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Function spectra

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0. *Introduction and statement of results*

Boardman's stable category (see [**5**]) is a closed category ([**4**], VII·7), and in the best of all possible worlds, the category of spectra underlying the stable category would be closed as well; this would make life considerably easier for those doing calculations in stable homotopy theory. Unfortunately none of the categories of spectra introduced to date are closed; only $\mathscr{S}$, the category introduced in [**2**], is even symmetric monoidal. The problem with making $\mathscr{S}$ closed is that it comes equipped with an augmentation to $\mathscr{I}$, the category of universes and linear isometries (called $\mathscr{U}_n$ in [**2**]), which preserves the symmetric monoidal structure. Since $\mathscr{I}$ is not closed, this makes it difficult to see how $\mathscr{S}$ might be closed.

The purpose of this paper is to show that there is a simple function spectrum construction in $\mathscr{S}$ that comes surprisingly close to making $\mathscr{S}$ closed. We will prove the following theorems:

THEOREM 1. *Given spectra X and Y in $\mathscr{S}$, there is a function spectrum $H(X, Y)$ in $\mathscr{S}$ which induces the function object in Boardman's stable category. There is a natural equivalence*

$$H(X, H(Y, Z)) \to H(X \wedge Y, Z).$$

Also $H(X, Y)$ is functorial in Y.

The extent to which $H(X, Y)$ is functorial in X will be discussed later in this paper. In order to explain the other properties of $H(X, Y)$, we need a little notation.

Notation 2. If X is a spectrum, let $\mathscr{U}_X$ be its underlying universe (so that $\mathscr{U}_X = \epsilon X$), and let S_X be the sphere spectrum (of dimension 0) indexed on $\mathscr{U}_X$ (see [**2**], p. 74, definition 4·6).

THEOREM 3. *The underlying universe of $H(X, Y)$ is $\mathscr{U}_Y$, and the diagram of functors*

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{H(X,-)} & \mathscr{S} \\ & {\scriptstyle\epsilon}\searrow \quad \swarrow{\scriptstyle\epsilon} & \\ & \mathscr{I} & \end{array}$$

commutes.

Our final theorem gives the extent to which $H(X, Y)$ admits a composition pairing. The special case in which $A = B = S^0$ below is of particular interest.

THEOREM 4. *Given spectra X, Y, Z, and A, there is a natural pairing map*

$$\mu: H(Y, Z) \wedge H(X, Y \wedge A) \to H(X, Z \wedge S_Y \wedge A)$$

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which is associative in the sense that given X, Y, Z, W, A, and B, the diagram

$$\begin{array}{ccc} H(Z,W) \wedge H(Y, Z \wedge B) \wedge H(X, Y \wedge A) & \xrightarrow{1 \wedge \mu} & H(Z,W) \wedge H(X, Z \wedge B \wedge S_Y \wedge A) \\ \downarrow{\scriptstyle \mu \wedge 1} & & \downarrow{\scriptstyle \mu} \\ H(Y, W \wedge S_Z \wedge B) \wedge H(X, Y \wedge A) & \xrightarrow{\mu} & H(X, W \wedge S_Z \wedge B \wedge S_Y \wedge A) \end{array}$$

commutes.

The further correctness of H as a function object is indicated by the fact, to be proved in a later paper, that $H(X,X)$ is always equivalent to an A_∞ ring spectrum.

1. *Definitions and proofs*

Since smash products in $\mathscr{S}$ are the same as May's external smash products, it seems reasonable to use his function spectra as well (see [3], section II·3). If $\mathscr{U}$ is any universe, recall that $\mathscr{S}\mathscr{U}$ is the subcategory of $\mathscr{S}$ whose objects are spectra indexed over $\mathscr{U}$ and whose maps are those in $\mathscr{S}$ whose underlying maps on universes are the identity ([2], theorem 0·4). Given spectra X and Y and a splitting

$$\mathscr{U}_Y \cong \mathscr{V} \oplus \mathscr{U}_X,$$

the function spectrum $F(X,Y)$ is defined as an object of $\mathscr{S}\mathscr{V}$. Further, we have the adjointness relations given in the following theorem.

THEOREM 5 (May). *Given spectra X, Y, and Z, together with a splitting*

$$\mathscr{U}_Z \cong \mathscr{V} \oplus \mathscr{U}_X \oplus \mathscr{U}_Y,$$

there is a natural isomorphism

$$F(X, F(Y,Z)) \cong F(X \wedge Y, Z)$$

in $\mathscr{S}\mathscr{V}$. If $\mathscr{V} = \{0\}$, this reduces to an isomorphism

$$\mathscr{S}\mathscr{U}_X(X, F(Y,Z)) \cong \mathscr{S}\mathscr{U}_Z(X \wedge Y, Z).$$

The difficulty with using $F(X,Y)$ in $\mathscr{S}$ is that it depends not only on X and Y, but also on the given splitting. Our approach to function spectra uses the trivial basic observation that $Y \wedge S_X$ is indexed over $\mathscr{U}_Y \oplus \mathscr{U}_X$, which already has a splitting specified.

Definition 6. Given spectra X and Y,

$$H(X,Y) \equiv F(X, Y \wedge S_X).$$

The first part of Theorem 3 follows immediately from this definition, and the second part is equivalent to Lemma 7 below. Note also that $H(X,Y)$ is clearly a function object in the stable category, since $F(X,Y)$ is a function object and Y is equivalent to $Y \wedge S_X$.

LEMMA 7. *$H(X,-)$ is a functor from $\mathscr{S}$ to itself that preserves underlying maps of universes.*

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Proof. This depends on the fact that a map $f: Y \to Z$ in $\mathscr{S}$ is equivalent to a map $Y \to f^*Z$ in $\mathscr{S}\mathscr{U}_Y$. (Here f is used for both the spectrum level and the universe level map and f^* is the pullback functor of [2], lemma 3·8 or [3], proposition I·2·5 and remark I·2·6; the two references are equivalent in the present case.) Given such an f, we produce a map

$$H(X, Y) = F(X, Y \wedge S_X) \to f^*F(X, Z \wedge S_X) = f^*H(X, Z)$$

as the composite

$$F(X, Y \wedge S_X) \to F(X, f^*Z \wedge S_X) \to F(X, (f \oplus 1)^*(Z \wedge S_X)) \cong f^*F(X, Z \wedge S_X).$$

The isomorphism is an easy consequence of the Yoneda lemma, the first map is induced by the given map, and the second one is adjoint to the composite

$$(f \oplus 1)_*(f^*Z \wedge S_X) \cong f_*f^*Z \wedge 1_*S_X \to Z \wedge S_X,$$

where the isomorphism follows from [3], lemma 5·1, and the map is the counit of the (f_*, f^*)-adjunction. The necessary check that composition is preserved is straightforward.

The next proposition completes the proof of Theorem 1.

Proposition 8. *There is a natural equivalence*

$$H(X, H(Y, Z)) \to H(X \wedge Y, Z).$$

Proof. The equivalence is given by the composite

$$\begin{aligned} H(X, H(Y, Z)) &= F(X, F(Y, Z \wedge S_Y) \wedge S_X) \\ &\to F(X, F(Y, Z \wedge S_X \wedge S_Y)) \cong F(X \wedge Y, Z \wedge S_X \wedge S_Y) \\ &= H(X \wedge Y, Z), \end{aligned}$$

where the arrow is induced by the natural equivalence

$$F(Y, Z \wedge S_Y) \wedge S_X \to F(Y, Z \wedge S_X \wedge S_Y)$$

whose adjoint is the composite

$$F(Y, Z \wedge S_Y) \wedge S_X \wedge Y \xrightarrow{1 \wedge \tau} F(Y, Z \wedge S_Y) \wedge Y \wedge S_X \xrightarrow{\epsilon \wedge 1} Z \wedge S_Y \wedge S_X \xrightarrow{1 \wedge \tau} Z \wedge S_X \wedge S_Y.$$

Here ϵ is the counit of the adjunction of Theorem 5, and τ is the twist map. Notice that the presence of two twists makes the underlying map on universes the identity, which is necessary in order to apply the adjunction.

Our last piece of business is the following proof.

Proof of Theorem 4. The pairing is given by the map

$$F(Y, Z \wedge S_Y) \wedge F(X, Y \wedge A \wedge S_X) \to F(X, Z \wedge S_Y \wedge A \wedge S_X)$$

whose adjoint is the composite

$$\begin{aligned} F(Y, Z \wedge S_Y) \wedge F(X, Y \wedge A \wedge S_X) \wedge X &\xrightarrow{1 \wedge \epsilon} F(Y, Z \wedge S_Y) \wedge Y \wedge A \wedge S_X \\ &\xrightarrow{\epsilon \wedge 1} Z \wedge S_Y \wedge A \wedge S_X. \end{aligned}$$

Associativity follows from the same formal techniques used to prove theorem 6 of [1].

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We conclude by observing the extent to which $H(X, Y)$ is functorial in X. Although not truly functorial, we can say precisely what weak equivalences need to be inverted in order to make it so.

LEMMA 9. *Given a spectrum W and a map of universes $\phi: \mathscr{U}_W \to \mathscr{U}_X$, there is a natural equivalence in $\mathscr{S}\mathscr{U}_Y$*

$$H(W, Y) \to H(\phi_* W, Y).$$

We have $H(X, Y)$ functorial in X up to inversion of all such weak equivalences.

Proof. The construction of the equivalence is a simple Yoneda lemma argument, using the observation that there is a natural equivalence $S_W \to \phi^* S_X$ adjoint to the isomorphism $\phi_* S_W \cong S_X$. Let Z be any spectrum in $\mathscr{S}\mathscr{U}_Y$; then we have

$$\begin{aligned}
&\mathscr{S}\mathscr{U}_Y(Z, H(W, Y)) = \mathscr{S}\mathscr{U}_Y(Z, F(W, Y \wedge S_W)) \to \mathscr{S}\mathscr{U}_Y(Z, F(W, Y \wedge \phi^* S_X)) \\
&\quad \cong \mathscr{S}(\mathscr{U}_Y \oplus \mathscr{U}_W)(Z \wedge W, Y \wedge \phi^* S_X) \to \mathscr{S}(\mathscr{U}_Y \oplus \mathscr{U}_W)(Z \wedge W, (1 \oplus \phi)^*(Y \wedge S_X)) \\
&\quad \cong \mathscr{S}(\mathscr{U}_Y \oplus \mathscr{U}_X)((1 \oplus \phi)_*(Z \wedge W), Y \wedge S_X) \cong \mathscr{S}(\mathscr{U}_Y \oplus \mathscr{U}_X)(Z \wedge \phi_* W, Y \wedge S_X) \\
&\quad \cong \mathscr{S}\mathscr{U}_Y(Z, F(\phi_* W, Y \wedge S_X)) = \mathscr{S}\mathscr{U}_Y(Z, H(\phi_* W, Y)).
\end{aligned}$$

The second arrow is induced by the natural equivalence

$$Y \wedge \phi^* S_X \to (1 \oplus \phi)^*(Y \wedge S_X)$$

whose adjoint is the composite

$$(1 \oplus \phi)_*(Y \wedge \phi^* S_X) \cong Y \wedge \phi_* \phi^* S_X \to Y \wedge S_X.$$

Next, given a map $f: W \to X$ in $\mathscr{S}$, let $\phi: \mathscr{U}_W \to \mathscr{U}_X$ be the underlying map on universes. We then factor f as

$$W \to \phi_* W \xrightarrow{f'} X,$$

where f' is a map in $\mathscr{S}\mathscr{U}_X$. Since $F(-, Y \wedge S_X)$ is a contravariant functor for maps in $\mathscr{S}\mathscr{U}_X$, we get an induced map in $\mathscr{S}\mathscr{U}_Y$

$$H(X, Y) = F(X, Y \wedge S_X) \to F(\phi_* W, Y \wedge S_X) = H(\phi_* W, Y).$$

We therefore have a natural diagram

$$H(X, Y) \to H(\phi_* W, Y) \leftarrow H(W, Y)$$

with the right-hand arrow a weak equivalence, as desired.

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