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ARTICLE

Strong complete minors in digraphs

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Abstract

Kostochka and Thomason independently showed that any graph with average degree $\Omega(r\sqrt{\log r})$ contains a K_r minor. In particular, any graph with chromatic number $\Omega(r\sqrt{\log r})$ contains a K_r minor, a partial result towards Hadwiger's famous conjecture. In this paper, we investigate analogues of these results in the directed setting. There are several ways to define a minor in a digraph. One natural way is as follows. A *strong* $\overrightarrow{K}_r$ *minor* is a digraph whose vertex set is partitioned into r parts such that each part induces a strongly connected subdigraph, and there is at least one edge in each direction between any two distinct parts. We investigate bounds on the dichromatic number and minimum out-degree of a digraph that force the existence of strong $\overrightarrow{K}_r$ minors as subdigraphs. In particular, we show that any tournament with dichromatic number at least $2r$ contains a strong $\overrightarrow{K}_r$ minor, and any tournament with minimum out-degree $\Omega(r\sqrt{\log r})$ also contains a strong $\overrightarrow{K}_r$ minor. The latter result is tight up to the implied constant and may be viewed as a strong-minor analogue to the classical result of Kostochka and Thomason. Lastly, we show that there is no function $f:\mathbb{N}\to\mathbb{N}$ such that any digraph with minimum out-degree at least $f(r)$ contains a strong $\overrightarrow{K}_r$ minor, but such a function exists when considering dichromatic number.

Keywords: strong minors, digraphs

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1. Introduction

The relationship between the chromatic number of a graph and the existence of certain types of substructures, like minors and topological minors, has a long history. As usual, we say that a graph is a K_r *minor* if there is a partition of its vertex set into r parts such that each part induces a connected subgraph, and there is at least one edge between any two distinct parts. One of the most famous examples of a problem of the aforementioned type is *Hadwiger's conjecture*: for every $r \geq 1$, any graph with chromatic number at least r contains a K_r minor.

There are a few partial results concerning this conjecture. It is known to be true for $r \leq 6$ (see [4, 8, 21, 24]). For general r, Kostochka [15] and Thomason [22] independently showed that any graph with average degree $\Omega(r\sqrt{\log r})$ contains a K_r minor. Of course, this implies that any graph with chromatic number $\Omega(r\sqrt{\log r})$ contains a K_r minor. This was the best general result towards Hadwiger's conjecture for some time. Recently, however, Norin, Postle, and Song [17, 19] improved this result. Building off of that work, Postle [20] proved that any graph with chromatic number $\Omega(r(\log\log r)^6)$ contains a K_r minor, and this is the best bound to date.

In this paper, we look at analogous problems in the directed setting. In particular, we investigate the existence of certain types of minors in digraphs under conditions such as large dichromatic

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number and large minimum out-degree. Before proceeding, let us introduce a bit of terminology. We remark that all digraphs we consider are simple and do not contain loops. A digraph D is *strongly connected* if for every ordered pair (u, v) of vertices in D there is a directed path in D from u to v. As usual, D is *weakly connected* if its underlying graph is connected. The complete directed graph on r vertices, denoted by $\overrightarrow{K}_r$, is a directed graph in which every pair of vertices is connected by an edge in each direction.

There are several ways one can define a minor in a digraph (e.g. the notion of a *butterfly minor*, originally introduced by Johnson, Robertson, Seymour, and Thomas; see section 5 of [13]). Here, we consider so-called *strong minors*. Given a digraph H, we say that D is a *strong H minor* if $V(D)$ admits a partition $\{X_v : v \in V(H)\}$ into non-empty sets (called *branch sets*) such that

- the digraph $D[X_v]$ induced by X_v is strongly connected for all $v \in H$, and
- $uv \in E(H)$ if and only if there is an edge in D from X_u to X_v.

A similar definition holds for *weak H minors*: we merely insist in this case that each branch set induces a weakly connected subdigraph. We say that a digraph D *contains* a strong (weak) H minor if it contains a strong (weak) H minor as a subdigraph. Equivalently, D contains a strong H minor whenever H can be obtained from a subdigraph of D by repeatedly contracting a strongly connected subdigraph to a vertex and removing loops and multiple edges. We remark that the notion of a strong minor has been investigated before by Kim and Seymour [14], where they showed that tournaments (more generally, semi-complete digraphs) are well-quasi-ordered under the strong minor relation.

Jagger [11], [12] investigated extensions of the classical result of Kostochka and Thomason to digraphs. In particular, he addressed the following question: How many edges must a digraph have in order to guarantee weak (or strong) complete minors as subdigraphs? In particular, he showed [11] that any digraph with average degree $\Omega(r\sqrt{\log r})$ contains a weak $\overrightarrow{K}_r$ minor. This is a weak-minor analogue of the aforementioned result of Kostochka and Thomason. A similar theorem, however, cannot hold for containing *strong* $\overrightarrow{K}_r$ minors: the transitive tournament on n vertices has $\binom{n}{2}$ edges, but does not even contain a strong $\overrightarrow{K}_2$ minor. Thus, in order for a digraph D to contain a strong $\overrightarrow{K}_r$ minor, it must have more than $\binom{|D|}{2}$ edges. Another way of saying this is the following: There is no function $f : \mathbb{N} \to \mathbb{N}$ such that any digraph with average degree at least $f(r)$ contains a strong $\overrightarrow{K}_r$ minor. This suggests that density is not the appropriate digraph parameter to force strong complete minors, but perhaps there are other parameters that do so.

A *k-colouring* of a digraph is a partition of its vertex set into k acyclic sets. The minimum k for which this is possible is the *dichromatic number* of D, which we shall denote by $\chi(D)$ (as we never consider the usual chromatic number in this paper, we hope this causes no confusion). This parameter was introduced by Neumann-Lara [16] and has garnered interest in recent years (e.g., see [2, 9, 10] for some interesting results concerning this parameter).

While the transitive tournament discussed above is dense, it contains no large complete minors. On the other hand, there are two parameters for which the transitive tournament is essentially trivial: it only has minimum out-degree 0 and dichromatic number 1. Therefore, one might hope that large dichromatic number/out-degree is sufficient to guarantee large complete minors in general digraphs. And, if this is not the case, perhaps it is true in the more restrictive class of tournaments. In general, which digraph parameters, if sufficiently large, force large strong complete minors? This is the kind of question we address in this paper, our focus being on large dichromatic number and large minimum out-degree. We remark that this type of problem has been addressed in the context of forcing *subdivisions* in digraphs, instead of minors; see Aboulker, Cohen, Havet, Lochet, Moura, and Thomassé [1], and the recent results of Gishboliner, Steiner, and Szabó [6, 7].

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1.1. Our results

We shall introduce the following terminology in order to state our results:

Definition 1.1. For a digraph D, let sm(D) denote the largest r such that D contains a strong $\overrightarrow{K_r}$ minor.

Our aim is to determine whether large dichromatic number or large out-degree is sufficient to force the existence of strong $\overrightarrow{K_r}$ minors in digraphs. We show that the large dichromatic number indeed guarantees the existence of strong $\overrightarrow{K_r}$ minors in digraphs. However, we show that the large minimum out-degree is not sufficient to force strong minors in general digraphs, but it is sufficient in tournaments. Specifically, we extend the classical results of Kostochka and Thomason to strong minors: any tournament with minimum out-degree $\Omega(r\sqrt{\log r})$ contains a strong $\overrightarrow{K_r}$ minor.

We first consider tournaments. The following theorem asserts that dichromatic number linear in r already forces strong $\overrightarrow{K_r}$ minors in tournaments.

Theorem 1.2. *Let $r \geq 1$ be an integer and suppose T is a tournament with $\chi(T) \geq 2r$. Then* $\mathrm{sm}(T) \geq r$. *Moreover, for every $r \geq 2$ there exists a tournament S_r such that $\chi(S_r) = r$ and* $\mathrm{sm}(S_r) \leq r - 1$.

The construction of S_r is done in Section 4. We believe that this construction is closer to the truth, concerning the correct dependence on χ for finding strong complete minors.

As is customary, for a digraph D we denote by $\delta^+(D)$ the minimum out-degree of D. Our next theorem investigates strong complete minors in tournaments with large minimum out-degree. It may be viewed as the appropriate analogue of the classical result of Kostochka and Thomason for strong complete minors.

Theorem 1.3. *There exists a constant $C > 0$ such that the following holds. If r is a positive integer and T is a tournament with $\delta^+(T) \geq Cr\sqrt{\log r}$, then* $\mathrm{sm}(T) \geq r$. *Moreover, this is tight up to the constant C.*

The tightness can be seen by considering a random tournament on $cr\sqrt{\log r}$ vertices, for some constant $c > 0$, and applying a standard argument (e.g. see Bollobás, Catlin and Erdős [3]). In fact, this argument shows that the random tournament on $cr\sqrt{\log r}$ vertices with high probability does not even contain r non-empty sets with an edge in each direction between each pair of sets. The analogous result to Theorem 1.3 for digraphs is false, however:

Theorem 1.4. *There is no function $f : \mathbb{N} \to \mathbb{N}$ such that every digraph D with $\delta^+(D) \geq f(r)$ satisfies* $\mathrm{sm}(D) \geq r$.

In particular, we show that a construction due to Thomassen [23] has large minimum out-degree, but does not even contain strong $\overrightarrow{K}_3$ minors; we prove this in Section 4 (see Proposition 4.3). On the other hand, we can show that large dichromatic number is sufficient to force strong minors in general digraphs.

Theorem 1.5. *Let $r \geq 1$ be an integer. If D is a digraph with $\chi(D) \geq r4^r$, then* $\mathrm{sm}(D) \geq r$.

Note that according to the second statement of Theorem 1.2, there is a digraph S_r such that $\chi(S_r) = r$ and $\mathrm{sm}(S_r) \leq r - 1$. The digraph S_r is a tournament. In addition, we show in Section 4 that for every $r \geq 2$ there exists a digraph G_r with $\chi(G_r) = r$ and $\mathrm{sm}(G_r) \leq r - 1$. In comparison with S_r, the digraph G_r is quite sparse. We believe the exponential dependence on r in the first part of this theorem is far from the truth.

1.2. Organisation

The remainder of this paper is organised as follows. Section 2 is devoted to additional definitions and notation. In Section 4, we analyse constructions of a tournament and digraph that have

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dichromatic number r, but do not contain strong clique minors on r vertices. We additionally prove Theorem 1.4 by analysing Thomassen's construction. In Section 5, we prove Theorem 1.2 and Theorem 1.3, which concern finding strong minors in tournaments under the assumption of large dichromatic number and large minimum out-degree, respectively. In Section 6, we prove our results concerning digraphs. First, we show that the assumption of large minimum out-degree is not sufficient to guarantee strong clique minors. Second, we prove Theorem 1.5, which shows that large dichromatic number is sufficient. Finally, we conclude in Section 7 with some remarks and open problems.

2. Notation and terminology

Here we provide some additional notation and terminology that will be used throughout the paper. Any further notation shall be introduced as necessary.

Let D be a digraph. We denote by $V(D)$ and $E(D)$ the vertex and edge set of D, respectively. For a vertex v in D, we write $N_D^+(v)$ and $N_D^-(v)$ for the out-neighbourhood and in-neighbourhood of v in D, respectively. We let $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$ denote the out-degree and in-degree of v. We let $\delta^+(D) = \min_{v \in V(D)} d_D^+(v)$ denote the minimum out-degree of D. We shall always omit the subscript 'D' when the digraph is clear from context. Given a subset $X \subseteq V(D)$, we denote by $D[X]$ the subdigraph of D induced by X. We say that a subset of vertices $X \subseteq V(D)$ is *acyclic* if the subdigraph $D[X]$ induced by X contains no directed cycle. Alternatively, we may say that X (or the induced subdigraph $D[X]$) is *transitive*.

For disjoint sets $X, Y \subseteq V(D)$, we write $X \to_D Y$ provided every edge of D with an endpoint in X and an endpoint in Y is oriented from X to Y. Thus, if D is a tournament, this means that every possible edge between X and Y is oriented from X to Y. If $X = \{x\}$, then we simply write $x \to_D Y$ (and similarly if $Y = \{y\}$). We shall omit the subscript 'D' when the digraph is understood from context.

A set $X \subset V(D)$ is a *cut-set* if $D - X$ is not strongly connected. A *strongly connected component* in a digraph D is a strongly connected subdigraph of D that is maximal with this property. Every digraph can be partitioned into its strongly connected components, say $V(D) = S_1 \cup \cdots \cup S_t$ where $D[S_i]$ is strongly connected for each $i \in [t]$ and $S_i \to S_j$ for all $1 \le i < j \le t$ (see Lemma 3.1). If $t \ge 2$, we then have $S := \bigcup_{i<t} S_i \to S_t$. We call S the *source set* and S_t the *sink set* of D. Moreover, if $X = \{x_1, \ldots, x_t\} \subseteq V(D)$ is a transitive set with $x_i \to x_j$ for all $1 \le i < j \le t$, we refer to x_1 and x_t as the *source* and *sink* of X, respectively.

Finally, given a positive integer k, a digraph is k-strongly connected if it has at least $k+1$ vertices and if it remains strongly connected upon the removal of any set of at most $k-1$ vertices.

3. Preliminaries

Here we collect some simple results that will be applied in several places throughout the paper. The first allows us to define source sets and sink sets of non-strongly connected digraphs. The second allows us to assume that our digraphs are strongly connected when considering dichromatic number.

Lemma 3.1. *The vertex set of every digraph D can be partitioned into non-empty sets $S_1, \ldots, S_t$ such that $D[S_i]$ is strongly connected for every i, and $S_i \to S_j$ for all $1 \le i < j \le t$.*

Proof. Say that two vertices x, y are strongly connected if there is a directed path from x to y and a directed path from y to x in D. This is clearly an equivalence relation, and we may take $S_1, \ldots, S_t$ to be the pairwise-disjoint equivalence classes. For the second claim, we may assume $t \ge 2$, since if $t = 1$ it holds vacuously. Define a digraph H whose vertex set is $[t]$ and we join i to j whenever there is an edge from S_i to S_j in D. If we cannot order the S_i's transitively as claimed, then H contains

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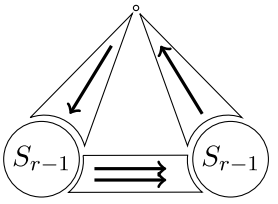


Figure 1. The tournament S_r.

a directed cycle $C = i_1 i_2 \dots i_k i_1$ for some $k \geq 2$. But then $D[\bigcup_{j \in V(C)} S_j]$ is strongly connected, which contradicts the maximality of $S_1, \dots, S_t$. □

Lemma 3.2. *If D is a digraph with $\chi(D) \geq r$, then D contains a strongly connected subdigraph D' with $\chi(D') \geq r$.*

Proof. If D itself is strongly connected, then we are done. Otherwise, apply Lemma 3.1 and let $V(D) = S_1 \cup \dots \cup S_t$ be a partition of the vertex set of D into strongly connected components, such that $S_i \to S_j$ for all $1 \leq i < j \leq t$, and $t \geq 2$. Assume that we have $\chi(D[S_i]) \leq r - 1$ for all $i \in [t]$. Colour the vertices in each S_i with colours in $[r-1]$. Then this produces an $(r-1)$-colouring of D, a contradiction. □

4. Constructions

In this section, we examine the following constructions: a tournament S_r with $\chi(S_r) = r$ and $\mathrm{sm}(S_r) \leq r - 1$, a digraph G_r with $\chi(G_r) = r$ and $\mathrm{sm}(G_r) \leq r - 1$, and a digraph D_k with out-degree k and no strong $\overrightarrow{K}_3$ minor.

4.1. Digraphs with large dichromatic number and no large strong minor

Given digraphs H, G, D, let $\Delta(H, G, D)$ be the digraph obtained by taking vertex-disjoint copies of H, G, and D, adding all possible undirected edges with at most one endpoint in each of $V(H)$, $V(G)$, and $V(D)$, and orienting the new edges so that $H \to G$, $G \to D$, and $D \to H$.

Construction 1. *Let $S_1 = K_1$ (i.e. the single vertex complete digraph with no edges). For $r \geq 2$ we set $S_r = \Delta(K_1, S_{r-1}, S_{r-1})$. We illustrate this construction in Figure 1.*

We remark that this construction has appeared before in [2]. It is not difficult to check by induction that $\chi(S_r) = r$ for all $r \geq 1$. The following shows that, in general, dichromatic number r is not sufficient to guarantee a strong $\overrightarrow{K}_r$ minor.

Proposition 4.1. *For every $r \geq 2$ we have $\chi(S_r) = r$ and $\mathrm{sm}(S_r) \leq r - 1$ for every $r \geq 2$.*

Proof. As mentioned, $\chi(S_r) = r$ is easy to verify by induction on r. We turn to determining $\mathrm{sm}(S_r)$. Note that S_2 is simply a directed triangle and so clearly $\mathrm{sm}(S_2) = 1$. Let $r \geq 3$ and suppose the result holds for smaller values. Since $S_r = \Delta(K_1, S_{r-1}, S_{r-1})$, let us write A and B for the two copies of S_{r-1}, and v for the copy of K_1. Clearly, at most one branch set from any strong minor in S_r contains v. If $X \subseteq V(S_r)$ is a strongly connected subset that does not contain v, then X cannot contain vertices from both A and B (since $A \to B$). Similarly, there cannot be one branch set contained in A and another contained in B. Since we can potentially add a new branch set in S_r containing v, we have

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$$\mathrm{sm}(S_r) \le \mathrm{sm}(S_{r-1}) + 1 \le r - 1,$$

as claimed. □

Construction 2. *Let $G_1 = K_1$ and suppose G_r has been constructed for some $r \ge 1$. To form G_{r+1}, consider first a transitive tournament T_{r+1} on $r+1$ vertices. For each directed edge e of T_{r+1} add pairwise disjoint copies of G_r, denoted by G_r^e, such that $V(G_r^e) \cap V(T_{r+1}) = \varnothing$ for all e. Finally, for each directed edge $e = vw$ in T_{r+1} and each vertex $u \in V(G_r^e)$ create a directed triangle uvw.*

Proposition 4.2. *For every $r \ge 2$, we have $\chi(G_r) = r$ and $\mathrm{sm}(G_r) \le r - 1$.*

Proof. Note that G_2 is a directed triangle, so we have $\chi(G_2) = 2$ and $\mathrm{sm}(G_2) = 1$. So suppose $r \ge 3$ and the result holds for smaller values. Since each copy of G_{r-1} in G_r can be $(r-1)$-coloured, we can r-colour G_r using one extra colour for the transitive tournament. On the other hand, we cannot $(r-1)$-colour G_r: assume there is an $(r-1)$-colouring of G_r, then some two vertices $x, y \in V(T_r)$ receive the same colour, say 1. Let $e = xy$ be the corresponding edge in T_r. As $\chi(G_{r-1}^e) = r - 1$, the colour 1 must appear in G_{r-1}^e, and so we obtain a directed triangle in colour 1, a contradiction.

Next, weshall show that $\mathrm{sm}(G_r) \le r - 1$. Let $t = \mathrm{sm}(G_r)$. Consider a strong $\overrightarrow{K}_t$ minor in G_r with branch sets $B_1, \dots, B_t$. Since T_r is not strongly connected, $B_i \not\subseteq V(T_r)$ for each $i = 1, \dots, t$. Observe also that if for some edge $e = xy$ of T_r, $B_i \cap V(G_{r-1}^e) \neq \emptyset$ and $B_i \setminus V(G_{r-1}^e) \neq \emptyset$, then B_i contains both x and y; otherwise, B_i does not induce a strongly connected digraph. There are two cases to consider.

Case 1. There is $i \in [t]$ such that $B_i \subseteq V(G_{r-1}^e)$ for some edge $e = xy$ of T_r.

Without loss of generality, assume $i = t$. Let $V' = V(G_{r-1}^e)$. Then any other B_j, $j = 1, \dots, t-1$ should either be contained in V' or contain x or y. If all B_i's are contained in V', then by induction $t \le r - 2$. Thus, we can assume that there is $j \in \{1, \dots, t-1\}$ such that B_j contains x or y. If B_j intersects V', we have that B_j contains both x and y by an observation before the statement of the case. If B_j does not contain vertices from V', it also must contain both x and y, otherwise the edges between B_t and B_j go in one direction only. Since the B_i's are pairwise disjoint, there is exactly one such index j, such that B_j is not contained in V'. The number of branch sets from $B_1, \dots, B_t$ that are contained in V' is by induction at most $r - 2$. Thus, $\mathrm{sm}(G_r) = t \le r - 2 + 1 = r - 1$.

Case 2. For each branch set B_i, there exists an edge e of T_r such that B_i contains both vertices of e.

Since the B_i's are pairwise disjoint, the respective edges e must also be pairwise disjoint. Since T_r has at most $r/2 \le r - 1$ pairwise disjoint edges, $t \le r - 1$. This completes the second case, and thus the proof. □

4.2. Digraphs with large out-degree and no large strong minor

Here we show that large out-degree is not sufficient to guarantee strong $\overrightarrow{K}_r$ minors in general digraphs, and thus prove Theorem 1.4. To this end, we need Thomassen's [23] construction of digraphs with large out-degree and no even cycle. The construction proceeds as follows.

Construction 3. *Let D_1 be a directed triangle. Suppose D_k is defined by induction. Now, add pairwise disjoint sets $A_x \cup \{x'\}$ where $|A_x| = k + 1$, for every vertex $x \in V(D_k)$, all of them disjoint from $V(D_k)$. We then add all directed edges between A_x and $N^+_{D_k}(x) \cup \{x\}$ such that $A_x \to \{x\} \cup N^+_{D_k}(x)$, and $x \to x'$. Finally, add all edges from x' to A_x, and denote by D_{k+1} the resulting digraph. We illustrate this construction in Figure 2.*

Then D_k clearly has minimum out-degree k, and one can check that it contains no even directed cycle (in particular, it contains no pair of vertices x, y with $xy, yx \in E(D_{k+1})$). The following asserts

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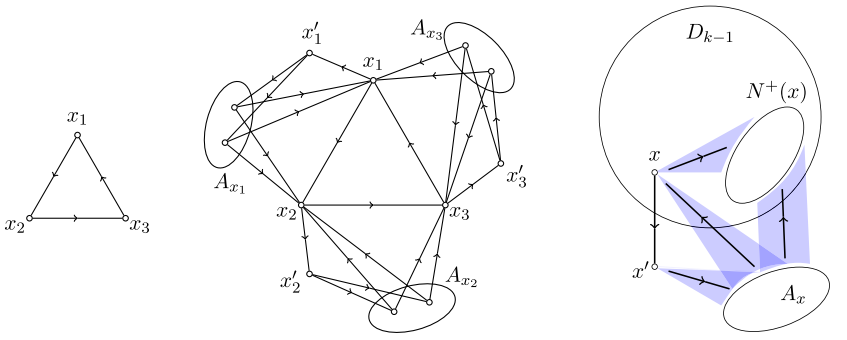


Figure 2. The digraph D_k for $k = 1, 2$ and sketch for general k.

that D_k does not contain large strong clique minors. In particular, large out-degree alone is not sufficient to guarantee strong minors in digraphs.

Proposition 4.3. *For every $k \geq 1$, D_k does not contain a strong $\overrightarrow{K}_3$ minor.*

Proof. We proceed by induction on k. This is clearly true for $k = 1$, so suppose $k \geq 2$ and the result holds for smaller values of k. Suppose, by way of contradiction, that D_{k+1} contains a strong $\overrightarrow{K}_3$ minor with branch sets U, W, X. Since D_k contains no strong $\overrightarrow{K}_3$ minor, it must be the case that at least one of these branch sets intersects $A_x \cup \{x'\}$ for some $x \in V(D_k)$. So suppose U intersects $A_x \cup \{x'\}$. We claim that $x \in U$.

Indeed, first suppose that $x' \in U$. Since $N^-(x') = \{x\}$, any nontrivial directed path to x' passes through x. As $D[U]$ is strongly connected, we either have $x \in U$ or $U = \{x'\}$. The latter cannot hold, since otherwise we must have both $x \in X$ and $x \in W$, a contradiction. Thus, we have $x \in U$.

Now suppose that $y \in U$ for some $y \in A_x$. Then since $N^-(y) = \{x'\}$, any nontrivial directed path to y includes x', so either $x' \in U$ or $U = \{y\}$. In the latter case, we must have both $x' \in X$ and $x' \in W$, a contradiction. In the former case, we have $x \in U$, by the above argument.

In any case, $x \in U$ as claimed. But then $U' = U \cap V(D_k)$ induces a strongly connected subdigraph in D_k. Indeed, if $U' = \{x\}$, then it is trivially strongly connected in D_k. Otherwise, let $u, v \in U'$ be distinct vertices. Then any path from u to v that uses vertices not in D_k, say from $A_x \cup \{x'\}$, passes through x and eventually an out-neighbour z of x. So there is a path from u to v contained in U', obtained by going from x directly to z and continuing outside of $A_x \cup \{x'\}$. Similarly, we have that $W' = W \cap V(D_k)$ and $X' = X \cap V(D_k)$ induce strongly connected subdigraphs in D_k.

Since U, W, and X are pairwise disjoint, then for any x at most one of these three sets intersects $A_x \cup \{x'\}$. Since there are no edges between $A_x \cup \{x'\}$ and $A_y \cup \{y'\}$ for distinct x and y, all edges between U, W, and X are the edges between U', W', and x'. Thus U', W', and x' form branch sets of a strong $\overrightarrow{K}_3$ minor in D_k, a contradiction. □

5. Strong minors in tournaments

5.1. Large dichromatic number in tournaments

The aim of this section is to prove Theorem 1.2. Before doing so, however, we shall prove the following warm-up result, which shows that dichromatic number linear in r is sufficient to guarantee strong $\overrightarrow{K}_r$ minors in tournaments.

Proposition 5.1. *Let $r \geq 1$ be an integer and suppose T is a tournament with $\chi(T) \geq 3r$. Then* $\mathrm{sm}(T) \geq r$.

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Proof. We proceed by induction on r. Clearly, this is true for $r = 1$, so we may assume $r \geq 2$ and the result holds for smaller values of r. Suppose T is a tournament with $\chi(T) \geq 3r$. We may assume T is strongly connected; otherwise, by Lemma 3.2, we can pass to a strongly connected component with dichromatic number at least $3r$. Let $A \subset V(T)$ be a largest transitive subset with source s and sink t. Since T is strongly connected, there exists a directed path in T from t to s. Pick a shortest directed path P from t to s. Note that $\chi(T[V(P)]) \leq 2$: by the minimality of P, all edges between vertices of P that have distance at least 2 on the path are oriented backwards. It follows that vertices at odd/even distances from the initial vertex produce a 2-colouring of P. Then $\chi(T[A \cup V(P)]) \leq 3$. Let $T' = T - (A \cup V(P))$. It follows that $\chi(T') \geq 3r - 3 = 3(r-1)$. By induction, we have that T' contains a strong $\overrightarrow{K}_{r-1}$ minor.

Observe that the maximality of A implies that for every vertex $v \in V(T) \setminus A$, v has at least one out-neighbour and at least one in-neighbour in A. In particular, for each branch set of our strong $\overrightarrow{K}_{r-1}$ minor in T', there are edges in both directions between A and this branch set. Finally, note that $T[A \cup V(P)]$ is strongly connected, so $A \cup V(P)$ can be taken as the rth branch set in a strong $\overrightarrow{K}_r$ minor in T, completing the induction and the proof. □

The proof of Proposition 5.1 follows by first considering a maximal transitive set, ensuring that this set is in and out 'dominating', and then making this set strongly connected by attaching a shortest path. Using this naive approach, we can only guarantee the resulting set has dichromatic number at most 3. We would like to reduce '3' to '2', and thus in order to prove Theorem 1.2, we must work a little harder. The following lemma, which may be viewed as the heart of the proof of Theorem 1.2, shows that one can find strongly connected dominating sets of dichromatic number 2 in any strongly connected tournament. This is clearly tight on the dichromatic number and it is a result of independent interest.

Before embarking on the proof, let us introduce some useful terminology. Let D be a digraph, let $S \subseteq V(D)$, and let $X \subseteq V(D) \setminus S$. We say that S *out-dominates* (*in-dominates*, resp.) X if every $x \in X$ has at least one in-neighbour (out-neighbour, resp.) in S. If $X = V(D) \setminus S$, we say that S is *out(in)-dominating*. If S is both out-dominating and in-dominating, we shall say that S is *dominating* (or, that it is a *dominating set*).

Lemma 5.2. *Let T be a strongly connected tournament. Then there exists a subset $R \subseteq V(T)$ satisfying the following:*

- *$T[R]$ is strongly connected,*
- *R is dominating, and*
- *$\chi(T[R]) = 2$.*

Proof. We illustrate the proof in Figure 3. Let x be a vertex in T with maximum out-degree in T. Let $S(x)$ be a largest possible transitive subtournament with sink x. Denote by y the source of $S(x)$. Observe that since $S(x)$ is maximum-sized, it must be out-dominating: for all $v \in V(T) \setminus S(x)$ there is $u \in S(x)$ such that $u \to v$. Moreover, as x has maximum out-degree in T, there exists a vertex $w^* \in V(T) \setminus S(x)$ such that $x \to w^*$ and $w^* \to y$, otherwise y would have a larger out-degree than x.

Now, let $F = \bigcap_{s \in S(x)} N^+(s)$. The set F consists of the vertices that are *not* in-dominated by $S(x)$. If $F = \varnothing$, then we are done, since then we may take $R = S(x) \cup \{w^*\}$. So we may assume that $F \neq \varnothing$. Let $F^+ \subseteq F$ be the set of vertices in F that have an out-neighbour in $V(T) \setminus F$. Since T is strongly connected, it follows that $F^+ \neq \varnothing$. We need to find a set (hopefully of small dichromatic number) that in-dominates the vertices in F. First, we prove the following claim showing that we may partition F in a useful way.

Claim 1. *For some $k \geq 1$ there is a partition $F = F_1 \cup \cdots \cup F_k$ such that $F_1 = F^+$ and for every $1 \leq i \leq k-1$*

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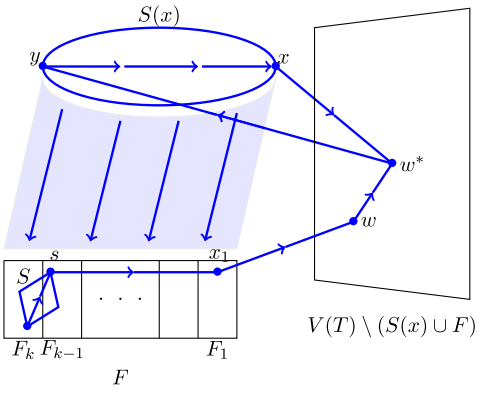


Figure 3. The proof of Lemma 5.2. The direction of ww^* is not important for the proof.

(1) for every $u \in F_{i+1}$ there is $z \in F_i$ such that $u \to z$, and
(2) for every $j, j > i+1$, $F_i \to F_j$.

Proof. Let $F_1 = F^+$. Suppose $F_1, \ldots, F_i$ have been constructed for some $i \geq 1$. Let $Y = F \setminus (\bigcup_{j\in[i]} F_j)$. If $Y = \varnothing$, then stop with $k := i$. Otherwise, let

$$F_{i+1} = \left(\bigcup_{u \in F_i} N^-(u) \right) \cap Y.$$

Note that F_{i+1} is non-empty due to the strong-connectivity of T. Indeed, if $F_{i+1} = \varnothing$, then $F_i \to Y$ and since $Y \cap F_1 = \varnothing$, no vertex of Y has an out-neighbour in $V(T) \setminus F$. Thus, there is no directed path from a vertex in Y to a vertex in $F_1 = F^+$, a contradiction to the strong-connectivity assumption.

Also, note that

$$\bigcup_{j\in[i]} F_j \to Y \setminus F_{i+1}.$$

By construction, every vertex in F_{i+1} has at least one out-neighbour in F_i. This completes the proof of the claim. □

Let S_k be a largest transitive set contained in $T[F_k]$. If F_{k-1} exists (i.e. $k \geq 2$), let $S = S_k \cup S_{k-1}$, where $S_{k-1} \subseteq F_{k-1}$ is as large as possible such that S is transitive, and all vertices of S_{k-1} lie *after* the vertices in S_k in the transitive order. If $k = 1$, let $S = S_1$. Let s denote the sink of S. In particular, either $s \in F_k$ or $s \in F_{k-1}$. Intuitively, S in-dominates F (which we shall show later), and it remains to 'connect' S to $S(x)$ to find a dominating set that is strongly connected.

To this end, consider a shortest directed path $P = s \ldots x_1$ contained in F from s to a vertex in F_1; such a path exists by Claim 1. Also, observe that since P is a shortest path in F to F_1, it has exactly one vertex in each F_i. Now, since $x_1 \in F_1 = F^+$, by definition of F^+ we may choose an out-neighbour $w \in V(T) \setminus (F \cup S(x))$ of x_1. It is possible that $w = w^*$, but this only makes the proof simpler. So we assume that $w \neq w^*$. Finally, let

$$R = S(x) \cup S \cup V(P) \cup W.$$

The following sequence of claims shows that R has all of the desired properties (see Figure 3).

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Claim 2. *T[R] is strongly connected.*

Proof. First, note that $T[S(x) \cup W]$ is strongly connected. Indeed, $T[S(x) \cup \{w^*\}]$ is strongly connected since $w^* \in N^-(y) \cap N^+(x)$. As $w \notin F$, it has an out-neighbour $v \in S(x)$, and the maximality of $S(x)$ implies that w has an in-neighbour $v' \in S(x)$. It follows that $T[R]$ is strongly connected, as claimed. □

Claim 3. *R is a dominating set in T.*

Proof. $S(x)$ dominates $V(T) \setminus (S(x) \cup F)$: the maximality of $S(x)$ implies that it is out-dominating, and it additionally in-dominates $V(T) \setminus F$ (since F is exactly the set of vertices *not* in-dominated by $S(x)$). So it suffices to show that R in-dominates $F \setminus (S \cup V(P))$. In particular, we claim that S in-dominates $F \setminus (S \cup V(P))$. Indeed, by (2) of Claim 1 we have

$$\bigcup_{i \le k-2} F_i \to S \cap F_k,$$

so it suffices to show that S in-dominates $(F_k \cup F_{k-1}) \setminus S$. Recall that $S = S_k \cup S_{k-1}$ where S_k is a largest transitive set in F_k, and S_{k-1} is a largest possible transitive extension of S_k in F_{k-1} such that all vertices of S_{k-1} lie after S_k in the transitive order. Thus, if there is some vertex $z \in (F_k \cup F_{k-1}) \setminus S$ such that $S \to z$, we obtain a contradiction with our choice of S. It follows that S in-dominates $(F_k \cup F_{k-1}) \setminus S$, as claimed. □

Claim 4. $\chi(T[R]) = 2$.

Proof. We produce a 2-colouring of R that depends on the parity of k and also on whether s (the sink of S) is in F_k or F_{k-1}. We deal with the case when k is even first, and sketch the proof when k is odd, as the proof is nearly identical. Recall that as $P = s \dots x_1$ is a shortest $s - F_1$ path in F, it has exactly one vertex in each F_i (except for maybe F_k).

Case 1: k is even. First, suppose that $k = 2$. If $s \in F_2$, then let one colour class be $S(x) \cup \{x_1\}$ and the other $S \cup W$. Note that the latter set is transitive since $S \subseteq F_2$ and so sends no out-edges to $V(T) \setminus F$. On the other hand, if $s \in F_1$, then let one colour class be $S(x) \cup (S \cap F_1)$ and the other $(S \cap F_2) \cup W$.

Now, suppose $k > 2$ and $x_k := s \in F_k$. Write $P = x_k x_{k-1} \dots x_1$ with $x_i \in F_i$ for all $i \in [k]$. Let $P_{\text{odd}} = \{x_1, x_3, \dots, x_{k-1}\}$ and $P_{\text{even}} = \{x_2, x_4, \dots, x_k\}$. Then we can let one colour class be $S(x) \cup P_{\text{odd}}$ and the other $S \cup P_{\text{even}} \cup W$. Indeed, $S(x)$ is transitive and $S(x) \to P_{\text{odd}}$. Moreover, $T[P_{\text{odd}}]$ is transitive by (2) of Claim 1. Similarly, since $x_k = s \in F_k$ and $k > 2$, we have $S \cup P_{\text{even}} \subseteq F \setminus F_1$ and so $W \to S \cup P_{\text{even}}$. As before, $T[S \cup P_{\text{even}}]$ is transitive and clearly so is W.

Lastly, we consider the case when $x_{k-1} := s \in F_{k-1}$, that is, when $P = x_{k-1}, \dots, x_1$ let $P_{\text{even}} = \{x_2, \dots, x_{k-2}\}$ and let $P_{\text{odd}} = V(P) \setminus P_{\text{even}}$. In this case, we let one colour class be $S(x) \cup P_{\text{odd}} \cup (S \cap F_{k-1})$ and the other be $(S \cap F_k) \cup P_{\text{even}} \cup W$. It is routine to check that this is indeed a 2-colouring using the same arguments as above.

Case 2: *k is odd.* If $k = 1$, let $S(x) \cup S$ be one colour class, and W the other. If $k > 1$, then an analogous argument as in the previous case yields a 2-colouring of R. □

Hence, R satisfies all of the claimed properties. This completes the proof of the lemma. □

We can now easily deduce Theorem 1.2.

Proof of Theorem 1.2. We have already established the second part of the statement in Proposition 4.1. It remains to prove the first part.

We proceed by induction on r. Clearly, the result holds for $r = 1$, so we suppose $r \ge 2$ and the theorem holds for smaller values. Let T be a tournament with $\chi(T) \ge 2r$, and, by Lemma 5.2, we may assume T is strongly connected. Lemma 5.2 grants a subset $R \subseteq V(T)$ such that $T[R]$ is strongly connected, R is dominating in T, and $\chi(T[R]) = 2$. Letting $T' = T - R$, we see that

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$\chi(T') \geq 2r-2 = 2(r-1)$, and so by induction T' contains a strong $\overrightarrow{K}_{r-1}$ minor. Taking R as the rth branch set yields a strong $\overrightarrow{K}_r$ minor in T. □

Perhaps the correct constant in Theorem 1.2 should be '1' instead of '2' as indicated by the construction S_r. In other words, it is possible that any tournament with dichromatic number at least $r+1$ contains a strong $\overrightarrow{K}_r$ minor. We can prove this in the first nontrivial case:

Proposition 5.3. *If T is a tournament with $\chi(T) \geq 3$, then* $\mathrm{sm}(T) \geq 2$.

Proof. Pick a directed triangle C_1 in T. Since $\chi(T) \geq 3$, $T - C_1$ is non-empty. If there is a vertex $v \notin C_1$ that sends edges in both directions to C_1, then we are done. Otherwise, $T - C_1$ can be partitioned into two sets A_1 and B_1 such that $A_1 \to C_1$ and $C_1 \to B_1$. If there is an edge e from B_1 to A_1, then we are done. Indeed, one branch set is the directed triangle formed from e and one vertex of C_1, and the other branch set is just one of the remaining vertices of C_1. So we may assume that $A_1 \to B_1$. In this case, we either have $\chi(A_1) \geq 3$ or $\chi(B_1) \geq 3$. Without loss of generality, $\chi(A_1) \geq 3$, and so we pass to this subtournament and repeat the argument.

In this process, we obtain a sequence of non-empty sets strictly decreasing in size, so it must eventually terminate with a strong $\overrightarrow{K}_2$ minor. □

5.2. Large out-degree in tournaments

In this section, we turn our attention to investigating the presence of strong minors in tournaments with a minimum out-degree condition. In particular, we prove Theorem 1.3. The strategy of our proof is as follows. First, we show that there is a constant C' such that any tournament that is $C'r\sqrt{\log r}$-strongly connected contains a strong $\overrightarrow{K}_r$ minor (see Lemma 5.8). Given a tournament T with $\delta^+(T) \geq Cr\sqrt{\log r}$ where $C \gg C'$, we may assume that no subtournament of T is $C'r\sqrt{\log r}$-strongly connected. This assumption together with the minimum out-degree condition allows us to iteratively construct a strong $\overrightarrow{K}_r$ minor in which every branch set is, in fact, a directed triangle. This last step is made precise in Lemmas 5.9 and 5.10.

Before proceeding, we need a few preliminary results. Given a positive integer k and a digraph D, we say that D is *k-linked* if $|V(D)| \geq 2k$ and for any two disjoint sets of vertices $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ of k vertices each, there are pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ such that P_i has initial vertex x_i and terminal vertex y_i for every $i \in [k]$. We shall use the following theorem of Pokrovskiy [18], showing that large enough strong-connectivity guarantees linkedness in tournaments.

Theorem 5.4. (Pokrovskiy [18]). *For every integer $k \geq 1$, any 452k-strongly connected tournament is k-linked.*

Recall that a weak minor has the same definition as a strong minor except that we only require that the branch sets induce connected (not necessarily strongly connected) subgraphs. As mentioned in the Introduction, Jagger [11] investigated average degree conditions for finding weak minors in digraphs. In the context of tournaments, we need the following:

Theorem 5.5. (Jagger [11]). *There exists an absolute constant $C > 0$ such that the following holds. If r is a positive integer and T is a tournament with at least $Cr\sqrt{\log r}$ vertices, then T contains a weak $\overrightarrow{K}_r$ minor.*

During the course of the proof of Theorem 1.3, we shall construct an auxiliary (undirected) graph and apply the following classical result:

Theorem 5.6. (Kostochka [15], Thomason [22]). *There exists a constant $c > 0$ such that for every positive integer r, every graph of average degree at least $cr\sqrt{\log r}$ contains a K_r minor.*

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Finally, we need one more preliminary result. We say that a subset $B \subset V(T)$ is *K-nearly-regular* if either $d_T^-(v) \le d_T^+(v) \le K d_T^-(v)$ for every $v \in B$, or $d_T^+(v) \le d_T^-(v) \le K d_T^+(v)$ for every $v \in B$. The following lemma appears in [5]. We reproduce the proof here for convenience.

Lemma 5.7. *Any tournament T contains a 4-nearly-regular subset of vertices R of size at least $\lfloor |T|/20 \rfloor$. In particular, any $v \in R$ has in and out-degree at least $|T|/6$.*

Proof. Let $|T| = n$; we may assume $n \ge 20$, otherwise the result is trivial. Let $R \subset V(T)$ be the set of vertices for which either the ratio between the out-degree and in-degree or vice-versa is between 1 and 4. If $|R| \ge n/10$, then we are done, as we may pass to a subset $A \subset R$ of at least half the size of R that is 4-nearly-regular. If $|R| < n/10$, then let $T' = T - R$, so that $|T'| > 9n/10$. Let T'_1 be the set of vertices $v \in V(T')$ for which $d_T^+(v) > 4d_T^-(v)$ and T'_2 be those vertices $v \in V(T')$ for which $d_T^-(v) > 4d_T^+(v)$. Suppose without loss of generality that $|T'_1| \ge |T'_2|$, so that $|T'_1| \ge \lceil 9n/20 \rceil$. Since each tournament on t vertices has a vertex of in-degree $\ge \lceil (t-1)/2 \rceil$, this implies that there is a vertex u in T'_1 which has in-degree inside T'_1 at least $(\lceil 9n/20 \rceil - 1)/2$. However,

$$d_T^-(u) < \frac{1}{4} d_T^+(u) < \frac{1}{4}(n - d_T^-(u)),$$

implying that $d_T^-(u) < n/5$. Thus, we obtain

$$(\lceil 9n/20 \rceil - 1)/2 \le d_T^-(u) < n/5,$$

which yields a contradiction for $n \ge 20$.

Finally, let $R \subseteq V(T)$ be 4-nearly-regular, and assume without loss of generality that $d_T^-(u) \le d_T^+(u) \le 4d_T^-(u)$ for all $u \in R$. If for some $v \in R$ we have $d_T^-(v) < n/6$, then $d_T^+(v) < 2n/3$, and so $n - 1 = d_T^+(u) + d_T^-(u) < 5n/6$, a contradiction since $n \ge 20$. □

With these preliminaries in place, we shall prove our first lemma, guaranteeing strong $\overrightarrow{K_r}$ minors in $C' r\sqrt{\log r}$-strongly connected tournaments. The strategy is to first embed a weak $\overrightarrow{K_r}$ minor in a 'nice' portion of the tournament (using Theorem 5.5), and then make each branch set strongly connected by linking appropriate pairs of vertices.

Lemma 5.8. *Let r be a positive integer. There exists a constant $C' > 0$ such that if T is a $C' r\sqrt{\log r}$-strongly connected tournament, then* $\mathrm{sm}(T) \ge r$.

Proof. Put $C_0 = \max\{C, 24\}$ where C is given by Theorem 5.5. Let $C' = 40C_0$, and suppose T is a tournament that is $C' r\sqrt{\log r}$-strongly connected. By Lemma 5.7, we can find a subset $S \subset V(T)$ of size at least $\lfloor |T|/20 \rfloor \ge C_0 r\sqrt{\log r}$ such that all vertices of S have in- and out-degree at least $|T|/6$. Take a subset $S' \subset S$ of size exactly $C_0 r\sqrt{\log r}$ (ignoring floors/ceilings) and let $T' = T - S'$. Find a weak $\overrightarrow{K_r}$ minor in S' with branch sets $B_1, \dots, B_r$, using Theorem 5.5.

Since T' is still $19C_0 r\sqrt{\log r}$-strongly connected, by Theorem 5.4, T′ is $(19C_0/452) r\sqrt{\log r}$-linked. Hence, T' is r-linked by our choice of C_0. We now construct a strong $\overrightarrow{K_r}$ minor as follows. For each $i \in [r]$, if $T[B_i]$ is already strongly connected, then we take B_i as a branch set. Otherwise, B_i decomposes into strongly connected components $X_1^i, \dots, X_t^i$ for some $t \ge 2$, such that $X_p^i \to X_q^i$ for all $1 \le p < q \le t$. Let s_i^1 be any vertex in X_1^i and let s_i^2 be any vertex of X_t^i.

Now, every vertex in S' has many in and out-neighbours in T'. More precisely, every vertex in S' has at least

$$|T|/6 - |S'| \ge \frac{20C_0 r\sqrt{\log r}}{6} - C_0 r\sqrt{\log r} > C_0 r\sqrt{\log r},$$

in-neighbours and out-neighbours in T'. Thus, for each s_i^1 we may choose an in-neighbour $u_i^1 \in V(T')$, and for each s_i^2 we may choose an out-neighbour $u_i^2 \in V(T')$, such that the vertices u_i^j are all

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pairwise distinct. In this way, we have disjoint sets $X = \{u_1^1, \dots, u_s^1\}$ and $Y = \{u_1^2, \dots, u_s^2\}$ where $s \le r$. Finally, use the fact that T' is r-linked to find pairwise vertex-disjoint directed paths P_i in T' from u_i^1 to u_i^2 for each $i \in [s]$. Clearly, $T[B_i \cup V(P_i)]$ is strongly connected for each $i \in [s]$, and thus we have found a strong $\overrightarrow{K}_r$ minor in T. □

Lemma 5.8 is tight up to the constant C, by considering a random tournament T on $cr\sqrt{\log r}$ vertices. As mentioned in the Introduction, following the argument of Bollobás, Catlin, and Erdős [3], w.h.p. it contains no strong $\overrightarrow{K}_r$ minor (in fact, it contains no r pairwise disjoint non-empty sets with edges in each direction between each pair of sets). Additionally, it is not hard to show that w.h.p. T is $(cr\sqrt{\log r})/10$-strongly connected. This can be seen by noting that w.h.p. every two vertices have at least approximately $|T|/10$ directed paths of length 2 in each direction.

The following two lemmas allow us to find some structure in a tournament with large minimum out-degree, under the additional assumption that it is not highly strongly connected. We use the following notation in their formulations. If $\mathcal{H}$ is a family of subdigraphs of D, let $\bigcup \mathcal{H}$ denote the set $\bigcup_{H \in \mathcal{H}} V(H)$.

Lemma 5.9. *Let $m \ge 2$ and $d \ge 1$ be integers and suppose T is a vertex-minimal tournament with the property that $\delta^+(T) \ge d$. Then there exists a positive integer $t \le m-1$ such that the following holds. If T is not m-strongly connected, then we can find a non-empty set $S = \{w_1v_1, \dots, w_tv_t\}$ of pairwise vertex-disjoint directed edges and a subtournament $T' \subseteq T$ with the following properties:*

- $V(T') \cap (\bigcup S) = \varnothing$.
- *For any $i \in [t]$ we have $|N^+(v_i) \cap T'| \ge |T'| - m + 1$, and $|N^-(w_i) \cap T'| \ge |T'| - d + 1$.*
- $\delta^+(T') \ge d - 2t$.

Proof. Assume that T is not m-strongly connected and let R be a smallest cut-set of size at most $m-1$. Observe that $R \ne \varnothing$: if T itself is not strongly connected, then consider the source set X and sink set Y of T. Since $X \to Y$, we must have $\delta^+(T[Y]) \ge d$, contradicting the minimality of T.

Let A and B denote the source set and sink set of $T - R$, respectively, and define

$$R' = \{x \in R : |N^+(x) \cap (B \cup R)| < d\}.$$

The minimality of T implies that $R' \ne \varnothing$: if $R' = \varnothing$, then $\delta^+(T[B \cup R]) \ge d$, a contradiction. Now, choose a maximum matching $M = \{w_1v_1, \dots, w_tv_t\}$ from R' to A and let $T' = T[(B \cup R) \setminus V(M)]$ and $S = M$. We claim that T' and S satisfy the required properties. Clearly, the first property is satisfied. To see the last property suppose that $x \in B \cup (R \setminus R')$, then its degree is at least $d - t$ in T'. On the other hand, if $x \in R' \setminus V(M)$, since we cannot extend the matching M, $N_T^+(x) \cap A \subset V(M)$, which implies $d_{T'}^+(x) \ge d - |V(M)| = d - 2t$. To see the second property, consider a vertex v_i. Then $v_i \to B$, and so there are at most $|R| \le m-1$ vertices in $|T'|$ that are in-neighbours of v_i. Similarly, consider a vertex w_i. Since $w_i \in R'$, by definition $|N^+(x) \cap (B \cup R)| < d$. Thus, w_i has at least $|T'| - d + 1$ in-neighbours in T'. □

Our final lemma is obtained from iteratively applying Lemma 5.9 under the assumption that no subtournament is highly strongly connected.

Lemma 5.10. *Suppose $m \ge 2$ and $d \ge 1$ are integers, and suppose T is a tournament such that $\delta^+(T) \ge d$ and no subtournament of T is m-strongly connected. Then there exists a set $S = \{w_iv_i : i \in [m]\}$ composed of pairwise vertex-disjoint directed edges and a set $F(S) \subseteq V(T)$ such that for every $i \in [m]$*

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(1) $F(S) \cap (\bigcup S) = \varnothing$ *and* $|F(S)| \geq 2(d-4m)$,
(2) $|N^+(v_i) \cap F(S)| \geq |F(S)| - m$, *and*
(3) $|N^-(w_i) \cap F(S)| \geq |F(S)| - d$.

Proof. We iteratively apply Lemma 5.9. Begin by removing vertices from T such that the remaining tournament T_0 is vertex-minimal with $\delta^+(T_0) \geq d$. Then, since T_0 is not m-strongly connected by assumption, Lemma 5.9 implies that there is a positive integer $t_1 < m$, a set of edges $S_1 = \{w_i^1 v_i^1 : i \in [t_1]\}$, and a subtournament $F_1 \subseteq T_0$ such that

- $V(F_1) \cap (\bigcup S_1) = \varnothing$,
- $\delta^+(F_1) \geq d - 2t_1$, and
- $|N^+(v_i^1) \cap F_1| \geq |F_1| - m$ and $|N^-(w_i^1) \cap F_1| \geq |F_1| - d$ for all $i \in [t_1]$.

We proceed by applying Lemma 5.9 inside F_1. Now assume we have already found subtournaments $F_1 \supset \cdots \supset F_k$ with $\delta^+(F_k) \geq d - 2\sum_{i=1}^k t_i$, and disjoint sets of edges $S_1, \ldots, S_k$ of sizes $t_1, \ldots, t_k$, such that for each $j \in [k]$ we have $|N^+(v_i^j) \cap F_k| \geq |F_k| - m$ and $|N^-(w_i^j) \cap F_k| \geq |F_k| - d$ for all $i \in [t_k]$.

Assume that $\sum_{i=1}^k t_i < m$. We first find a vertex-minimal subtournament T_k of F_k such that $\delta^+(T_k) \geq d - 2\sum_{i=1}^k t_i$. Since by assumption T_k is not m-strongly connected, Lemma 5.9 implies that there is $t_{k+1} < m$, a non-empty set of edges $S_{k+1} = \{w_i^{k+1} v_i^{k+1} : i \in [t_{k+1}]\}$, and a subtournament $F_{k+1} \subseteq T_k$ such that

- $V(F_{k+1}) \cap (\bigcup S_i) = \varnothing$ for $i \in [k+1]$,
- $\delta^+(F_{k+1}) \geq \delta^+(T_k) - 2t_{k+1} \geq d - 2\sum_{i=1}^{k+1} t_i$,
- $|N^+(v_i^j) \cap F_{k+1}| \geq |F_{k+1}| - m$ and $|N^-(w_i^j) \cap F_{k+1}| \geq |F_{k+1}| - d$ for $j \in [k+1]$ and $i \in [t_j]$.

We stop this process at a step k_0 when $\sum_{i=1}^{k_0} t_i \geq m$, and let

$$S = \bigcup_{i=1}^{k_0} S_i \quad \text{and} \quad F(S) = F_{k_0}.$$

Then clearly properties (2) and (3) of the claim are satisfied, and we have a set S of at least m edges which are all disjoint from $F(S)$. We just need to show that $|F(S)| \geq 2(d-4m)$. Since $\sum_{i=1}^{k_0-1} t_i < m$ and $t_{k_0} < m$, we have that $\sum_{i=1}^{k_0} t_i < 2m$, and so $\delta^+(F_{k_0}) \geq d - 4m$. Thus,

$$|F(S)| = |F_{k_0}| \geq 2\delta^+(F_{k_0}) \geq 2(d-4m),$$

as claimed. □

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. As we already mentioned, the tightness can be seen by considering a random tournament on $\Omega(r\sqrt{\log r})$ vertices.

Now, let C' be the constant given by Lemma 5.8 and let c be the constant given by Theorem 5.6. Put $C_0{}' = \max\{C', 62c\}$ and $C = 1600C_0{}'$. Define $m = C'r\sqrt{\log r}$ and $d = Cr\sqrt{\log r}$, and suppose T is a tournament with $\delta^+(T) \geq d$. Note that $C \geq 1600C'$. If T, or any subtournament of T, is m-strongly connected, then we are done by Lemma 5.8; so we may assume otherwise. In this

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situation, we may apply Lemma 5.10. Indeed, let S and $F(S)$ be given as in Lemma 5.10, and let $N_i = N^-(w_i) \cap F(S)$, $M_i = N^+(v_i) \cap F(S)$, and $L_i = M_i \cap N_i$. We first observe that for each $i \in [m]$

$$|L_i| \geq |F(S)| - d - m \geq d/2 - m \geq d/4, \tag{5.1}$$

using the fact that $|F(S)| \geq 2(d - 4m)$. For every $i \in [m]$ choose a vertex z_i from L_i uniformly at random with replacement and let L denote the resulting random set of vertices. Note that $\Delta_i := \{w_i, v_i, z_i\}$ forms a directed triangle. Let X denote the random variable recording the size of a largest subset of L in which all vertices are pairwise distinct. Further, let x' count the number of pairs $\{z_i, z_j\}$ with $i \neq j$ and $z_i = z_j$. We have that $\mathbb{P}(z_i = z_j) = |L_i \cap L_j|/|L_i||L_j| \leq 4/d$ using (5.1). It follows that $\mathbb{E}[X'] \leq \binom{m}{2}\frac{4}{d} \leq 2m^2/d$, and hence

$$\mathbb{E}[X] \geq m - \mathbb{E}[X'] \geq m - 2m^2/d = (C' - 2C'^2/C)r\sqrt{\log r} \geq (C'/2)r\sqrt{\log r} = m/2, \tag{5.2}$$

since $C \geq 1600C'$.

We say that a pair $\{\Delta_i, \Delta_j\}$ of directed triangles is *good* if there are edges between them in both directions; otherwise, we say this pair is *bad*. Observe that

$$\begin{aligned}\mathbb{P}(\{\Delta_i, \Delta_j\} \text{ is bad }) &\leq \mathbb{P}(z_j \notin N^+(v_i) \cap F(S) \text{ or } z_i \notin N^+(v_j) \cap F(S)) \\ &\leq \frac{m}{|L_j|} + \frac{m}{|L_i|} \leq \frac{8m}{d} = \frac{8C'}{C} \leq \frac{1}{200},\end{aligned}$$

using (5.1) and since $C \geq 1600C'$. Hence, letting Y denote the random variable counting the number of bad pairs of directed triangles, we have that

$$\mathbb{E}[Y] \leq \binom{m}{2}\frac{1}{200} \leq m^2/400. \tag{5.3}$$

Combining (5.2) and (5.3), and using the Cauchy–Schwarz inequality, we have

$$\mathbb{E}\left[X^2 - 40Y - \frac{m^2}{9}\right] \geq \mathbb{E}[X]^2 - 40\mathbb{E}[Y] - \frac{m^2}{9} \geq \frac{m^2}{4} - \frac{m^2}{10} - \frac{m^2}{9} > 0.$$

Accordingly, there is a choice of vertices from $\bigcup_{i=1}^m L_i$ such that $X^2 - 40Y - m^2/9 > 0$. It follows that $X^2 \geq m^2/9$, and so $X \geq m/3$; that is, there is a set U of at least $m/3$ vertices and they are all pairwise distinct. By possibly passing to a subset, let us assume $|U| = \lfloor m/3 \rfloor$. Moreover, we must have $40Y \leq X^2 \leq m^2$, and so the number of bad pairs of triangles is at most $m^2/40$.

Our final aim is to define a suitable auxiliary graph on our directed triangles and apply Theorem 5.6. Define a graph G in the following way: put $V(G) = \{\Delta_i : z_i \in U\}$ and join Δ_i to Δ_j if and only if the pair $\{\Delta_i, \Delta_j\}$ is good. By our above analysis, we have $|V(G)| = \lfloor m/3 \rfloor$, and there are at most $m^2/40$ non-edges in G. Thus, there are at least

$$\binom{\lfloor m/3 \rfloor}{2} - m^2/40 \geq m^2/36 - m^2/40 \geq .0027m^2,$$

edges in G, and so the average degree is at least $.0162m \geq .0162(62cr\sqrt{\log r}) > cr\sqrt{\log r}$. Here we are using the fact that $m = C'r\sqrt{\log r}$ and $C' \geq 62c$ by definition. Applying Theorem 5.6 to G, we have that G contains a K_r minor. It is easy to see that this K_r minor corresponds to a strong $\overrightarrow{K_r}$ minor in the original tournament. This completes the proof of Theorem 1.3. □

6. Strong minors in digraphs

We already know that large out-degree is, in general, not sufficient to guarantee large complete minors in digraphs. In this section, we give a positive result in digraphs for dichromatic number.

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More precisely, we prove Theorem 1.5, showing that there is a function $f(r)$ such that any digraph with dichromatic number at least $f(r)$ contains a strong $\overrightarrow{K}_r$ minor. First, we need a few definitions. A $\overrightarrow{K}_r$*-template* is a digraph D such that $V(D)$ admits a partition into sets $X_1, \dots, X_r$ such that for every ordered pair (i, j) with $i \neq j \in [r]$, there is an edge from X_i to X_j. We say that a digraph contains a $\overrightarrow{K}_r$-template if it contains a $\overrightarrow{K}_r$-template as a subdigraph. It is easy to find $\overrightarrow{K}_r$-templates in digraphs with high dichromatic number:

Lemma 6.1. *Let $r \geq 1$ be an integer and suppose D is a digraph with $\chi(D) \geq r$. Then D contains a $\overrightarrow{K}_r$-template.*

Proof. Let A be a maximum sized transitive set in D and let $D' = D - A$. Then every vertex in D' has an out-neighbour and an in-neighbour in A (otherwise, we could create a larger transitive set in D). Moreover, $\chi(D') \geq r - 1$. The result follows by induction on r, by taking the union of a $\overrightarrow{K}_{r-1}$-template in D' together with the set A. □

We are now ready to prove Theorem 1.5. The proof follows the ideas of Aboulker et al. [?] for embedding subdivisions of $\overrightarrow{K}_r$ in digraphs of high dichromatic number.

For a strongly connected digraph D, we define an *out-BFS-tree* $T^+ = T_v^+$ in D with root v as a subdigraph of D spanning $V(D)$, such that T_v^+ is an oriented tree and for every $w \in V(D)$ we have $\text{dist}_{T^+}(v, w) = \text{dist}_D(v, w)$. Similarly, an *in-BFS-tree* $T^- = T_v^-$ rooted at v is a subdigraph of D spanning $V(D)$ which is an oriented tree, such that for every $w \in V(D)$ we have $\text{dist}_{T^-}(w, v) = \text{dist}_D(w, v)$. For every integer i and for a vertex $v \in V(D)$, let $L_i^{+,v}$ denote the ith out-layer from v in D, that is, $L_i^{+,v} = \{w \in V(D) : d_D(v, w) = i\}$. Similarly, $L_i^{-,v}$ denotes the ith in-layer from v in D, that is, $L_i^{-,v} = \{w \in V(D) : d_D(w, v) = i\}$.

We need the following simple fact:

Lemma 6.2. *Let D be a strongly connected digraph and let T be an in- or out-BFS-tree in D. Let X be any subset of vertices of D. Then there is a layer L of T such that $\chi(D[L \cap X]) \geq \chi(D[X])/2$.*

Proof. Without loss of generality, suppose that T is an out-BFS-tree in D with layers $L_1, \dots, L_m$. Let D_1 and D_2 be the subdigraphs of D induced by the odd and even layers of T, respectively. Since there is no arc from L_i to L_j for $j > i + 1$, the strongly connected components of D_1 and D_2 must be contained within the layers. It follows that $\chi(D_1[X]) = \max_{i \text{ odd}} \chi(D[L_i \cap X])$ and $\chi(D_2[X]) = \max_{i \text{ even}} \chi(D[L_i \cap X])$. Since D is strongly connected, we have $V(D) = V(D_1) \cup V(D_2)$ and thus, $\chi(D[X]) \leq \chi(D_1[X]) + \chi(D_2[X]) \leq 2 \max_{i \in [m]} \chi(L_i \cap X)$. □

Proof of Theorem 1.5. Let D be a digraph such that $\chi(D) \geq r4^r$. We shall show that $\text{sm}(D) \geq r$. We may assume that D is strongly connected, otherwise, by Lemma 3.2, pass to a strongly connected component of high dichromatic number. For an integer m with $0 \leq m \leq r$, an *m- partial strong* $\overrightarrow{K}_r$ minor in a digraph D is a collection of pairwise disjoint subsets $X_1, \dots, X_r \subseteq V(D)$ such that

- for every pair (i, j) with $i \neq j \in [r]$, there exists an edge from X_i to X_j, and
- $D[V_i]$ is strongly connected for at least m sets V_i.

Note that a $\overrightarrow{K}_r$-template is a 0-partial strong $\overrightarrow{K}_r$ minor. We define $f_r(m)$ to be the smallest k such that any digraph with $\chi(D) \geq k$ contains an m-partial strong $\overrightarrow{K}_r$ minor. Note that by Lemma 6.1, $f_r(0) \leq r$.

Claim 1. *For any m, $0 \leq m \leq r$, $f_r(m) \leq r4^m$.*

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Claim 1 implies in particular that any digraph with dichromatic number at least $r4^r$ has a strong $\overrightarrow{K}_r$ minor. We shall prove Claim 1 by induction on m. Let $\chi(D) \geq r4^m$, $0 \leq m \leq r$. If $m = 0$, then the claim follows by Lemma 6.1. So assume that $1 \leq m \leq r$ and that the claim holds for smaller m. Let $v \in D$, let T^-, T^+ be the in- and out-BFS-trees rooted at v, respectively.

Apply Lemma 6.2 to D with $T = T^-$ and $X = V(D)$ first to find a layer L in $T = T^-$ such that $\chi(D[L]) \geq \chi(D)/2$. Then apply Lemma 6.2 to D with $T = T^+$ and $X = L$ to find a layer L' in T^+ such that $\chi(D[L' \cap L]) \geq \chi(D[L])/2$. Thus, we have that $\chi(D[L' \cap L]) \geq \chi(D)/4 \geq r4^{m-1}$. By induction, we have that $f_r(m-1) \leq r4^{m-1}$, thus we can find an $(m-1)$-partial strong $\overrightarrow{K}_r$ minor K contained in $L' \cap L$ with branch sets $V_1, \ldots, V_r$. If all sets V_i induce a strongly connected digraph, we have that K is a strong $\overrightarrow{K}_r$-minor and $\mathrm{sm}(D) \geq r$, so we are done. Otherwise assume without loss of generality that $D[V_1]$ is not strongly connected. Let $V_1 = \{x_1, \ldots, x_s\}$. Now for each $\ell \in [s]$ let P_ℓ^+ be the directed $v - x_\ell$ path in T^+ and P_ℓ^- the directed $x_\ell - v$ path in T^-. Note that these paths 'leave' the respective layers in T^+ and T^- immediately. Then for $\ell \in [s]$, $\{x_\ell\} \cup (P_\ell^-) \cup (P_{\ell+1}^+) \cup \{x_{\ell+1}\}$ (with addition modulo s) induces an $x_\ell - x_{\ell+1}$ walk, which contains an $x_\ell - x_{\ell+1}$ path. This path intersects K exactly in the two points $\{x_\ell, x_{\ell+1}\}$, since $V(K)$ is contained in a layer L of T^- and in a subset L' of a layer of T^+. Thus, letting $V_1' = V_1 \cup \bigcup_{\ell=1}^{s} P_\ell^+ \cup P_\ell^-$, we see that $D[V_k']$ is strongly connected and intersects K only in V_1. Hence, we obtain an m-partial strong $\overrightarrow{K}_r$ minor K' with branch sets $V'_1, V_2, \ldots, V_r$. This proves the claim and the theorem. □

7. Concluding remarks and open problems

We have investigated several relationships between large dichromatic number and the presence of strong complete minors in digraphs. Many problems remain. Regarding Theorem 1.2, the most obvious question is whether or not the bound on dichromatic number can be decreased.

Question 7.1 Is it true that any tournament T with $\chi(T) \geq (1 + o(1))r$ satisfies $\mathrm{sm}(T) \geq r$?

For general digraphs, our bounds are far apart. To take a more general view, we may address the following question: Which digraph parameters force the existence of large strong complete minors? To formalise this, for a digraph H and a digraph parameter ϕ, we let $\mathrm{sm}_\phi(H)$ denote the smallest integer such that any digraph D with $\phi(D) \geq \mathrm{sm}_\phi(H)$ contains a strong H minor; if such an integer does not exist, then we define $\mathrm{sm}_\phi(H) = \infty$. This is analogous to the work of Aboulker et al. [1], where they introduced the parameter $\mathrm{mader}_\phi(H)$ as the smallest integer such that any digraph D with $\phi(D) \geq \mathrm{mader}_\phi(H)$ contains a subdivision of H. Theorem 1.4 can be expressed as saying that for any r, $\mathrm{sm}_{\delta^+}(\overrightarrow{K}_r) = \infty$, and our Theorem 1.5 can be expressed succinctly as

$$r + 1 \leq \mathrm{sm}_\chi(\overrightarrow{K}_r) \leq r4^r.$$

It would be very interesting to obtain better bounds on this function.

In Lemma 5.8, we showed that strong-connectivity is sufficient to guarantee large complete minors in tournaments. Is the same true in general digraphs? Letting $\kappa(D)$ denote the strong-connectivity of a digraph D, we pose the following problem:

Problem 7.2. *Determine whether or not* $\mathrm{sm}_\kappa(\overrightarrow{K}_r) < \infty$.

For a given natural digraph parameter ϕ, it would be interesting to determine $\mathrm{sm}_\phi(H)$, for some other digraphs H, for example, oriented trees, cycles, or transitive tournaments.

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