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A note on stochastic integrators

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1. *Introduction*

The object of this paper is to show that some important classes of stochastic processes can be proved to be stochastic integrators, in the sense of Bichteler[**1**], by entirely elementary methods. Let us recall what this means.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ that satisfies the 'usual conditions', namely we suppose that $(\mathscr{F}_t)$ is right-continuous and that $\mathscr{F}_0$ contains all the P-null sets. Let T_0, T_∞ be the stopping times that are respectively identically zero and identically $+\infty$. Denote by $\mathscr{E}_1$ the set of all processes of the form $V = Z1_{[T_0]}$, where Z is a bounded real $\mathscr{F}_0$-measurable random variable, and by $\mathscr{E}_2$ the set of all processes of the form $V = Z1_{(T, T_\infty]}$, where T is a stopping time and Z is a bounded real $\mathscr{F}_T$-measurable random variable. If X is another stochastic process we define a new process, denoted by $V.X$, in each of the above two cases as follows: (i) if $V = Z1_{[T_0]}$ then

$$(V.X)_t = ZX_{t\wedge T_0} \qquad (t\in\mathbb{R}_+)$$

and (ii) if $V = Z1_{(T, T_\infty]}$ then

$$(V.X)_t = Z(X_{t\wedge T_\infty} - X_{t\wedge T}) \quad (t\in\mathbb{R}_+).$$

We denote by $\mathscr{E}$ the (real) linear span of $\mathscr{E}_1 \cup \mathscr{E}_2$, and call $\mathscr{E}$ the space of *elementary previsible processes*. The map $V \mapsto V.X$ extends uniquely to a linear map (for which we use the same notation) from $\mathscr{E}$ into the space of all real processes. In this situation $V.X$ is called the *elementary integral* of V with respect to X. If X_∞ is defined we can, if need be, make sense of $(V.X)_\infty$ by an obvious extension of the above discussion.

Suppose that $0 \leqslant p < \infty$. The process X is called a *stochastic L^p-integrator* if it is càdlàg, adapted and such that (a) $X_t \in L^p$ for all $t\in\mathbb{R}_+$, (b) for each $t\in\mathbb{R}_+$ the map $V \mapsto (V.X)_t$ is continuous from $\mathscr{E}$ into L^p when $\mathscr{E}$ has the topology of uniform convergence on $\mathbb{R}_+ \times \Omega$ and L^p has its usual topology. This differs somewhat from Bichteler's definition, but is fairly easily seen to be equivalent to it, as he shows in [**1**]. Note also that, by a deep theorem of Bichteler[**1**] and Dellacherie, Mokobodzki and Meyer[**6**], stochastic L^0-integrators are the same thing as semimartingales, a fact which led Dellacherie[**3**] later to *define* semimartingales in this fashion.

The concept of stochastic integrator is the foundation of Bichteler's [**1**] elegant account of stochastic integration theory, and he shows that many major properties of the stochastic integral can be derived directly from it, before it becomes necessary to examine special classes of integrators (L^2-martingales, continuous martingales etc.). This means that once the processes of a given class can be shown to be stochastic integrators, a great deal of the basic general theory in Bichteler's account becomes available for them without more ado. Bichteler [**1**] of course also gives due

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attention to the standard examples of integrators, as does Dellacherie [3]. However, both authors when proving that certain fundamentally important types of process are stochastic integrators have occasion to use sophisticated arguments and techniques that require the full force of the (continuous-parameter) Doob–Meyer decomposition. The present paper shows that such arguments and techniques may sometimes be replaced by completely elementary ones.

The main things proved below in this fashion are the following: (i) martingales and quasimartingales are L^0-integrators (Theorem 3 and Corollary 2), (ii) bounded positive supermartingales are L^2-integrators (Theorem 4), (iii) quasimartingales are locally L^1-integrators (Theorem 5). The only continuity assumption is that the processes in question should be càdlàg.

2. *Some inequalities from discrete-parameter theory*

We gather here, without proof, some discrete-parameter results that will be useful in the sequel. They all have elementary proofs, for which we give appropriate references.

Let $(Y_n, \mathscr{G}_n)_{0 \leqslant n \leqslant N}$, where N is finite, be a supermartingale on $(\Omega, \mathscr{F}, P)$ and let $(W_n)_{0 \leqslant n \leqslant N}$ be a sequence of bounded random variables such that each W_n is $\mathscr{G}_{n-1}$-measurable (where we set $\mathscr{G}_{-1} = \mathscr{G}_0$). For $0 \leqslant n \leqslant N$ let

$$(W * Y)_n = W_0 Y_0 + \sum_{r=1}^{n} W_r (Y_r - Y_{r-1}). \tag{1}$$

Let $$\|W\| = \sup\{|W_n(\omega)| : 0 \leqslant n \leqslant N, \omega \in \Omega\}$$

and suppose that $\alpha > 0$. It follows that

(i) if $(Y_n, \mathscr{G}_n)_{0 \leqslant n \leqslant N}$ is a positive supermartingale then

$$\alpha P[|(W * Y)_N| > \alpha] \leqslant 9 \|W\| \, \|Y_0\|_{L^1}; \tag{2}$$

(ii) if $(Y_n, \mathscr{G}_n)_{0 \leqslant n \leqslant N}$ is a martingale then

$$\alpha P[|(W * Y)_N| > \alpha] \leqslant 18 \|W\| \, \|Y_N\|_{L^1}. \tag{3}$$

Burkholder [2] was the first to prove such inequalities, but his proofs were rather difficult. There now exist short elementary proofs (for which see [5, 7] or, for a similar account with larger constants, [9]).

Much more elementary than (3) is

(iii) if $(Y_n, \mathscr{G}_n)_{0 \leqslant n \leqslant N}$ is a sequence in L^2 that is either a martingale or positive and increasing (i.e. such that $Y_{n-1} \leqslant Y_n$ when $1 \leqslant n \leqslant N$) then

$$\|(W * Y)_N\|_{L^2} \leqslant \|W\| \, \|Y_N\|_{L^2}. \tag{4}$$

The martingale case of (4) will be evident e.g. from a perusal of section 3·5 in [1]. The case where Y is increasing follows from the obvious estimate

$$|(W * Y)_N| \leqslant \|W\| \, Y_N.$$

For the final inequalities of this section we need another construction. Suppose that $(Y_n, \mathscr{G}_n)_{0 \leqslant n \leqslant N}$ is a positive supermartingale, and write $\mathscr{G}_n = \mathscr{F}, Y_n = 0$ for all $n > N$. Then $(Y_n, \mathscr{G}_n)_{n \geqslant 0}$ is also a positive supermartingale and we can form its Doob

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decomposition $Y = M - A$, where $(M_n, \mathscr{G}_n)_{n \geqslant 0}$ is a martingale and $(A_n)_{n \geqslant 0}$ is an increasing sequence of random variables with $A_0 = 0$, and where each A_n is $\mathscr{G}_{n-1}$-measurable. By inspection of the defining formulae we see that, because $Y_n = 0$ for all $n > N$,

$$M_{N+k} = A_{N+k} = M_N \quad (k \geqslant 1).$$

We shall use the following result:

(iv) if $(Y_n, \mathscr{G}_n)_{0 \leqslant n \leqslant N}$ is a positive supermartingale bounded above by a real constant c then $M_n = A_n \in L^2$ for all n and

$$\mathbb{E}(M_{N+1}^2) = \mathbb{E}(A_{N+1}^2) \leqslant 2c\mathbb{E}(Y_0). \tag{5}$$

It follows that

$$\|(W * Y)_N\|_{L^2} \leqslant 2\|W\|(2c\mathbb{E}(Y_0))^{\frac{1}{2}}. \tag{6}$$

A convenient proof of (5) is given in [5]. To deduce (6) note that

$$\mathbb{E}(|(W*M)_N|^2) = \mathbb{E}(W_0^2 M_0^2 + \sum_{n=1}^{N} W_n^2(M_n^2 - M_{n-1}^2)) \leqslant \|W\|^2\mathbb{E}(M_N^2) \leqslant \|W\|^2\mathbb{E}(M_{N+1}^2)$$

and that

$$|(W*A)_N| \leqslant \|W\| A_N \leqslant \|W\| A_{N+1}.$$

The inequality (6) is now an obvious consequence of (5).

The above results have been presented in the order most convenient for subsequent use, rather than in the most intrinsically logical one.

3. *Some L^0-integrators*

Now let us return to the situation of Section 1. If $V \in \mathscr{E}$ it is easy to show that it admits a representation of the form

$$V = Z_0 1_{[T_0]} + \sum_{n=1}^{N} Z_n 1_{(T_{n-1}, T_n]}, \tag{7}$$

where $N \geqslant 1$, $(T_n)_{0 \leqslant n \leqslant N}$ is a sequence of stopping times such that

$$0 = T_0 \leqslant T_1 \leqslant \ldots \leqslant T_N = \infty,$$

and each Z_n is a bounded $\mathscr{F}_{T_{n-1}}$-measurable random variable (where we set $T_{-1} = T_0$). Suppose also that $(X_t)_{t \in \mathbb{R}_+}$ is a càdlàg adapted process. Then the elementary integral $V.X$, defined in Section 1, is given by

$$(V.X)_t = Z_0 X_0 + \sum_{n=1}^{N} Z_n(X_{t \wedge T_n} - X_{t \wedge T_{n-1}}) \tag{8}$$

for all $t \in \mathbb{R}_+$. For all $V \in \mathscr{E}$ we write

$$\|V\| = \sup\{|V_t(\omega)| : t \in \mathbb{R}_+, \quad \omega \in \Omega\}.$$

Now we can and do suppose that in the representation (7) no two T_n are equal. With that assumption we see that

$$\|V\| = \sup\{|Z_n(\omega)| : 0 \leqslant n \leqslant N, \quad \omega \in \Omega\}. \tag{9}$$

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THEOREM 1. *Let $V \in \mathscr{E}$, let X be a càdlàg positive supermartingale, let T be a stopping time, and suppose that $\alpha > 0$. Then*

$$\alpha P[|(V.X)_T| > \alpha] \leqslant 9\|V\|\,\|X_0\|_{L^1}, \tag{10}$$

where it is to be understood that $(X_\infty, \mathscr{F})$ closes $(X_t, \mathscr{F}_t)_{t\in\mathbb{R}_+}$. Consequently X is a stochastic L^0-integrator.

The closure hypothesis is of course not a restriction, since we can for instance take $X_\infty = 0$.

First recall that if $Y \in L^1$ and S_1, S_2 are stopping times then

$$\mathbb{E}(\mathbb{E}(Y\,|\,\mathscr{F}_{S_1})\,|\,\mathscr{F}_{S_2}) = \mathbb{E}(Y\,|\,\mathscr{F}_{S_1\wedge S_2}).$$

(For this see e.g. p. 82 of [**4**].) Moreover, for any stopping time S we have $X_S \in L^1(\mathscr{F}_S)$, so that if $0 \leqslant t_1 \leqslant t_2 \leqslant \infty$ then, by Doob's stopping theorem,

$$X_{t_1\wedge T} \geqslant \mathbb{E}(X_{t_2\wedge T}\,|\,\mathscr{F}_{t_1\wedge T}) = \mathbb{E}(\mathbb{E}(X_{t_2\wedge T}\,|\,\mathscr{F}_T)\,|\,\mathscr{F}_{t_1}) = \mathbb{E}(X_{t_2\wedge T}\,|\,\mathscr{F}_{t_1}).$$

Thus the stopped process $(X_t^T, \mathscr{F}_t)_{0\leqslant t\leqslant\infty}$ is a positive supermartingale. By another application of Doob's stopping theorem we deduce in like fashion that if S_1, S_2 are stopping times such that $S_1 \leqslant S_2$ then

$$X_{S_1\wedge T} \geqslant \mathbb{E}(X_{S_2\wedge T}\,|\,\mathscr{F}_{S_1\wedge T}) = \mathbb{E}(X_{S_2\wedge T}\,|\,\mathscr{F}_{S_1}).$$

It is therefore clear that $(X_{T_n}^T, \mathscr{F}_{T_n})_{0\leqslant n\leqslant N}$ is a positive supermartingale.

Now by (7) we have

$$(V.X)_T = Z_0 X_{T_0}^T + \sum_{n=1}^{N} Z_n(X_{T_n}^T - X_{T_{n-1}}^T). \tag{11}$$

Taking $\mathscr{G}_n = \mathscr{F}_{T_n}$, $Y_n = X_{T_n}^T$, $W_n = Z_n$ we see that $(V.X)_T$ is just $(W * Y)_N$ as defined by (1), and that we have exactly the right conditions for inequality (2). But inequality (10) is now merely (2) clothed in different notation. The proof of Theorem 1 is thus complete.

COROLLARY 2. *Every càdlàg quasimartingale is an L^0-integrator.*

By the Rao–Stricker theorem [**9**, **11**] a quasimartingale can always be written as the difference of two positive supermartingales. The corollary is therefore clear.

Corollary 2 includes the special case of L^1-bounded martingales. But martingales can be treated more directly as follows.

THEOREM 3. *Let $V \in \mathscr{E}$, let X be a càdlàg martingale, let T be a stopping time, and suppose that $\alpha > 0$. If either X is closed or T is bounded then*

$$\alpha P[|(V.X)_T| > \alpha] \leqslant 18\|V\|\,\|X_T\|_{L^1}. \tag{12}$$

Consequently every càdlàg martingale is an L^0-integrator.

We take V to have the same representation (7) as before.

Take first the case where X is closed at infinity by $X_\infty \in L^1$, and let $\mathscr{F}_\infty = \mathscr{F}$. Then the proof of Theorem 1, mutatis mutandis, shows that $(X_{T_n}^T, \mathscr{F}_{T_n})_{0\leqslant n\leqslant N}$ is a martingale. Inspection of the formula (11) for $(V.X)_T$ shows that we are now (apart from notation) in the situation of the inequality (3) in Section 2; in other words (12) follows immediately from (3).

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Next we drop the assumption that X is closed, and suppose instead that $T \leqslant k$ for some real constant k. But then the stopped process X^k is a closed martingale and moreover $(X^k)^T = X^T$, $(V.X^k)_T = (V.X)_T$. The desired inequality for $V.X$ therefore follows directly from the result for closed martingales. This completes the proof.

4. *Some L^p-integrators*

The final conclusion in Theorem 1 can be strengthened if we suppose that X is bounded.

THEOREM 4. *Let $V \in \mathscr{E}$, suppose that X is a càdlàg positive supermartingale bounded above by a real constant c, and let T be a stopping time. Then*

$$\|(V.X)_T\|_{L^2} \leqslant 2\|V\|\,(2c\mathbb{E}(X_0))^{\frac{1}{2}}, \tag{13}$$

where it is to be understood that $(X_\infty, \mathscr{F})$ closes X. Consequently X is a stochastic L^2-integrator.

Using the notation of Section 3 we see from the proof of Theorem 1 that $(X^T_{T_n}, \mathscr{F}_{T_n})_{0 \leqslant n \leqslant N}$ is now a positive supermartingale bounded above by c. Proceeding now as for Theorem 1 we see that the formula (11) for $(V.X)_T$ allows us to invoke the inequality (6) to obtain (13). This completes the proof of Theorem 4.

THEOREM 5. *Every càdlàg positive supermartingale is locally a stochastic L^1-integrator.*

Let X be the process. As before we may take it to be closed by $(X_\infty, \mathscr{F})$. Suppose that $c > 0$ and let $S = \inf\{t : X_t > c\}$. Then S is a stopping time and

$$X^S = X1_{[0,S)} + X_S\,1_{[S,\infty)}.$$

Since $X_S \in L^1_+(\mathscr{F}_S)$, the term $A \equiv X_S\,1_{[S,\infty)}$ is a càdlàg positive adapted L^1-bounded process such that $A_s \leqslant A_t$ whenever $s \leqslant t$. In particular, it is a càdlàg submartingale. But (see Section 3) X^S is a càdlàg supermartingale. It follows that

$$X1_{[0,S)} = X^S - A$$

a positive càdlàg supermartingale. Since $X1_{[0,S)}$ is also evidently bounded above by c, Theorem 4 tells us that it is an L^2-integrator, and hence an L^1-integrator. As for A, we deduce from the trivial inequality

$$|(V * A)_T| \leqslant \|V\|\,A^T_{T_n} \leqslant \|V\|\,A_T$$

that

$$\|(V * A)_T\|_{L^1} \leqslant \|V\|\,\|A_T\|_{L^1},$$

where of course $A_\infty = \lim_{t\to\infty} A_t \in L^1$. Hence A is an L^1-integrator, and therefore so is X^S.

Now write $S_n = \inf\{t : X_t > n\}$ for each integer $n \geqslant 1$. Then X^{S_n} is an L^1-integrator for each n, and (S_n) is an increasing sequence of stopping times. Since X is càdlàg we have $\lim_{n\to\infty} S_n = \infty$. So (S_n) is a localizing sequence for the purposes of Theorem 5, and the proof is complete.

COROLLARY 6. *Every càdlàg quasimartingale is locally an L^1-integrator.*

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This follows from Theorem 5 by the Rao–Stricker [**9**, **11**] decomposition of the quasimartingale into the difference of two positive submartingales.

5. *Concluding comments*

In constructing the stochastic integral some authors commence with a smaller class of elementary integrands than the class $\mathscr{E}$ we have used here. For example, one can confine attention to the case where all the stopping times T_n are constant [**4**]. One can also suppose that T is constant. If those restrictions are adopted then some of the above proofs can be shortened. I have worked here with what seems to be the largest reasonable $\mathscr{E}$, making almost the same choice as Rogers and Williams [**10**] (in fact they take the linear span of $\mathscr{E}_2$ rather than of $\mathscr{E}_1 \cup \mathscr{E}_2$). Bichteler [**1**] adopts a position intermediate between that of [**4**] and the present paper.

The theorems stated above deliberately follow [**1**] rather closely in formulation except for a greater emphasis here on supermartingales, which leads conveniently to corollaries about quasimartingales.

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