frac{i}{n}\bigl(\tan\Delta_{jn}-\tan\Delta_{jk}\bigr) = O(n^{-3/2+\epsilon}), & \text{for } k\neq j,\\ \frac{i}{n}\bigl(2\tan\Delta_{jn}+\sum_{r=1}^{n-1}\tan\Delta_{jr}\bigr) = O(n^{-1/2+\epsilon}), & \text{for } k = j.\end{cases}$$

Using the matrix norm $\|\Phi\| = \max_j\sum_k|\Phi_{jk}|$, we have $\|\Phi\| = O(n^{-1/2+\epsilon})$, so Lemma 3.1 tells us that

$$|I+\Phi| = \exp\bigl(\operatorname{tr}\Phi - \tfrac{1}{2}\operatorname{tr}\Phi^2 + O(n^{-1/2+3\epsilon})\bigr). \tag{3.2}$$

By direct calculation using $\Delta_{jk} = -\Delta_{kj}$ and $\Delta_{jj} = 0$ for all j,k, we obtain

$$\operatorname{tr}\Phi = \frac{2i}{n}\sum_{j=1}^{n-1}\tan\Delta_{jn} = O(n^{-1/2+\epsilon}).$$

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A less trivial contribution comes from

$$\operatorname{tr}\Phi^2 = \sum_{j=1}^{n-1} \Phi_{jj}^2 + \sum_{1\leqslant j\neq r\leqslant n-1} \Phi_{jr}\Phi_{rj}. \tag{3.3}$$

After writing $\tan\Delta_{jk} = \Delta_{jk} + O(n^{-3/2+3\epsilon})$, the first term on the right simplifies to

$$\begin{aligned}\sum_{j=1}^{n-1} \Phi_{jj}^2 &= -\frac{1}{n^2}\sum_{j=1}^{n-1}\left(\sum_{r=1}^{n-1}\Delta_{jr}\right)^2 + O(n^{-1+4\epsilon}) \\ &= -\frac{n-1}{n^2}\sum_{1\leqslant j<k\leqslant n-1}\Delta_{jk}^2 + O(n^{-1+4\epsilon}) \\ &= -\frac{1}{n}\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^2 + O(n^{-1+4\epsilon}). \end{aligned} \tag{3.4}$$

We have added some extra terms at the last step for later convenience, noting that they are covered by the error term.

Since the second term on the right of (3.3) is clearly $O(n^{-1+2\epsilon})$, we can apply (3.2)–(3.4) to obtain

$$|I+\Phi| = \exp\left(\frac{1}{2n}\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^2 + O(n^{-1/2+3\epsilon})\right).$$

□

To complete the estimation of S_0, we need the following general result adapted from [11].

Lemma 3.3. *Suppose that $a = a(n)$ is a positive real function which tends to a positive limit as $n\to\infty$, and b is a real constant. For $n \geqslant 2$ define*

$$J(a,b,n) = \int_{U_{n-1}(n^{-1/2+\epsilon})} \exp\left(-a\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^2 + b\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^4\right) d\boldsymbol{\theta}',$$

where $\boldsymbol{\theta}' = (\theta_1,\theta_2,\ldots,\theta_{n-1})$ and $\theta_n = 0$. Then, as $n\to\infty$,

$$J(a,b,n) = n^{1/2}\left(\frac{\pi}{an}\right)^{(n-1)/2}\exp\left(\frac{3b}{2a^2} + O(n^{-1/2+4\epsilon})\right).$$

Proof. This would be a specialization of Theorem 2.1 of [11] except that $a(n)$ was there required to be constant. However, the same proof holds without modification. □

Theorem 3.2.

$$S_0 = 2^{(n-1)/2}\pi^{(n+1)/2}n^{n/2-1}\left(1+O(n^{-1/2+4\epsilon})\right).$$

Proof. By Taylor's theorem, we have

$$\prod_{1\leqslant j<k\leqslant n}\cos\Delta_{jk} = \exp\left(-\tfrac{1}{2}\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^2 - \tfrac{1}{12}\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^4 + O(n^{-1+6\epsilon})\right).$$

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Combining this with Lemma 3.2, we obtain an integrand matching the general form of Lemma 3.3. The validity of taking the error term outside the integrand is due to the fact that the remaining integrand is positive over the region of integration. □

4. The insignificant parts of the integral

In this section we will prove that S_0 contributes almost all of S, even though it involves only a tiny part of the region of integration. We will continue using the same value of ϵ as in the previous section.

It will be convenient to define $E'T = \{jk, kj : jk \in ET\}$, and express the integrand of (2.1) as

$$F(\boldsymbol{\theta}) = \sum_{T \in \mathcal{T}_n} \prod_{1 \leqslant j < k \leqslant n} f_{jk}(T, \boldsymbol{\theta}),$$

where

$$f_{jk}(T, \boldsymbol{\theta}) = \begin{cases} \cos\Delta_{jk}\left(1 + i\tan\Delta_{jk}\right), & \text{for } jk \in ET, \\ \cos\Delta_{jk}\left(1 - i\tan\Delta_{jk}\right), & \text{for } kj \in ET, \\ \cos\Delta_{jk}, & \text{otherwise.} \end{cases}$$

Note that $|f_{jk}(T, \boldsymbol{\theta})| \leqslant 1$ for all values of the parameters.

Lemma 4.1. *For real numbers x with $|x| \leqslant \frac{9}{16}\pi$, we have $|\cos(x)| \leqslant \exp(-\frac{1}{2}x^2)$.*

Proof. This follows easily from Taylor's theorem for $|x| \leqslant \frac{1}{2}\pi$, and by elementary computations for larger $|x|$. □

Lemma 4.2. *The number of labelled trees on n nodes with first node having degree greater than d is less than $2n^{n-2}/d!$ for all $d \geqslant 0$.*

Proof. As is well known, the generating function by degree sequence for labelled trees on n nodes is

$$f(\boldsymbol{x}) = x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2}.$$

This gives that the number of trees with first node of degree d_1 is

$$\binom{n-2}{d_1 - 1}(n-1)^{n-d_1-1} \leqslant \frac{n^{n-2}}{(d_1 - 1)!}.$$

For $d > 0$, sum over $d_1 > d$ to obtain the desired result. For $d = 0$, note that the number of trees altogether is n^{n-2}. (The constant is larger than necessary, but for our purposes any constant will do.) □

Divide the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ mod π into 32 equal intervals $H_0, \ldots, H_{31}$ such that $H_0 = [-\frac{1}{64}\pi, \frac{1}{64}\pi]$. For each j, define the region $W_j \subseteq U_n(\pi/2)$ as the set of points having at least $\frac{1}{32}n$ coordinates in H_j. Clearly the W_js cover $U_n(\pi/2)$, and also each W_j can be mapped to W_0 by a uniform translation of the θ_j mod π. This mapping preserves the

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integrand of (2.1) and also maps V_0 to itself, so we have that $\int_{U_n(\pi/2)-V_0} |F(\boldsymbol{\theta})|\,d\boldsymbol{\theta} \leqslant 32Z$, where

$$Z = \int_{W_0-V_0} |F(\boldsymbol{\theta})|\,d\boldsymbol{\theta}.$$

We proceed by defining integrals $S_1,\ldots,S_4$ in such a way that Z is obviously bounded by their sum. We then show that $S_j = o(S_0)$ for $j = 1,2,3,4$ separately. Write $F(\boldsymbol{\theta}) = F_a(\boldsymbol{\theta}) + F_b(\boldsymbol{\theta})$, where $F_a(\boldsymbol{\theta})$ and $F_b(\boldsymbol{\theta})$ are defined by restricting the sum to trees with maximum degree greater than $\frac{1}{4}n$ and no more than $\frac{1}{4}n$, respectively. Also define regions V_1 and V_2 as follows.

$$\begin{aligned} V_1 &= \{\boldsymbol{\theta} \in W_0 : |\theta_j| \geqslant \tfrac{1}{32}\pi \text{ for fewer than } n^\epsilon \text{ values of } j\}, \\ V_2 &= \{\boldsymbol{\theta} \in V_1 : |\theta_j| \geqslant \tfrac{1}{16}\pi \text{ for at least one value of } j\}. \end{aligned}$$

Then our four integrals can be defined as

$$\begin{aligned} S_1 &= \int_{W_0-V_1} |F(\boldsymbol{\theta})|\,d\boldsymbol{\theta}, \\ S_2 &= \int_{V_1} |F_a(\boldsymbol{\theta})|\,d\boldsymbol{\theta}, \\ S_3 &= \int_{V_2} |F_b(\boldsymbol{\theta})|\,d\boldsymbol{\theta}, \\ S_4 &= \int_{V_1-V_2-V_0} |F_b(\boldsymbol{\theta})|\,d\boldsymbol{\theta}. \end{aligned}$$

We begin with S_1. If $|\theta_j| \leqslant \frac{1}{64}\pi$ and $|\theta_k| \geqslant \frac{1}{32}\pi$ or vice versa, but $jk \notin E'T$, we have $|f_{ij}(T,\boldsymbol{\theta})| \leqslant \cos(\frac{1}{64}\pi)$. This includes more than $\frac{1}{32}n^{1+\epsilon} - n$ pairs jk, so we have

$$S_1 \leqslant n^{n-2}(2\pi)^n \cos(\tfrac{1}{64}\pi)^{n^{1+\epsilon}/32-n} = O\big(\exp(-cn^{1+\epsilon})\big)S_0$$

for some $c > 0$.

To bound S_2, first note from Lemma 4.2 that the number of trees with maximum degree greater than $\frac{1}{4}n$ is less than $2n^{n-1}/(n/4)!$. Using Lemma 4.1, we see that $|f_{ij}(T,\boldsymbol{\theta})| \leqslant \exp\big(-\frac{1}{2}\Delta_{jk}^2\big)$ except for at most $n^{2\epsilon}$ pairs jk with $|\Delta_{jk}| \geqslant \frac{1}{16}\pi$ and fewer than n pairs in $E'T$. In those excluded cases the value $\exp\big(-\frac{1}{2}\Delta_{jk}^2\big)$ may be high by a factor $\exp(\frac{1}{2}\pi^2)$. Hence, allowing π for the placement of θ_n, we have

$$\begin{aligned} S_2 &\leqslant \frac{2\pi n^{n-1}}{(n/4)!}\exp\big(\tfrac{1}{2}\pi^2(n+n^{2\epsilon})\big)\int_{U_{n-1}(\infty)} \exp\Big(-\tfrac{1}{2}\sum_{1\leqslant j<k\leqslant n}\Delta_{jk}^2\Big)\,d\boldsymbol{\theta}' \\ &= O(n^{-cn})S_0 \end{aligned}$$

for some $c > 0$. (The exact value of the integral is $n^{1/2}(2\pi/n)^{(n-1)/2}$.)

For $1 \leqslant r \leqslant n^\epsilon$ and $1 \leqslant d_{\max} \leqslant \frac{1}{4}n$, let $S_3(r,d_{\max})$ denote the contribution to S_3 of those trees T with maximum degree $d_{\max}$, and $\boldsymbol{\theta} \in V_2$ such that $|\theta_j| \geqslant \frac{1}{16}\pi$ for exactly r values of j. If $|\theta_j| \leqslant \frac{1}{32}\pi$ and $|\theta_k| \geqslant \frac{1}{16}\pi$ or vice versa, we have $|f_{ij}(T,\boldsymbol{\theta})| \leqslant \cos(\frac{1}{32}\pi)$ unless $jk \in E'T$. This includes at least $r(n - d_{\max} - n^\epsilon)$ pairs jk. For $|\theta_j|,|\theta_k| \leqslant \frac{1}{16}\pi$, but $jk \notin E'T$, we have $|f_{ij}(T,\boldsymbol{\theta})| \leqslant \exp\big(-\frac{1}{2}\Delta_{jk}^2\big)$. Put $\boldsymbol{\theta}'' = (\theta_1,\ldots,\theta_{m-1})$, where $m = n - r$.

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Then, allowing n^r for the choice of those values of j for which $|\theta_j| \geqslant \frac{1}{16}\pi$,

$$S_3(r, d_{\max}) \leqslant O(1)n^r \cos(\tfrac{1}{32}\pi)^{r(n-d_{\max}-n^\epsilon)} \sum_T \int_{U_{m-1}(\pi/8)} \exp\left(-\tfrac{1}{2}\sum_{j,k}^{(T)} \Delta_{jk}^2\right) d\boldsymbol{\theta}'', \tag{4.1}$$

where the first sum is over trees with maximum degree $d_{\max}$, the second sum is over $1 \leqslant j < k \leqslant m-1$ except for $jk \in E'T$, and $\theta_m = 0$.

For $1 \leqslant j \leqslant m-1$, apply the transformation $\phi : \theta_j \mapsto y_j + \mu/(m^{1/2}+1)$, where $\mu = y_1 + \cdots + y_{m-1}$. As noted in [11], ϕ transforms the quadratic form $\sum_{1\leqslant j<k\leqslant m} \Delta_{jk}^2$ with $\theta_m = 0$ to $m\sum_{j=1}^{m-1} y_j^2$, so we proceed by adjusting it for the terms belonging to $E'T$. For $1 \leqslant j,k \leqslant m-1$, ϕ maps Δ_{jk}^2 to $(y_j - y_k)^2$, which is at most $2(y_j^2 + y_k^2)$. For $j \leqslant m-1$, we have $\Delta_{jm}^2 \leqslant \frac{1}{64}\pi^2$. Thus, for each tree T,

$$-\tfrac{1}{2}\sum_{j,k}^{(T)} \Delta_{jk}^2 \leqslant \tfrac{1}{128}\pi^2 d_m + \sum_{j=1}^{m-1} -(\tfrac{1}{2}m - d_j)y_j^2,$$

where d_j is the total degree of node j. Since the determinant of the transformation ϕ is $m^{1/2}$, the value of the integral in (4.1) is bounded above by

$$m^{1/2}\pi^{(m-1)/2} \exp(\tfrac{1}{128}\pi^2 d_m) \prod_{j=1}^{m-1} \left(\tfrac{1}{2}m - d_j\right)^{-1/2}.$$

Furthermore, $\prod_{j=1}^{m-1}(\frac{1}{2}m - d_j)^{-1/2} = O(1)(2/m)^{(m-1)/2}$, since $\sum_{j=1}^{m-1} d_j \leqslant 2n$, $m \geqslant n - n^\epsilon$ and $d_{\max} \leqslant \frac{1}{4}n$. Applying Lemma 4.2, we find that

$$S_3(r, d_{\max}) \leqslant$$
$$O(1)n^{n+r-1/2}(2\pi/m)^{(m-1)/2} \cos(\tfrac{1}{32}\pi)^{r(n-d_{\max}-n^\epsilon)} \exp(\tfrac{1}{128}\pi^2 d_{\max})/(d_{\max}-1)!,$$

from which we can calculate that

$$S_3 = \sum_{r=1}^{n^\epsilon} \sum_{d_{\max}=1}^{n/4} S_3(r, d_{\max}) = O(c^{-n})S_0$$

for some $c > 1$.

The region $V_1 - V_2 - V_0$ which defines S_4 is covered by the subregion of W_0 defined by

(i) $|\Delta_{jn}| \leqslant \frac{1}{8}\pi$ for $1 \leqslant j \leqslant n-1$,
(ii) $|\Delta_{jn}| \geqslant n^{-1/2+\epsilon}$ for at least one j.

Applying the same argument as used for S_3, with $m = n$, we obtain

$$S_4 \leqslant 2\pi n^{1/2} \sum_{d_n=1}^{n/4} \exp(\tfrac{1}{128}\pi^2 d_n) \sum_{T\in\mathscr{T}_n(d_n)} \int \exp\left(-\tfrac{1}{2}\sum_{j=1}^{n-1}(n-2d_j)y_j^2\right) d\boldsymbol{y}, \tag{4.2}$$

where $\mathscr{T}_n(d_n)$ contains those trees with node n having degree d_n and the region of integration is the image of the region defined by (i) and (ii). From Lemma 4.2, we know that $|\mathscr{T}_n(d_n)| \leqslant 2n^{n-2}/(d_n-1)!$. The integrand in (4.2) is that of an $(n-1)$-dimensional Gaussian with covariances zero and variances close to $1/n$ (since $d_j \leqslant \frac{1}{4}n$ by assumption). Thus all but a fraction less than $O\left(\exp(-cn^{2\epsilon})\right)$ (some $c > 0$) of the integral over all $\boldsymbol{y}$ lies

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in the region defined by $|y_j| \leqslant \frac{1}{4}n^{-1/2+\epsilon}$ for all j and $|\mu| \leqslant \frac{1}{4}n^{\epsilon}$. However, these conditions imply that $|\Delta_{jn}| \leqslant n^{-1/2+\epsilon}$ for all j, contrary to (ii). Applying the same calculation as used for S_3, we find that $S_4 = O\big(\exp(-c'n^{2\epsilon})\big)S_0$ for some $c' > 0$.

Combining the bounds on $S_1, \ldots, S_4$, we obtain the desired result.

Lemma 4.3. *For some constant $c > 0$, we have $S = \big(1 + O(\exp(-cn^{2\epsilon}))\big)S_0$.* □

5. The major results

Theorem 5.1. *As $n \to \infty$ with n odd,*

$$\begin{aligned} \mathrm{Eul}(K_n) &= 2^{(n+1)/2}\pi^{1/2}e^{-n^2/2+11/12}n^{(n-2)(n+1)/2}\big(1 + O(n^{-1/2+\epsilon})\big) \\ \textit{and} \\ \mathit{Exp}_{\mathscr{R}_n}\big(\mathrm{Eul}(R)\big) &= 2^{-(n-2)(n+1)/2}\pi^{n/2}e^{-n^2/2+17/12}n^{(n^2-4)/2}\big(1 + O(n^{-1/2+\epsilon})\big), \end{aligned}$$

for any $\epsilon > 0$, where $\mathit{Exp}_{\mathscr{R}_n}$ denotes expectation in the space of random regular tournaments with uniform distribution.

Proof. The value of $\mathrm{Eul}(K_n)$ is now immediate from (2.1), Theorem 3.2 and Lemma 4.3. To obtain $\mathit{Exp}_{\mathscr{R}_n}\big(\mathrm{Eul}(R)\big)$, we need only divide by $|\mathscr{R}_n|$. The estimate

$$|\mathscr{R}_n| = \left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2} n^{1/2}e^{-1/2}\big(1 + O(n^{-1/2+\epsilon})\big)$$

was proved in [11]. □

A statement equivalent to the second part of Theorem 5.1 is that a random regular tournament on n nodes has on average $2^{-n+1}e^{1/2}n^{n-2}\big(1 + O(n^{-1/2+\epsilon})\big)$ directed spanning trees rooted at the nth node. Note that this is $e^{1/2}$ more than the average for a random tournament which is not necessarily regular.

These results have several interesting probabilistic interpretations. Suppose we have an Eulerian graph or digraph G and, starting at some node v, walk at random subject to each edge being used at most once. At each step, the choice between the available next edge is made uniformly at random. This random walk eventually ends at the starting node v, as a circuit that cannot be further extended. What is the probability, $P_1(G, v)$, that every edge of G is used by this circuit?

Theorem 5.2. *Let G be an Eulerian graph (or digraph) with degrees (out-degrees) $d_1, d_2, \ldots, d_n$. Let $m = |EG|$ and let v be any node, with d its degree (out-degree).*

(i) *For undirected G,*

$$P_1(G, v) = 2^{m-d-1}d\binom{d}{d/2}\mathrm{Eul}(G)\prod_{j=1}^{n}\frac{(d_j/2)!}{d_j!}.$$

(ii) *For directed G,*

$$P_1(G, v) = \frac{d\,\mathrm{Eul}(G)}{\prod_{j=1}^{n} d_j!}.$$

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Proof. The probability that a particular Eulerian circuit is traced by our random walk is independent of the circuit in each case.

For an undirected graph, the first step has chance $\frac{1}{2}$, as any edge in the correct direction will do. The second time v is reached there is one chance in $d-2$ of choosing the right edge out, one in $d-4$ the time after, and so on. For nodes other than v, the first time they are reached there is one chance in d_j-1 of choosing the right edge out, one in d_j-3 the time after, and so on, where d_j is the degree. This gives (i).

For directed graphs, the chances of choosing the right edge out of the starting node are 1, one in $d-1$, one in $d-2$, and so on; for other nodes, one in d_j, one in d_j-1 and so on. This gives (ii). □

Another related random process involves choosing a random pairing of the edges at each node. For an undirected graph, at each node divide the incident edges into unordered pairs, with each possible pairing being equally likely and independent at each node. For a directed graph, randomly pair each incoming edge with an outgoing edge. Now we can ask what is the probability $P_2(G)$ that this random process makes an Eulerian circuit.

Theorem 5.3. *Let G be an Eulerian graph (or digraph) with degrees (out-degrees) $d_1, d_2, \ldots, d_n$ and m edges.*

(i) *For undirected G,*

$$P_2(G) = 2^{m-1}\,\mathrm{Eul}(G)\prod_{j=1}^{n}\frac{(d_j/2)!}{d_j!}.$$

(ii) *For directed G,*

$$P_2(G) = \frac{\mathrm{Eul}(G)}{\prod_{j=1}^{n} d_j!}.$$

Proof. The ideas are very similar to those used to prove Theorem 5.2, so we will omit the details. It is necessary to note that in the undirected case an Eulerian circuit and its reverse correspond to the same pairing. □

Theorem 5.4. *As $n \to \infty$ with n odd, we have*

$$\begin{aligned} P_1(K_n) &\sim e^{3/4}n^{-1/2}, \\ P_2(K_n) &\sim 2^{-1/2}e^{3/4}\pi^{1/2}n^{-1}, \\ Exp_{\mathscr{R}_n}\big(P_1(R)\big) &\sim e^{3/2}n^{-1}, \quad \textit{and} \\ Exp_{\mathscr{R}_n}\big(P_2(R)\big) &\sim 2e^{3/2}n^{-2}. \end{aligned}$$

Proof. In each case these follow on substituting the estimate from Theorem 5.1 into Theorems 5.2 and 5.3. □

The values of P_1 and P_2 can be computed for directed graphs using Theorems 2.1, 3.1, 5.2 and 5.3. For example, if DK_n is the complete directed graph with n nodes, then

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Table 1 Exact values of e_n and the error term in Theorem 5.1

n	e_n	Error term
3	2	0.88639
5	264	0.97772
7	10 15440	0.99697
9	9 04492 51200	1.00374
11	169 10704 34783 65440	1.00638
13	62674 16821 16507 92035 99360	1.00739
15	4435 71127 63059 05572 69512 76764 67200	1.00770
17	5839 30527 51308 54565 39291 38771 58038 68245 19680	1.00767
19	1 40217 72793 55129 76955 93332 91385 68841 53315 25419 02716 92800	1.00750
21	604 98832 13879 13576 98014 78838 38038 42810 83283 62622 45623 80312 39839 74400	1.00725

$P_1(DK_n) = n^{n-2}/(n-1)^{n-1} \sim e/n$ and $P_2(DK_n) = n^{n-2}/(n-1)^n \sim e/n^2$, as communicated by Janson [5].

6. Exact results

In our paper [10], we presented the exact values of $\mathrm{Eul}(K_n)$ for $n \leqslant 17$ but did not explain the method of computation. In this section we will explain the technique used (slightly improved), and also extend the computation by two values. We will consider the number

$$e_n = \mathrm{Eul}(K_n)/(\tfrac{1}{2}n - \tfrac{3}{2})!^n = \sum_{T \in \mathscr{T}_n} RT(T),$$

as explained in Section 2. As noted there, e_n is the constant term in the function

$$f(x_1, \ldots, x_n) = \prod_{1 \leqslant j < k \leqslant n} \left(x_j^{-1} x_k + x_j x_k^{-1}\right) \sum_{T \in \mathscr{T}_n} \prod_{jk \in ET} \frac{x_j x_k^{-1}}{x_j^{-1} x_k + x_j x_k^{-1}}. \tag{6.1}$$

The degree of each x_j in each term in the expansion of $f(x_1, \ldots, x_n)$ is an even number in the interval $[-n+1, n-1]$. Moreover, the total degree of each term is 0, so e_n is in fact the constant term in the function $f(x_1, \ldots, x_{n-1}, 1)$. This coefficient can be extracted using the technique given in [10]. Let m be the odd member of $\{\frac{1}{2}(n+1), \frac{1}{2}(n+3)\}$. Since 0 is the only even integer in $[-n+1, n-1]$ that is divisible by m, we have

$$e_n = m^{-n+1} \sum_{i_1=0}^{m-1} \cdots \sum_{i_{n-1}=0}^{m-1} f(\omega^{i_1}, \ldots, \omega^{i_{n-1}}, 1), \tag{6.2}$$

where ω is a primitive mth root of unity. Instead of a complex root we can use an mth root of unity in a field of prime order p, and the result is the congruence class of e_n modulo p. Repeating this for a sufficient number of primes enables us to infer the exact value of e_n by means of the Chinese Remainder Theorem.

To make the computation feasible, we replaced the sum over $T \in \mathscr{T}_n$ in (6.1) by a determinant, using Theorem 3.1. We also grouped together all terms of the summation in (6.2) that are equivalent under permutations of $i_1, \ldots, i_{n-1}$, as these have the same value. The set of primes must have product greater than e_n, which we can check using

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Theorem 5.2 and the obvious bound $P_1(K_n) \leqslant 1$. We added several extra primes to the set for checking purposes. The results are presented in Table 1, together with values of the error term $1 + O(n^{-1/2+\epsilon})$ for $\mathrm{Eul}(K_n)$ in Theorem 5.1. Numerical experiments suggest that the real asymptotic size of the error term might be $O(n^{-1})$.

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