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THE SECOND-ORDER REGULAR VARIATION OF ORDER STATISTICS

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Let $X_1, \ldots, X_n$ be non-negative, independent and identically distributed random variables with a common distribution function F, and denote by $X_{1:n} \le \cdots \le X_{n:n}$ the corresponding order statistics. In this paper, we investigate the second-order regular variation (2RV) of the tail probabilities of $X_{k:n}$ and $X_{j:n} - X_{i:n}$ under the assumption that $\overline{F}$ is of the 2RV, where $1 \le k \le n$ and $1 \le i < j \le n$.

1. INTRODUCTION

Regular variation (RV) is an asymptotic property of functions and measures with power law growth. The one-variable theory is well-established, and has found numerous applications in both pure and applied mathematics. RV is a flexible description of power law behavior. A measurable function $h : \Re_+ \to \Re_+$ is said to be regularly varying at infinity with index $\alpha \in \Re$, denoted by $h \in \mathrm{RV}_\alpha$, if

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\alpha \tag{1.1}$$

for all $x > 0$. A regularly varying function with index zero is called slowly varying function. If $h \in \mathrm{RV}_\alpha$, it is easy to check that $h(x) = x^\alpha L(x)$ for some slowly varying function $L(x)$. We refer to Bingham et al. [3], Embrechts et al. [7], and Jessen and Mikosch [11] for more details. It is well-known that the functions $L(x) \equiv C$ and $L(x) = (\log x)^\gamma$, $\gamma \in \Re$, are both slowly varying. This means that regularly varying functions with the same index can differ a lot from each other and the difference between them may go to infinity as pointed by Hua & Joe [10]. This is why the second-order regular variation (2RV) was recommended; see de Haan and Resnick [5], Geluk et al. [8], Resnick and Stărică [15], Degen et al. [6], Mao and Hu [13, 14]. 2RV was originally studied in the extreme value theory, which refers to the rates of convergence in (1.1). A measurable function $h : \Re_+ \to \Re_+$ is said to be second-order regularly varying at infinity with the first-order index $\alpha \in \Re$ and the second-order index $\rho \le 0$, denoted by $h \in 2\mathrm{RV}_{\alpha,\rho}$, if there exists an eventually positive or negative

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function $A(t)$ with $A(t) \to 0$ as $t \to \infty$ such that

$$\lim_{t\to\infty} \frac{\frac{h(tx)}{h(t)} - x^{\alpha}}{A(t)} = x^{\alpha}\frac{x^{\rho}-1}{\rho} \tag{1.2}$$

for all $x > 0$. Here, $(x^{\rho}-1)/\rho$ is interpreted as $\log x$ for $\rho = 0$, and $A(t)$ is referred as an auxiliary function. For a general theory of 2RV, we refer to de Haan and Ferreira [4].

As we know, RV can be used to characterize the maximum domain of the extreme value distribution—Fréchet distribution. On the other hand, extreme value distributions are the limiting distributions of normalized largest order statistics. It is well-known that order statistics are among the most fundamental tools in non-parametric statistics and inference, such as robust location estimates, detection of outliers, censored sampling, goodness of fit and etc (see Balakrishnan and Rao [1, 2]). Let $X_1, \ldots, X_n$ be non-negative, independent and identically distributed (iid) random variables with a common distribution function F, and denote by $X_{1:n} \le \cdots \le X_{n:n}$ the corresponding order statistics. It is easy to verify that if $\overline{F} \in \mathrm{RV}_{-\alpha}$, $\alpha > 0$, then

$$X_{k:n} \in \mathrm{RV}_{-(n-k+1)\alpha}, \quad k = 1, \ldots, n$$

and

$$X_{k:n} - X_{\ell:n} \in \mathrm{RV}_{-(n-k+1)\alpha}, \quad 1 \le \ell < k \le n.$$

Moreover, we have

$$\mathsf{P}(X_{k:n} > t) \sim \mathsf{P}(X_{k:n} - X_{\ell:n} > t) \sim \binom{n}{k-1}\overline{F}^{n-k+1}(t), \quad t \to \infty.$$

Here and henceforth, the notation $\sim$ means asymptotic equivalence, that is, for functions g and h, $g(t) \sim h(t)$ as $t \to t_0$, means $\lim_{t\to t_0} g(t)/h(t) = 1$.

The purpose of this paper is to investigate the closure properties of 2RV for order statistics $X_{k:n}$ and generalized spacings $X_{k:n} - X_{\ell:n}$ if the underlying survival function $\overline{F}$ possesses the 2RV property.

This paper is organized as follows. Section 2 is the preliminary, where we present some lemmas that are crucial to the proofs of our main results. Sections 3 and 4 establish the 2RV closure properties for order statistics and generalized spacings, respectively.

2. PRELIMINARIES

First we give a lemma on 2RV, which will be used in the proof of the main result of Section 3.

LEMMA 1: *If $h \in 2\mathrm{RV}_{-\alpha,\rho}$ with auxiliary function $A(t)$, $\alpha \in \Re$, $\rho \le 0$, then $h^r \in 2\mathrm{RV}_{-r\alpha,\rho}$ with auxiliary function $rA(t)$ for $r > 0$.*

PROOF: By the definition of 2RV, we have

$$\frac{h(tx)}{h(t)} = x^{-\alpha}\left[1 + \frac{x^{\rho}-1}{\rho}A(t)(1+o(1))\right].$$

Then

$$\begin{aligned}\frac{h^r(tx)}{h^r(t)} &= x^{-\alpha r}\left[1 + \frac{x^{\rho}-1}{\rho}A(t)(1+o(1))\right]^r \\ &= x^{-\alpha r}\left[1 + \frac{x^{\rho}-1}{\rho}rA(t)(1+o(1))\right].\end{aligned}$$

This implies $h \in 2\mathrm{RV}_{-r\alpha,\rho}$ with auxiliary function $rA(t)$. ■

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Let F be the distribution function of a non-negative random variable, and introduce its truncated mean function as

$$\mu_F(t) = \int_0^t x dF(x), \quad t > 0.$$

Define

$$V_\beta(t) = \int_0^\infty \left(1 - \left(1 + \frac{x}{t}\right)^{-\beta}\right) dF(x).$$

Assume that $\overline{F} \in \mathrm{RV}_{-\alpha}$. The following two lemmas state that $V_\beta(t)$ is asymptotically equivalent to $\beta\mu_F(t)/t$ for $\alpha \geq 1$ and $c\overline{F}(t)$ for $0 < \alpha < 1$, respectively, where the constant c can be determined explicitly.

LEMMA 2: *Let F be the distribution function of a non-negative random variable satisfying $\overline{F} \in \mathrm{RV}_{-\alpha}$ with $\alpha \geq 1$. For $\beta > 0$, we have*

$$\lim_{t\to\infty} \frac{V_\beta(t)}{t^{-1}\mu_F(t)} = \beta.$$

PROOF: If the mean of F, μ_F, is finite, then

$$\lim_{t\to\infty} \mu_F(t) = \mu_F.$$

Since

$$t\left[1 - \left(1 + \frac{x}{t}\right)^{-\beta}\right] \leq \beta x, \quad x > 0,$$

the desired result for this case follows from the dominated convergence theorem. So, we only need to prove the lemma for the case $\mu_F = \infty$. Using integration by parts,

$$tV_\beta(t) = \beta \int_0^\infty \overline{F}(x) \left(1 + \frac{x}{t}\right)^{-\beta-1} dx. \tag{2.1}$$

Define

$$V_{1,\beta}(t) = \beta \int_0^t \overline{F}(x) \left(1 + \frac{x}{t}\right)^{-\beta-1} dx$$

and $V_{2,\beta}(t) = tV_\beta(t) - V_{1,\beta}(t)$. As pointed out by Mao and Hu [14], the following statements hold for $\alpha \geq 1$:

$$\mu_F(t) \sim \int_0^t \overline{F}(x)dx, \quad t \to \infty$$

and

$$\frac{t\overline{F}(t)}{\int_0^t \overline{F}(x)dx} \to 0, \quad t \to \infty.$$

Note that, for $0 \leq v \leq 1$,

$$1 - (\beta + 1)v \leq (1 + v)^{-\beta-1} \leq 1.$$

Hence, we have

$$\lim_{t\to\infty} \frac{V_{1,\beta}(t)}{\mu_F(t)} = \lim_{t\to\infty} \frac{\beta \int_0^t \overline{F}(x) \left(1 + x/t\right)^{-\beta-1} dx}{\int_0^t \overline{F}(x)dx} \leq \beta$$

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and

$$\lim_{t\to\infty}\frac{V_{1,\beta}(t)}{\mu_F(t)} \geq \beta-\beta(\beta+1)\lim_{t\to\infty}\frac{\int_0^t \overline{F}(x)xdx}{t\int_0^t \overline{F}(x)dx}$$
$$= \beta-\beta(\beta+1)\lim_{t\to\infty}\frac{t\overline{F}(t)}{\int_0^t \overline{F}(x)dx+t\overline{F}(t)} = \beta.$$

These two inequalities imply that

$$\lim_{t\to\infty}\frac{V_{1,\beta}(t)}{\mu_F(t)} = \beta.$$

Then the desired result follows by observing that

$$\lim_{t\to\infty}\frac{V_{2,\beta}(t)}{\mu_F(t)} \leq \lim_{t\to\infty}\frac{t\overline{F}(t)}{\mu_F(t)} = 0.$$

So we complete the proof of the lemma. ■

LEMMA 3: *Let F be the distribution function of a non-negative random variable satisfying $\overline{F}\in \mathrm{RV}_{-\alpha}$ with $0<\alpha<1$. For $\beta>0$, define*

$$I(\beta,\alpha) = \alpha\int_0^\infty \left(1-(1+x)^{-\beta}\right)x^{-\alpha-1}dx. \tag{2.2}$$

Then

$$\lim_{t\to\infty}\frac{V_\beta(t)}{\overline{F}(t)} = I(\beta,\alpha).$$

PROOF: Set $E=(0,\infty]$ and denote

$$\mathscr{M}_+(E) = \{\mu : \text{Radon, non-negative measures on the Borel } \sigma\text{-algebra } \mathfrak{B} \text{ of } E\}.$$

Define a measure μ on E as $\mu([y,\infty]) = y^{-\alpha}$, $y\in E$. Clearly, μ is Radon and every set $[y,\infty]$ is a μ-continuity set. Define

$$\mu_t([y,\infty]) = \int_{ty}^\infty \frac{dF(x)}{\overline{F}(t)} = \frac{\overline{F}(ty)}{\overline{F}(t)}, \quad y\in E.$$

Since $\mu_t([y,\infty]) \to \mu([y,\infty])$ as $t\to\infty$ for any $y\in E$, it follows that $\mu_t \xrightarrow{v} \mu$ as $t\to\infty$ (see Lemma 4.11 in Hua [9]), where $\xrightarrow{v}$ stands for the vague convergence of Radon measures.

For any $\delta>0$, $[\delta,\infty]$ is a μ-continuity set. Thus, we have

$$\int_{\delta t}^\infty \left(1-\left(1+\frac{x}{t}\right)^{-\beta}\right)\frac{1}{\overline{F}(t)}dF(x) = \int_\delta^\infty \left(1-(1+x)^{-\beta}\right)\mu_t(dx)$$
$$\longrightarrow \int_\delta^\infty \left(1-(1+x)^{-\beta}\right)\mu(dx),$$

where the last term converges to $I(\beta,\alpha)$ as $\delta\downarrow 0$. It remains to prove that

$$\lim_{\delta\downarrow 0}\limsup_{t\to\infty}\int_0^\delta \left(1-(1+x)^{-\beta}\right)\mu_t(dx) = 0. \tag{2.3}$$

Noting that $1-(1+x)^{-\beta}\leq \beta x$ for $x>0$, we can prove (2.3) by the same method as in Liu et al. [12]. Hence, we complete the proof of the lemma. ■

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PROPOSITION 1: *Let X and Y be two independent non-negative random variables with respective distribution functions F and G. If $\overline{F} \in 2\mathrm{RV}_{-\alpha,\rho}$ with auxiliary function $A(t)$ and $\overline{G} \in \mathrm{RV}_{-\beta}$, $\alpha > 0$, $\beta > 0$, $\rho \le 0$, then, as $t \to \infty$,*

$$\frac{\mathsf{P}(X - Y > t)}{\overline{F}(t)} - 1 = \begin{cases} -\alpha t^{-1}\mu_G(t)(1 + o(1)) + o(A(t)), & \beta \ge 1, \\ -I(\alpha,\beta)\overline{G}(t)(1 + o(1)) + o(A(t)), & 0 < \beta < 1. \end{cases}$$

PROOF: For $t \ge 0$,

$$\begin{aligned} \frac{\mathsf{P}(X - Y > t)}{\overline{F}(t)} - 1 &= \frac{1}{\overline{F}(t)} \int_0^\infty \overline{F}(t + x) dG(x) - 1 \\ &= \int_0^\infty \left(\frac{\overline{F}(t + x)}{\overline{F}(t)} - \left(1 + \frac{x}{t}\right)^{-\alpha} \right) dG(x) \\ &\quad - \int_0^\infty \left(1 - \left(1 + \frac{x}{t}\right)^{-\alpha}\right) dG(x) \\ &\stackrel{\mathrm{def}}{=} \Delta_1(t) - \Delta_2(t). \end{aligned}$$

Choose $0 < \delta < \alpha - \rho$. Since $\overline{F} \in 2\mathrm{RV}_{-\alpha,\rho}$ with auxiliary function $A(t)$, by Drees-type inequality, for any $\epsilon > 0$, there exist $A_0(t) \sim A(t)$ as $t \to \infty$ and $t_0 = t_0(\epsilon, \delta) > 0$, such that for all $t \ge t_0$ and $x \ge 0$,

$$\left| \frac{\frac{\overline{F}(x+t)}{\overline{F}(t)} - \left(1 + \frac{x}{t}\right)^{-\alpha}}{A_0(t)} - \left(1 + \frac{x}{t}\right)^{-\alpha} \frac{\left(1 + \frac{x}{t}\right)^{\rho} - 1}{\rho} \right| \le \epsilon \left(1 + \frac{x}{t}\right)^{-\alpha+\rho+\delta} \le \epsilon.$$

Integrating both sides of the above inequality with respect to $G(x)$ and taking $t \to \infty$, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{\Delta_1(t)}{A(t)} &= \lim_{t\to\infty} \int_0^\infty \frac{1}{A_0(t)} \left(\frac{\overline{F}(x + t)}{\overline{F}(t)} - \left(1 + \frac{x}{t}\right)^{-\alpha} \right) dG(x) \\ &= \lim_{t\to\infty} \int_0^\infty \left(1 + \frac{x}{t}\right)^{-\alpha} \frac{(1 + x/t)^{\rho} - 1}{\rho} dG(x) = 0, \end{aligned} \tag{2.4}$$

where the last equality follows from the dominated convergence theorem since

$$\sup_{x\ge 0} \left| \left(1 + \frac{x}{t}\right)^{-\alpha} \frac{(1 + x/t)^{\rho} - 1}{\rho} \right| \le \begin{cases} -1/\rho, & \rho > 0, \\ 1/(e\alpha), & \rho = 0. \end{cases}$$

Applying Lemmas 2 and 3 to $\Delta_2(t)$ yields that

$$\lim_{t\to\infty} \frac{\Delta_2(t)}{t^{-1}\mu_G(t)} = \alpha \quad \text{for } \beta \ge 1, \tag{2.5}$$

and

$$\lim_{t\to\infty} \frac{\Delta_2(t)}{\overline{G}(t)} = I(\alpha,\beta) \quad \text{for } 0 < \beta < 1. \tag{2.6}$$

Combining (2.4)–(2.6) yields the desired result. This completes the proof of the proposition. ■

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3. ORDER STATISTICS

The main result in this section states that if the underlying survival function of a random sample is of 2RV, then the survival functions of the corresponding order statistics preserve the 2RV property.

THEOREM 1: *Let $X_1, \ldots, X_n$ be iid non-negative random variables with a common survival function $\overline{F} \in 2\mathrm{RV}_{-\alpha,\rho}$ with auxiliary function $A(t)$, $\alpha > 0, \rho \le 0$. Then*

$$\mathsf{P}(X_{1:n} > \cdot) \in 2\mathrm{RV}_{-n\alpha,\rho}$$

with auxiliary function $nA(t)$, and for $2 \le k \le n$,

$$\mathsf{P}(X_{k:n} > \cdot) \in 2\mathrm{RV}_{-\alpha(n-k+1),\rho\vee(-\alpha)}$$

with auxiliary function $B(t)$ given by

$$B(t) = \begin{cases} (n-k+1)A(t), & \rho > -\alpha, \\ (n-k+1)A(t) + \dfrac{\alpha(k-1)(n-k+1)}{n-k+2}\overline{F}(t), & \rho = -\alpha, \\ \dfrac{\alpha(k-1)(n-k+1)}{n-k+2}\overline{F}(t), & \rho < -\alpha. \end{cases}$$

PROOF: Since $\mathsf{P}(X_{1:n} > t) = \overline{F}^n(t)$, the result for $X_{1:n}$ follows directly from Lemma 1. We only need to prove the theorem for $X_{k:n}$ with $k \ge 2$. Since

$$F^r(t) = \left(1 - \overline{F}(t)\right)^r = 1 - r\overline{F}(t) + o\left(\overline{F}(t)\right), \quad t \to \infty,$$

for any $r > 0$, we have

$$\begin{aligned} \mathsf{P}(X_{k:n} > t) &= \sum_{r=n-k+1}^{n} \binom{n}{r} \overline{F}^r(t) F^{n-r}(t) \\ &= \binom{n}{n-k+1} \overline{F}^{n-k+1}(t) F^{k-2}(t) \left[F(t) + \frac{k-1}{n-k+2}\overline{F}(t) + o(\overline{F}(t))\right] \\ &= \binom{n}{n-k+1} \overline{F}^{n-k+1}(t) \left[1 - (k-2)\overline{F}(t) + o(\overline{F}(t))\right] \\ &\quad \times \left[1 + \left(\frac{k-1}{n-k+2} - 1\right)\overline{F}(t) + o(\overline{F}(t))\right] \\ &= \binom{n}{n-k+1} \overline{F}^{n-k+1}(t) \left(1 - \frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t) + o(\overline{F}(t))\right). \end{aligned}$$

By Lemma 1, $\overline{F}^r \in 2\mathrm{RV}_{-r\alpha,\rho}$ with auxiliary function $rA(t)$. For $x > 0$, we have

$$\frac{\mathsf{P}(X_{k:n} > tx)}{\mathsf{P}(X_{k:n} > t)} = \left(\frac{\overline{F}(tx)}{\overline{F}(t)}\right)^{n-k+1} \cdot \frac{1 - \frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(tx) + o(\overline{F}(tx))}{1 - \frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t) + o(\overline{F}(t))}$$

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$$
\begin{aligned}
&= \left(\frac{\overline{F}(tx)}{\overline{F}(t)}\right)^{n-k+1} \cdot \left(1-(x^{-\alpha}-1)\frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t)(1+o(1))\right) \\
&= x^{-\alpha(n-k+1)}\left(1+\frac{x^{\rho}-1}{\rho}(n-k+1)A(t)(1+o(1))\right) \\
&\quad \times \left(1-(x^{-\alpha}-1)\frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t)(1+o(1))\right) \\
&= x^{-\alpha(n-k+1)}\left[1+\frac{x^{\rho}-1}{\rho}(n-k+1)A(t)\right. \\
&\quad \left. + \frac{x^{-\alpha}-1}{-\alpha}\cdot\frac{\alpha(k-1)(n-k+1)}{n-k+2}\overline{F}(t)+o(|A(t)|+\overline{F}(t))\right].
\end{aligned}
$$

This implies the desired result for $2 \le k \le n$. Hence, we complete the proof of theorem. ■

4. SPACINGS

In this section, we first give a result on the 2RV closure property for simple spacings $X_{k+1:n} - X_{k:n}$ with $1 \le k < n$, and then give a more general result for generalized spacings $X_{k:n} - X_{\ell:n}$ with $1 \le \ell < k \le n$.

THEOREM 2: *Suppose that $X_1, \ldots, X_n$ are iid non-negative random variables with a common survival function $\overline{F} \in 2\mathrm{RV}_{-\alpha,\rho}$ with auxiliary function $A(t)$, $\alpha > 0$ and $\rho \le 0$. Then, for each $1 \le k < n$,*

$$\mathsf{P}(X_{(k+1):n} - X_{k:n} > \cdot) \in 2\mathrm{RV}_{-\alpha(n-k),\gamma}$$

with $\gamma = \rho \vee [-(1 \wedge \alpha)]$ and auxiliary function $B(t)$ given by

$$
B(t) = \begin{cases}
(n-k)\alpha t^{-1}\mathsf{E}[X_{k:k}], & \alpha \ge 1, \rho < -1 \text{ and } \mathsf{E}[X_1] < \infty; \\
(n-k)\alpha k t^{-1}\mu_F(t), & \alpha = 1, \rho < -1 \text{ and } \mathsf{E}[X_1] = \infty; \\
\alpha k I((n-k)\alpha, \alpha)\overline{F}(t), & 0 < \alpha < 1, \rho < -\alpha; \\
(n-k)A(t), & \rho > -(1 \wedge \alpha),
\end{cases}
$$

where $I(\cdot,\cdot)$ is defined by (2.2).

PROOF: Fix $1 \le k < n$, and denote by $\overline{G}_k$ the survival function of $X_{(k+1):n} - X_{k:n}$. It is easy to see that

$$
\begin{aligned}
\overline{G}_k(t) &= \iint_{\{y-x>t\}} \frac{n!}{(k-1)!(n-k-1)!}F^{k-1}(x)\overline{F}^{n-k-1}(y)dF(x)dF(y) \\
&= \frac{n!}{(k-1)!(n-k-1)!}\int_0^{\infty} F^{k-1}(x)dF(x)\int_{x+t}^{\infty}\overline{F}^{n-k-1}(y)dF(y) \\
&= \frac{n!}{(k-1)!(n-k)!}\int_0^{\infty} F^{k-1}(x)\overline{F}^{n-k}(x+t)dF(x) \\
&= \binom{n}{k}\int_0^{\infty}\overline{F}^{n-k}(x+t)dF^k(x).
\end{aligned}
$$

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Let Z_1 and Z_2 be two independent random variables with distribution functions $F_{Z_1}(t) = 1 - \overline{F}^{n-k}(t)$ and $F_{Z_2}(t) = F^k(t)$, respectively. Then

$$\overline{G}_k(t) = \binom{n}{k} \mathsf{P}(Z_1 - Z_2 > t).$$

By Theorem 1, $\overline{F}_{Z_1} \in 2\mathrm{RV}_{-(n-k)\alpha,\rho}$ with auxiliary function $(n-k)A(t)$, and $\overline{F}_{Z_2} \in \mathrm{RV}_{-\alpha}$ with survival function $1 - F^k(t) \sim k\overline{F}(t)$ as $t \to \infty$. Then by Proposition 1, we have

$$\frac{\overline{G}_k(t)}{\overline{F}^{n-k}(t)} - \binom{n}{k} = \begin{cases} -\alpha(n-k)\binom{n}{k} t^{-1} \mu_{F^k}(t)(1+o(1)) + o(A(t)), & \alpha \geq 1, \\ -I((n-k)\alpha, \alpha)\binom{n}{k} k\overline{F}(t)(1+o(1)) + o(A(t)), & 0 < \alpha < 1. \end{cases} \tag{4.1}$$

Three cases arise.

Case 1: $\alpha \geq 1$ and $\rho < -1$. If $\mathsf{E}[X_1] < \infty$, then

$$\lim_{t\to\infty} \frac{\mu_{F^k}(t)}{\mu_F(t)} = \frac{\mathsf{E}[X_{k:k}]}{\mathsf{E}[X_1]}, \tag{4.2}$$

while if $\mathsf{E}[X_1] = \infty$ (which implies $\alpha = 1$), then

$$\lim_{t\to\infty} \frac{\mu_{F^k}(t)}{\mu_F(t)} = \lim_{t\to\infty} \frac{\int_0^t (1 - F^k(x))dx}{\int_0^t \overline{F}(x)dx} = k. \tag{4.3}$$

Define

$$\eta = \begin{cases} \mathsf{E}[X_{k:k}]/\mathsf{E}[X_1], & \mathsf{E}[X_1] < \infty, \\ k, & \mathsf{E}[X_1] = \infty. \end{cases}$$

Substituting (4.2) and (4.3) into (4.1) and noting the fact that $A(t) = o(\mu_F(t)/t)$ as $t \to \infty$, we have

$$\overline{G}_k(t) = \binom{n}{k} \overline{F}^{n-k}(t) \left[1 - (n-k)\alpha\eta \frac{\mu_F(t)}{t}(1+o(1))\right], \quad t \to \infty.$$

Hence,

$$\begin{aligned} \frac{\overline{G}_k(tx)}{\overline{G}_k(t)} &= \frac{\overline{F}^{n-k}(tx)}{\overline{F}^{n-k}(t)} \cdot \frac{1 - (n-k)\alpha\eta(tx)^{-1}\mu_F(tx) + o((tx)^{-1}\mu_F(tx))}{1 - (n-k)\alpha\eta t^{-1}\mu_F(t) + o(t^{-1}\mu_F(t))} \\ &= x^{-(n-k)\alpha}\left(1 + \frac{x^\rho - 1}{\rho}(n-k)A(t)(1+o(1))\right) \\ &\quad \times \left(1 - (x^{-1}-1)(n-k)\alpha\eta\frac{\mu_F(t)}{t}(1+o(1))\right) \\ &= x^{-(n-k)\alpha}\left(1 - (x^{-1}-1)(n-k)\alpha\eta\frac{\mu_F(t)}{t}(1+o(1))\right). \end{aligned}$$

This implies that $\overline{G}_k \in 2\mathrm{RV}_{-(n-k)\alpha,-1}$ with auxiliary function $(n-k)\alpha\eta\mu_F(t)/t$. In particular, when $\mathsf{E}[X_1] < \infty$, we have

$$\eta\frac{\mu_F(t)}{t} = \frac{\mathsf{E}[X_{k:k}]}{\mathsf{E}[X_1]}\frac{\mu_F(t)}{t} = \frac{1}{t}\mathsf{E}[X_{k:k}]\ (1+o(1)), \quad t \to \infty.$$

This proves the result for this case.

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Case 2: $0 < \alpha < 1$ and $\rho < -\alpha$. Since $A(t) = o(\overline{F}(t))$ as $t \to \infty$, substituting this into (4.1) yields that

$$\overline{G}_k(t) = \binom{n}{k}\overline{F}^{n-k}(t)[1 - kI((n-k)\alpha, \alpha)\overline{F}(t)(1 + o(1))], \quad t \to \infty.$$

So, we have, as $t \to \infty$,

$$\begin{aligned}\frac{\overline{G}_k(tx)}{\overline{G}_k(t)} &= \left(\frac{\overline{F}(tx)}{\overline{F}(t)}\right)^{n-k} \cdot \frac{1 - kI((n-k)\alpha, \alpha)\overline{F}(tx)(1 + o(1))}{1 - kI((n-k)\alpha, \alpha)\overline{F}(t)(1 + o(1))} \\ &= x^{-(n-k)\alpha}\left[1 + \frac{x^\rho - 1}{\rho}(n-k)A(t)(1 + o(1))\right] \\ &\quad \times [1 - (x^{-\alpha} - 1)kI((n-k)\alpha, \alpha)\overline{F}(t)(1 + o(1))] \\ &= x^{-(n-k)\alpha}(1 - (x^{-\alpha} - 1)kI((n-k)\alpha, \alpha)\overline{F}(t)(1 + o(1))).\end{aligned}$$

This means that $\overline{G}_k \in 2\mathrm{RV}_{-(n-k)\alpha, -\alpha}$ with auxiliary function $\alpha k I((n-k)\alpha, \alpha)\overline{F}(t)$.

Case 3: $\rho > -(1 \wedge \alpha)$. In this case, we have $t^{-1}\mu_F(t) = o(A(t))$ and $\overline{F}(t) = o(A(t))$ as $t \to \infty$. Substituting this into (4.1) yields that

$$\overline{G}_k(t) = \binom{n}{k}\overline{F}^{n-k}(t)(1 + o(A(t))), \quad t \to \infty.$$

Hence, we have

$$\begin{aligned}\frac{\overline{G}_k(tx)}{\overline{G}_k(t)} &= \frac{\overline{F}^{n-k}(tx)}{\overline{F}^{n-k}(t)} \cdot \frac{1 + o(A(tx))}{1 + o(A(t))} \\ &= x^{-(n-k)\alpha}\left(1 + \frac{x^\rho - 1}{\rho}(n-k)A(t)(1 + o(1))\right) \cdot (1 + o(A(t))) \\ &= x^{-(n-k)\alpha}\left(1 + \frac{x^\rho - 1}{\rho}(n-k)A(t)(1 + o(1))\right), \quad t \to \infty.\end{aligned}$$

This implies that $\overline{G}_k \in 2\mathrm{RV}_{-(n-k)\alpha, \rho}$ with auxiliary function $(n-k)A(t)$. Therefore, we complete the proof of the theorem. ■

Before we state and prove the next main result, we need the following proposition which gives the second-order expansions of the tail probabilities of generalized spacings under the assumption that the underlying survival function possesses the 2RV property.

PROPOSITION 2: *Let $X_1, \ldots, X_n$ be iid random variables with a common survival function $\overline{F} \in 2\mathrm{RV}_{-\alpha, \rho}$ with auxiliary function $A(t)$. Then, for $k \geq \ell + 2 \geq 3$,*

$$\begin{aligned}&\frac{\mathsf{P}(X_{k:n} - X_{\ell:n} > t)}{\overline{F}^{n-k+1}(t)} - \binom{n}{k-1} \\ &= \begin{cases} -k\binom{n}{k}\alpha t^{-1}\mathsf{E}[X_{\ell:(k-1)}](1 + o(1)) + o(A(t)), & \alpha > 1 \\ -k\binom{n}{k}\alpha t^{-1}\mathsf{E}[X_{\ell:(k-1)}](1 + o(1)) - \eta_k\overline{F}(t)(1 + o(1)) + o(A(t)), & \alpha = 1, \\ -\eta_k\overline{F}(t)(1 + o(1)) + o(A(t)), & 0 < \alpha < 1, \end{cases}\end{aligned}$$

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where

$$\eta_k = \binom{n}{k}\frac{k(k-1)}{n-k+2}.$$

PROOF: For $k \geq \ell+2 \geq 3$, denote by $\overline{G}_{\ell,k}$ the survival function of $X_{k:n} - X_{\ell:n}$, and define

$$\zeta_{\ell,k} = \frac{n!}{(\ell-1)!(k-\ell-1)!(n-k)!}.$$

Then it is easy to see

$$\begin{aligned}
\overline{G}_{\ell,k}(t) &= \zeta_{\ell,k}\iint_{\{y-x>t\}} F^{\ell-1}(x)[\overline{F}(x)-\overline{F}(y)]^{k-\ell-1}\overline{F}^{n-k}(y)\,dF(x)\,dF(y)\\
&= \zeta_{\ell,k}\sum_{j=0}^{k-\ell-1}(-1)^j\binom{k-\ell-1}{j}\iint_{\{y-x>t\}} F^{\ell-1}(x)\overline{F}^{k-\ell-j-1}(x)\\
&\quad\times\overline{F}^{n-k+j}(y)\,dF(x)dF(y)\\
&= \zeta_{\ell,k}\sum_{j=0}^{k-\ell-1}\frac{(-1)^j\binom{k-\ell-1}{j}}{n-k+j+1}\int_0^\infty F^{\ell-1}(x)\overline{F}^{k-\ell-j-1}(x)\overline{F}^{n-k+j+1}(x+t)\,dF(x)\\
&\stackrel{\text{def}}{=} \zeta_{\ell,k}\sum_{j=0}^{k-\ell-1}(-1)^j\binom{k-\ell-1}{j}\frac{(\ell-1)!(k-\ell-j-1)!}{(n-k+j+1)(k-j-1)!}\Delta_j(t),
\end{aligned}$$

where

$$\Delta_j(t) = \frac{(k-j-1)!}{(\ell-1)!(k-\ell-j-1)!}\int_0^\infty F^{\ell-1}(x)\overline{F}^{k-\ell-j-1}(x)\overline{F}^{n-k+j+1}(x+t)dF(x).$$

For each $j \in \{0,\ldots,k-\ell-1\}$, let Y_j and Z_j be two independent random variables satisfying

$$Y_j \stackrel{\text{d}}{=} X_{1:(n-k+j+1)} \text{ and } Z_j \stackrel{\text{d}}{=} X_{\ell:(k-j-1)}.$$

Then

$$\Delta_j(t) = \mathsf{P}(Y_j - Z_j > t).$$

From Theorem 1, it follows that $\mathsf{P}(Y_j>\cdot) \in 2\mathrm{RV}_{-(n-k+j+1)\alpha,\,\rho}$ with auxiliary function $(n-k+j+1)A(t)$, and $\mathsf{P}(Z_j>\cdot)\in \mathrm{RV}_{-(k-\ell-j)\alpha}$ with

$$\mathsf{P}(Z_j>t) \sim \binom{k-j-1}{\ell-1}(\overline{F}(t))^{k-\ell-j}, \quad t\to\infty.$$

Define $a_j = n-k+j+1$ and $b_j = k-\ell-j$. Then, by Proposition 1, we have

$$\frac{\Delta_j(t)}{\overline{F}^{n-k+j+1}(t)} - 1 = \begin{cases} -a_j\alpha t^{-1}\mu_{Z_j}(t)(1+o(1)) + o(A(t)), & b_j\alpha \geq 1,\\ -I(a_j\alpha, b_j\alpha)\,\mathsf{P}(Z_j>t)(1+o(1)) + o(A(t)), & 0<b_j\alpha<1. \end{cases} \tag{4.4}$$

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As special cases, we have, as $t \to \infty$,

$$\frac{\Delta_0(t)}{\overline{F}^{n-k+1}(t)} = \begin{cases} 1 - a_0\alpha t^{-1}\mu_{z_0}(t)(1+o(1)) + o(A(t)), & b_0\alpha \geq 1, \\ 1 - o(\overline{F}(t)) + o(A(t)), & 0 < b_0\alpha < 1, \end{cases}$$

$$\frac{\Delta_1(t)}{\overline{F}^{n-k+1}(t)} = \overline{F}(t)(1+o(1)),$$

$$\frac{\Delta_j(t)}{\overline{F}^{n-k+1}(t)} = o(\overline{F}(t)), \quad j \geq 2.$$

Combining these, we conclude that, for $b_0\alpha \geq 1$,

$$\begin{aligned} \frac{\overline{G}_{\ell,k}(t)}{\overline{F}^{n-k+1}(t)} &= \binom{n}{k-1}\frac{\Delta_0(t)}{\overline{F}^{n-k+1}(t)} - \binom{n}{k}\frac{k(k-1)}{n-k+2}\frac{\Delta_1(t)}{\overline{F}^{n-k+1}(t)} + o(\overline{F}(t)) \\ &= \binom{n}{k-1}\left(1 - a_0\alpha\frac{\mu_{z_0}(t)}{t}(1+o(1)) + o(A(t))\right) \\ &\quad - \binom{n}{k}\frac{k(k-1)}{n-k+2}\overline{F}(t)(1+o(1)) \\ &= \begin{cases} \binom{n}{k-1} - k\binom{n}{k}\alpha t^{-1}\mathsf{E}\,[X_{\ell:(k-1)}](1+o(1)) + o(A(t)), & \alpha > 1, \\ \binom{n}{k-1} - \binom{n}{k}\frac{k(k-1)}{n-k+2}\overline{F}(t)(1+o(1)) + o(A(t)), & \alpha < 1. \end{cases} \end{aligned} \tag{4.5}$$

If $\alpha = 1$, then

$$\begin{aligned} \frac{\overline{G}_{\ell,k}(t)}{\overline{F}^{n-k+1}(t)} &= \binom{n}{k-1} - k\binom{n}{k}\alpha t^{-1}\mathsf{E}\,[X_{\ell:(k-1)}](1+o(1)) \\ &\quad - \binom{n}{k}\frac{k(k-1)}{n-k+2}\overline{F}(t)(1+o(1)) + o(A(t)). \end{aligned} \tag{4.6}$$

For $b_0\alpha < 1$, which implies $\alpha < 1$, we have

$$\frac{\overline{G}_{\ell,k}(t)}{\overline{F}^{n-k+1}(t)} = \binom{n}{k-1} - \binom{n}{k}\frac{k(k-1)}{n-k+2}\overline{F}(t)(1+o(1)) + o(A(t)), \quad t \to \infty. \tag{4.7}$$

The desired result now follows from (4.5)–(4.7). Hence, we complete the proof of the proposition. ■

THEOREM 3: *Let $X_1, \ldots, X_n$ be iid random variables with a common survival function $\overline{F} \in 2\mathrm{RV}_{-\alpha,\rho}$ with auxiliary function $A(t)$, $\alpha > 0$, $\rho \leq 0$ and $n \geq 3$. Then, for $k \geq \ell + 2 > 2$,*

$$\mathsf{P}(X_{k:n} - X_{\ell:n} > \cdot) \in 2\mathrm{RV}_{-(n-k+1)\alpha,\gamma}$$

with $\gamma = \rho \vee [-(1 \wedge \alpha)]$ and auxiliary function $B(t)$ given by

$$B(t) = \begin{cases} (n-k+1)\alpha t^{-1}\mathsf{E}\,[X_{\ell:(k-1)}], & \alpha > 1, \quad \rho < -1; \\ (n-k+1)t^{-1}\mathsf{E}\,[X_{\ell:(k-1)}] + \dfrac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t), & \alpha = 1, \quad \rho < -1; \\ \alpha\dfrac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t), & 0 < \alpha < 1, \quad \rho < -\alpha; \\ (n-k+1)A(t), & \rho > -(1 \wedge \alpha). \end{cases}$$

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PROOF: Fix $k \geq \ell + 2 > 2$, and denote by $\overline{G}_{\ell,k}$ the survival function of $X_{k:n} - X_{\ell:n}$. Three cases arise.

Case 1: $\alpha \geq 1$ and $\rho < -1$. For $\alpha > 1$, by Proposition 2, we have

$$\frac{\overline{G}_{\ell,k}(t)}{\overline{F}^{n-k+1}(t)} = \binom{n}{k-1} - k\binom{n}{k}\alpha t^{-1}\mathsf{E}\left[X_{\ell:(k-1)}\right](1+o(1)), \quad t \to \infty.$$

Then, as $t \to \infty$,

$$\begin{aligned}\frac{\overline{G}_{\ell,k}(tx)}{\overline{G}_{\ell,k}(t)} &= \frac{\overline{F}^{n-k+1}(tx)}{\overline{F}^{n-k+1}(t)} \cdot \frac{1-(n-k+1)\alpha(tx)^{-1}\mathsf{E}\left[X_{\ell:(k-1)}\right](1+o(1))}{1-(n-k+1)\alpha t^{-1}\mathsf{E}\left[X_{\ell:(k-1)}\right](1+o(1))} \\ &= x^{-(n-k+1)\alpha}\left(1+(n-k+1)\frac{x^{\rho}-1}{\rho}A(t)(1+o(1))\right) \\ &\quad \times \left(1-(x^{-1}-1)(n-k+1)\alpha\frac{\mathsf{E}\left[X_{\ell:(k-1)}\right]}{t}(1+o(1))\right) \\ &= x^{-(n-k+1)\alpha}\left(1-(x^{-1}-1)(n-k+1)\alpha\frac{\mathsf{E}\left[X_{\ell:(k-1)}\right]}{t}(1+o(1))\right).\end{aligned}$$

This implies that $\overline{G}_{\ell,k} \in 2\mathrm{RV}_{-(n-k+1)\alpha,-1}$ with auxiliary function $(n-k+1)\alpha t^{-1}\mathsf{E}\left[X_{\ell:(k-1)}\right]$.

For $\alpha = 1$, by Proposition 2, we have

$$\frac{\overline{G}_{\ell,k}(t)}{\overline{F}^{n-k+1}(t)} = \binom{n}{k-1} - \binom{n}{k-1}h_{\ell,k}(t)(1+o(1)), \quad t \to \infty,$$

where

$$h_{\ell,k}(t) = (n-k+1)\frac{\mathsf{E}\left[X_{\ell:(k-1)}\right]}{t} + \frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t) \in \mathrm{RV}_{-1}.$$

Then, as $t \to \infty$,

$$\begin{aligned}\frac{\overline{G}_{\ell,k}(tx)}{\overline{G}_{\ell,k}(t)} &= \frac{\overline{F}^{n-k+1}(tx)}{\overline{F}^{n-k+1}(t)} \cdot \frac{1-h_{\ell,k}(tx)(1+o(1))}{1-h_{\ell,k}(t)(1+o(1))} \\ &= x^{-(n-k+1)\alpha}\left(1+(n-k+1)\frac{x^{\rho}-1}{\rho}A(t)(1+o(1))\right) \\ &\quad \times (1-(x^{-1}-1)h_{\ell,k}(t)(1+o(1))) \\ &= x^{-(n-k+1)\alpha}(1-(x^{-1}-1)h_{\ell,k}(t)(1+o(1))).\end{aligned}$$

This implies that $\overline{G}_{\ell,k} \in 2\mathrm{RV}_{-(n-k+1)\alpha,-1}$ with auxiliary function $h_{\ell,k}(t)$. This proves Case 1.

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Case 2: $0 < \alpha < 1$ and $\rho < -\alpha$. By Proposition 2, we have

$$\frac{\overline{G}_{\ell,k}(t)}{\overline{F}^{n-k+1}(t)} = \binom{n}{k-1} - \binom{n}{k}\frac{k(k-1)}{n-k+2}\overline{F}(t)(1+o(1)), \quad t \to \infty,$$

since $A(t) = o(\overline{F}(t))$ as $t \to \infty$. Then, as $t \to \infty$,

$$\begin{aligned}\frac{\overline{G}_{\ell,k}(tx)}{\overline{G}_{\ell,k}(t)} &= \frac{\overline{F}^{n-k+1}(tx)}{\overline{F}^{n-k+1}(t)} \cdot \frac{1 - \frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(tx)(1+o(1))}{1 - \frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t)(1+o(1))} \\ &= x^{-(n-k+1)\alpha}\left(1 + (n-k+1)\frac{x^\rho - 1}{\rho}A(t)(1+o(1))\right) \\ &\quad \times \left(1 - (x^{-\alpha}-1)\frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t)(1+o(1))\right) \\ &= x^{-(n-k+1)\alpha}\left(1 - (x^{-\alpha}-1)\frac{(k-1)(n-k+1)}{n-k+2}\overline{F}(t)(1+o(1))\right),\end{aligned}$$

which implies that $\overline{G}_{\ell,k} \in 2\mathrm{RV}_{-(n-k+1)\alpha,-\alpha}$ with auxiliary function $\alpha(((k-1)(n-k+1))/(n-k+2))\overline{F}(t)$. This proves Case 2.

Case 3: $\rho > -(1 \wedge \alpha)$. In this case, it is easy to see that, as $t \to \infty$,

$$\overline{F}(t) = o(A(t)) \quad \text{and} \quad \frac{\mathsf{E}\left[X_{\ell:(k-1)}\right]}{t} = o(A(t)).$$

Then, by Proposition 2, we have

$$\frac{\overline{G}_{\ell,k}(t)}{\overline{F}^{n-k+1}(t)} = \binom{n}{k-1} + o(A(t)), \quad t \to \infty.$$

Then, as $t \to \infty$,

$$\begin{aligned}\frac{\overline{G}_{\ell,k}(tx)}{\overline{G}_{\ell,k}(t)} &= \frac{\overline{F}^{n-k+1}(tx)}{\overline{F}^{n-k+1}(t)} \cdot \frac{1+o(A(tx))}{1+o(A(t))} \\ &= x^{-(n-k+1)\alpha}\left(1 + \frac{x^\rho - 1}{\rho}(n-k+1)A(t)(1+o(1))\right)(1+o(A(t))) \\ &= x^{-(n-k+1)\alpha}\left(1 + \frac{x^\rho - 1}{\rho}(n-k+1)A(t)(1+o(1))\right),\end{aligned}$$

implying that $\overline{G}_{\ell,k} \in 2\mathrm{RV}_{-(n-k+1)\alpha,\rho}$ with auxiliary function $(n-k+1)A(t)$. Therefore, we complete the proof of the theorem. ■

A special consequence of Theorem 3 is the following corollary concerning the 2RV property of the sample range $X_{n:n} - X_{1:n}$.

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COROLLARY 1: *Under the condition of* Theorem 3 *with* $n \geq 3$*, we have*

$$\mathsf{P}(X_{n:n} - X_{1:n} > \cdot) \in 2\mathrm{RV}_{-\alpha,\gamma}$$

with $\gamma = \rho \vee [-(1 \wedge \alpha)]$ *and auxiliary function* $B(t)$ *given by*

$$B(t) = \begin{cases} \alpha t^{-1} \mathsf{E}\,[X_{1:(n-1)}], & \alpha > 1, \rho < -1, \\ t^{-1} \mathsf{E}\,[X_{1:(n-1)}] + \dfrac{n-1}{2} \overline{F}(t), & \alpha = 1, \rho < -1, \\ \alpha \dfrac{n-1}{2} \overline{F}(t), & 0 < \alpha < 1, \rho < -\alpha, \\ A(t), & \rho > -(1 \wedge \alpha). \end{cases}$$

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