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Efficient Graph Packing via Game Colouring

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The game colouring number $\mathrm{gcol}(G)$ of a graph G is the least k such that, if two players take turns choosing the vertices of a graph, then either of them can ensure that every vertex has fewer than k neighbours chosen before it, regardless of what choices the other player makes. Clearly $\mathrm{gcol}(G) \leqslant \Delta(G)+1$. Sauer and Spencer [20] proved that if two graphs G_1 and G_2 on n vertices satisfy $2\Delta(G_1)\Delta(G_2) < n$ then they pack, *i.e.*, there is an embedding of G_1 into the complement of G_2. We improve this by showing that if $(\mathrm{gcol}(G_1)-1)\Delta(G_2)+(\mathrm{gcol}(G_2)-1)\Delta(G_1) < n$ then G_1 and G_2 pack. To our knowledge this is the first application of colouring games to a non-game problem.

1. Introduction

The purpose of this article is to demonstrate that game colouring can be a useful tool for attacking a non-game problem. We will use optimal strategies for Alice in the marking game to obtain an efficient packing of two graphs. It is our expectation that there will be other applications of this natural technique. A secondary goal is to illustrate a paradigm for generalizing results concerning graphs with bounded maximum degree.

The paper is organized as follows. In Sections 2, 3 and 4 we define and briefly discuss generalized colouring numbers, chromatic and marking games, and graph packing. In Section 5 we state and prove our main theorem. In Section 6 we make some concluding remarks.

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The following graph Q_t will be used as an example several times. It is obtained from K_t by first duplicating edges so that every original edge has multiplicity t and then subdividing every edge exactly once. The t original vertices of K_t are called *branch vertices* of Q_t; the remaining $t\binom{t}{2}$ vertices are called *subdivision vertices.*

2. Generalized colouring numbers

Fix a graph $G = (V, E)$ and denote the set of linear orderings on V by Π. Recall that the *colouring number* $\mathrm{col}(G)$ of G is the least integer k such that, for some $L \in \Pi$, every vertex has fewer than k neighbours that precede it in L. It is well known that $\mathrm{col}(G) \leqslant d$ if and only if every subgraph has a vertex with degree less than d. This observation provides a polynomial-time algorithm for determining $\mathrm{col}(G)$.

Starting with Chen and Schelp, several authors [6, 13, 11, 14] have considered useful variations of colouring number, with names such as arrangeability, admissibility and rank. Kierstead and Yang [16] standardized these ideas in terms of a hierarchy of colouring numbers as follows. Let $L \in \Pi$. A vertex z is *k-reachable* from a vertex x *with respect to* L if and only if $z <_L x$ and there exists an x,z-path P of length at most k such that $x <_L y$ for all interior vertices y of P. Let $R_{L,k}(x)$ denote the set of vertices that are k-reachable from x with respect to L. The *k-colouring number* was defined in [16] to be

$$\mathrm{col}_k(G) = 1 + \min_{L\in\Pi} \max_{x\in V} R_{L,k}(x).$$

The 1-colouring number of G is just the ordinary colouring number. The following easy inequalities hold for $j \leqslant k$:

$$\chi(G) \leqslant \mathrm{col}(G) \leqslant \mathrm{col}_j(G) \leqslant \mathrm{col}_k(G) \leqslant \Delta(G)(\Delta(G)-1)^{k-1} + 1.$$

The graph Q_t satisfies $\mathrm{col}(Q_t) = 3$. Each of arrangeability, admissibility and rank can be polynomially bounded in terms of 2-colouring number and *vice versa*, but for specific applications each may have its own advantage. However, the 2-colouring number cannot be bounded in terms of the colouring number, since $\mathrm{col}_2(Q_t) = t + 1$. Ordering all branch vertices before all subdivision vertices witnesses the upper bounds on both $\mathrm{col}_2(Q_t)$ and $\mathrm{col}(Q_t)$. For the lower bound on $\mathrm{col}_2(Q_t)$, note that for every ordering L, if x is the last branch vertex then $|R_{L,2}(x)| \geqslant t$.

The 2-colouring number will have special importance for us. It – or slight variations of it – has been previously used in several different contexts. In a typical application one would like to show that a conclusion about the class of graphs with bounded maximum degree holds for the larger class of graphs with bounded colouring number. When we are unable to do this we may still be able to derive the conclusion for the intermediate class of graphs with bounded 2-colouring number. Here is the original example. Burr and Erdős [4] made the following conjecture.

Conjecture 2.1 ([4]). *For every positive integer k there exists an integer C such that, for every graph G on n vertices with $\mathrm{col}(G) \leqslant k$ and every graph H on Cn vertices, either $G \subseteq H$ or $G \subseteq \overline{H}$.*

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Chvátal, Rödl, Szemerédi and Trotter [7] proved a weaker version of the conjecture with $\Delta(G) \leqslant k$ instead of $\mathrm{col}(G) \leqslant k$. Then Chen and Schelp [6] introduced the original precursor of the 2-colouring number, and showed (essentially) that the conjecture holds with $\mathrm{col}_2(G) \leqslant k$. They demonstrated the significance of their result by showing (essentially) that the 2-colouring number of planar graphs is bounded, even though their maximum degree is not. Kierstead and Trotter [13] showed that it is bounded by 10, and gave an example to show that it is at least 8. A simpler example is the 'buckyball', *i.e.*, the dual of the graph formed by a 32-panel football, or soccer ball (drawn with the stitching). To see this, fix an order L on the vertices and let x be the largest vertex that has a larger neighbour. After noting that if y and x have a common neighbour larger than x then $y \in R_{L,2}(x)$, it easily follows that $|R_{L,2}(x)| \geqslant 7$. Very recently Kierstead, Mohar, Špacapan, Yang and Zhu [12] proved the following theorem.

Theorem 2.2 ([12]). *Every planar graph G satisfies* $\mathrm{col}_2(G) \leqslant 9$.

A *vertex colouring* of G is degenerate if the union U of any k colour classes satisfies $\mathrm{col}(G[U]) \leqslant k$. In particular, setting $k = 1$ we see that a degenerate colouring is a proper colouring; moreover, setting $k = 2$ we see that it is an acyclic colouring. The degenerate chromatic number $\chi_d(G)$ of G is the least k such that G has a degenerate colouring with k colours. While $\mathrm{col}(Q_t) = 3$, we have $\chi_d(G) \geqslant \chi_a(Q_t) = t$. If we try to use fewer colours then two branch vertices will have the same colour, and two of their common neighbours will also share a colour, creating a bicoloured 4-cycle. However, $\chi_d(G) \leqslant \Delta(G)(\Delta(G) - 1) + 1$. Fix an order L, and colour sequentially so that each new vertex x gets a colour that has has not been used on $|R_{L,2}(x)|$. Then each vertex y will have at most one neighbour that precedes it in each colour class. This idea leads to the following observation [12], as well as its list colouring counterpart.

Theorem 2.3. *Every graph G satisfies $\chi_d(G) \leqslant \mathrm{col}_2(G)$. In particular, $\chi_d(G) \leqslant 9$, if G is planar.*

Finally, we remark that generalized colouring numbers are related to the concept of bounded expansion classes (see Nešetřil and Ossona de Mendez [18]). Zhu [24] has very recently proved that a graph class $\mathcal{K}$ has bounded expansion if and only if there exists a function f such that every graph $G \in \mathcal{K}$ satisfies $\mathrm{col}_k(G) \leqslant f(k)$ for all natural numbers k.

3. Games

The *chromatic game* is played on G by two players, Alice and Bob, with a fixed set of colours C. The game begins with Bob deciding which player will make the first move. Once this decision is made, the players take turns colouring the vertices with colours from C with *legal* colours, *i.e.*, so that adjacent vertices receive distinct colours. Alice wins if the graph is eventually properly coloured; Bob wins if at some time there is an uncoloured vertex that cannot be legally coloured. The game chromatic number $\chi_g(G)$ of G is the least integer k such that Alice has a winning strategy when $|C| = k$; for a class $\mathcal{C}$ of graphs

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$\chi_g(\mathcal{C}) = \max_{G\in\mathcal{C}}\chi_g(G)$. This game was originally introduced by Brams and published by Gardner [10] for planar graphs. It was re-invented and introduced to the mathematical community by Bodlaender [1]. Faigle, Kern, Kierstead and Trotter [9] showed that the game chromatic number of the class of forests is 4. Kierstead and Trotter [13] used a variant of the 2-colouring number to show that the game chromatic number of planar graphs is at most 33. This result was improved by a series of papers [8, 21, 11, 23]. The following bound obtained by Zhu is currently the best.

Theorem 3.1 ([23]). *Every planar graph G satisfies $\chi_g(G) \leqslant 17$.*

Proofs in [9, 21, 11, 23] are all obtained by bounding a new game parameter (implicitly in [9]), the *game colouring number*. The *marking game* is played by Alice and Bob on G. Bob decides who goes first and then the players take turns choosing unchosen vertices until all vertices have been chosen. Let L be the order in which the vertices are chosen, where the first vertex is smallest. The score of the game is the maximum k such that in L some vertex has $k-1$ smaller neighbours. Alice's goal is to keep the score low, while Bob's goal is to force it high. The *game colouring number* $\mathrm{gcol}(G)$ is the least k such that Alice can always obtain a score of at most k. The game k-colouring number $\mathrm{gcol}_k(G)$ of G is defined analogously. Clearly $\mathrm{gcol}(G) = \Delta(G)+1$ if G is regular. Moreover, if Alice interprets Bob's moves in the chromatic game as choices of vertices in the marking game, then she can choose vertices to colour so that no vertex has $\mathrm{gcol}(G)$ coloured neighbours. It follows that:

$$\chi(G) \leqslant \mathrm{col}(G) \leqslant \mathrm{gcol}(G) \leqslant \Delta(G)+1 \quad \text{and} \quad \chi_g(G) \leqslant \mathrm{gcol}(G).$$

However, $\mathrm{gcol}(G)$ cannot be bounded in terms of $\mathrm{col}(G)$, since $\mathrm{col}(Q_t) = 3$, but $\mathrm{gcol}(Q_t) > s = \log_2(t)$ for $t = 2^s$. Argue by induction on s and observe that the base step $s = 0$ is trivial. First Bob chooses $\frac{1}{2}t$ subdivision vertices that dominate all branch vertices. Alice can respond by choosing at most $\frac{1}{2}t$ branch vertices, and so there exists a set U of $\frac{1}{2}t$ unchosen branch vertices, each of which is adjacent to a chosen subdivision vertex. Now apply the induction hypothesis to $G[U]$. The following theorem is the third example of our paradigm.

Theorem 3.2 ([11]). *Every graph G satisfies $\mathrm{gcol}(G) \leqslant 3\mathrm{col}_2(G) - 1$.*

Kierstead and Trotter [14] showed that if G is planar then $\mathrm{col}_4(G)$ is bounded. They used this to show that $\mathrm{gcol}_2(G)$ is bounded, which was key in showing that the *game oriented chromatic number* of G is bounded. Kierstead and Yang [16] showed that the *game k-colouring number* is bounded in terms of the $2k$-colouring number and that the k-colouring number of planar graphs is bounded.

4. Packing

Two n-vertex graphs G_1 and G_2 *pack* if there exists an edge-disjoint placement of these graphs onto the same set of n vertices. By definition, G_1 and G_2 pack if G_1 is a subgraph of the complement $\overline{G}_2$ of G_2. A number of basic graph-theoretic problems can be expressed in a unified and symmetric form as graph packing problems. Among such problems are

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Turán-type, Ramsey-type, colouring and equitable colouring problems. Since even partial cases of graph packing problems are NP-hard, study of extremal problems was initiated in 1970s by seminal papers of Bollobás and Eldridge [2], Sauer and Spencer [20], and Catlin [5]. Their conditions involved maximum degrees of graphs. For example, Sauer and Spencer [20] proved the following result.

Theorem 4.1 ([20]). *If G_1 and G_2 are graphs on n vertices and $2\Delta(G_1)\Delta(G_2) < n$, then G_1 packs with G_2.*

There have been several refinements of Theorem 4.1. One of them involves the *maximum edge degree* $\theta(G) := \max_{xy\in E(G)}(d(x) + d(y))$. Clearly, $\theta(G)$ is closely related to the maximum degree of the line graph $L(G)$,

$$\theta(G) = \Delta(L(G)) + 2,$$

and $\theta(G) \leqslant 2\Delta(G)$ for every graph G. Kostochka and Yu [17] showed that $2\Delta(G_1)$ could be replaced by $\theta(G_1)$ in Theorem 4.1, as well as the following result.

Theorem 4.2 ([17]). *If two n-vertex graphs G_1 and G_2 satisfy the inequality $\theta(G_1)\Delta(G_2) \leqslant n$, then G_1 and G_2 pack, with the following exceptions:*

(I) *G_1 is a perfect matching and G_2 either is $K_{n/2,n/2}$ with $n/2$ odd or contains $K_{n/2+1}$,*
(II) *G_2 is a perfect matching, and G_1 either is $K_{r,n-r}$ with r odd or contains $K_{n/2+1}$.*

Bollobás, Kostochka and Nakprasit [3] showed that if one of the two graphs has small colouring number then much weaker conditions on $\Delta(G_1)$ and $\Delta(G_2)$ imply the existence of a packing.

Theorem 4.3 ([3]). *Let $\mathrm{col}(G_1) \geqslant 3$. If G_1 and G_2 are graphs on n vertices with*

$$40\Delta(G_1)\ln\Delta(G_2) < n \quad \textit{and} \quad 40\mathrm{col}(G_1)\Delta(G_2) < n,$$

then G_1 packs with G_2.

Let G_1 and G_2 be graphs on n vertices. We could write the hypothesis of Theorem 4.1 as a two-term sum. Then motivated by Theorem 4.3 and our paradigm, we might consider strengthening Theorem 4.1 to

$$\exists C\, C\mathrm{col}(G_1)\Delta(G_2) + C\mathrm{col}(G_2)\Delta(G_1) < n \ \text{ implies } \ G_1 \ \text{ packs with } \ G_2.$$

This is too rash to call a conjecture, and we certainly cannot prove it, but the following weakening according to our paradigm is a corollary to our main result, Theorem 5.1.

Corollary 4.4. *Let G_1 and G_2 be graphs on n vertices. If*

$$3\mathrm{col}_2(G_1)\Delta(G_2) + 3\mathrm{col}_2(G_2)\Delta(G_1) < n$$

then G_1 packs with G_2.

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5. The main result

Our main result, Theorem 5.1, proved in this section, not only is a natural example of our paradigm, but also, to our knowledge, is the first use of game colouring to prove a non-game result. The statement of the theorem is stronger than that of the Sauer–Spencer theorem, since $\mathrm{gcol}(G) - 1 \leqslant \Delta(G)$ for every graph G. The proof is motivated by the classical back-and-forth construction, showing that any two countable dense linear orders without endpoints are isomorphic.

Theorem 5.1. *If G_1 and G_2 are n-vertex graphs and*

$$(\mathrm{gcol}(G_1) - 1)\Delta(G_2) + (\mathrm{gcol}(G_2) - 1)\Delta(G_1) < n, \tag{5.1}$$

then G_1 and G_2 pack.

Proof. For $i \in [2]$ set $G_i := (V, E_i)$, $d_i(v) := d_{G_i}(v)$, $g_i := \mathrm{gcol}(G_i) - 1$ and $\Delta_i = \Delta(G_i)$. A packing of G_1 with G_2 will be viewed as a bijection $f : V \to V$ such that every $uv \in E_1$ satisfies $f(u)f(v) \notin E_2$.

Our plan is to construct a packing of G in stages. At the end of stage s we will have a partial packing f^s of $H_1^s \subseteq G_1$ onto $H_2^s \subseteq G_2$, where $|H_i^s| = s$. We will maintain the invariant $H_i^s \subseteq H_i^t$, but not necessarily $f^s \subseteq f^t$, for $s < t$. At stage s we will choose two new vertices x_1^s and x_2^s with $x_i^s \in V(G_i - H_i)$. These choices will be interpreted as moves in two colouring games, Game_1, played on G_1, and Game_2, played on G_2. The players for Game_i are Alice_i and Bob_i, although as we shall see it would perhaps be more informative to change the names to Algorithm and Black Box. We assume that each Alice_i plays with an optimal strategy witnessing the game colouring number of G_i. We will provide each Bob_i with a strategy designed to produce a packing.

Recall that Bob_i decides who plays first. For our purposes, we have Bob_i choose to play first if and only if $i = 2$. Let $x_i^1, x_i^2, \ldots, x_i^n$ be the sequence of moves in Game_i; so Bob_i plays x_i^s if and only if $i + s$ is odd. The optimal strategy of Alice_i must be able to handle this situation. At stage 1 of the construction Alice_1 chooses x_1^1 in accordance with her optimal strategy. Our construction will generate $f^1(x_1^1) = x_2^1$, which is interpreted as Bob_2's first play in Game_2. At stage 2, Alice_2 responds to Bob_2 by choosing x_2^2 in accordance with her optimal strategy. Then the construction will generate $(f^2)^{-1}(x_2^2) = x_1^2$, which is interpreted as Bob_1's response to x_1^1. Continuing in this back-and-forth fashion, we will generate the two sequences $x_i^1, x_i^2, \ldots, x_i^n$, $i \in [2]$. By the definition of game colouring number, Alice_i's optimal strategy ensures that the set $N_i^+(w)$ of neighbours of w preceding w in H_i^s has cardinality at most g_i. It remains to show how the construction will generate x_i^s for $i + s$ odd.

Argue by induction on s. The base step $s = 0$ is trivial, so consider the induction step $s + 1$. By the induction hypothesis, f^s is a packing of H_1^s with H_2^s. Assume $s + 1$ is odd as the even case is symmetrical. So Alice_1 has just chosen $x := x_1^{s+1}$. For $x' \in V(G_2 - H_2^s)$ define the set $C(x')$ by

$$C(x') := \{y \in H_1^s : xy \in E_1 \text{ and } x'f^s(y) \in E_2\}.$$

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Case 1. *$C(x') = \emptyset$ for some $x' \in V(G_2 - H_2^s)$.*

Setting $x_2^{s+1} := x'$ and $f^{s+1} = f^s \cup \{(x, x')\}$, we are done.

Case 2. *$C(x') \neq \emptyset$ for all $x' \in V(G_2 - H_2^s)$.*

Fix $x' \in V(G_2 - H_2^s)$; set $x_2^{s+1} := x'$ and $C := C(x')$. We will show that for some *good* $z \in V(H_1^{s+1}) \backslash C$, the mapping $\varphi := \varphi_z$ obtained from $f := f^s \cup \{(x, x')\}$ by switching the images of x and z is a packing of H_1^{s+1} with H_2^{s+1}. Suppose $z \in V(H_1^{s+1}) \backslash C$ is *bad*, *i.e.*, not good. Then there exists an edge $uv \in E_1$ with $\varphi(u)\varphi(v) \in E_2$. Since f^s packs H_1^s with H_2^s, at least one, say v, of u, v is in $\{x, z\}$. Since $z \notin C$, we have $u \notin \{x, z\}$.

Suppose $v = x$. Then $xu \in E_1$ and $\varphi(x)\varphi(u) = f^s(z)f^s(u) \in E_2$. Thus $f^s(z) \in P$, where

$$P := \bigcup_{u \in N_1^+(x)} N_2(f^s(u)).$$

Since $|N_1^+(x)| \leqslant g_1$ and $|N_2(w)| \leqslant \Delta_2$ for all $w \in V$, we have $|P| \leqslant g_1\Delta_2$. But this is an overcount, since we count $x' \in N_2(f^s(u))$ for each $u \in C$. With this observation we obtain

$$|P| \leqslant g_1\Delta_2 - |C| + 1.$$

Moreover, by definition, $V(G - H_2^s) \subseteq P$.

Suppose $v = z$. Then $zu \in E_1$ and $\varphi(z)\varphi(u) = x'f^s(u) \in E_2$. Then $z \in Q$, where

$$Q := \bigcup_{u' \in N_2^+(x')} N_1((f^s)^{-1}(u')).$$

Since $N_2^+(x') \leqslant g_2$ and $|N_1((f^s)^{-1}(u')| \leqslant \Delta_1$ for all $u' \in V$, we have $|Q| \leqslant g_2\Delta_1$. Again, this is an overcount, since we count x for each $u' \in f^s(C)$. Thus

$$|Q| \leqslant g_2\Delta_1 - |C| + 1.$$

Note that $x' \in P$ and $x \in Q$, since $C \neq \emptyset$. Thus

$$|P \cup f(Q)| \leqslant g_1\Delta_2 + g_2\Delta_1 - 2|C| + 2 - 1 < n - |C|.$$

It follows that there exists $z' \in V - (P \cup f(Q) \cup f(C))$. By definition, $z' \in V(H_2^s)$. So $z := f^{-1}(z')$ is good, and thus $f^{s+1} := \varphi_z$ is a packing of H_1^{s+1} with H_2^{s+1}. This completes the induction step and the proof. □

6. Conclusion

Recall from Section 2 that there is a polynomial-time algorithm for determining $\mathrm{col}(G)$. Given the many applications of the 2-colouring number, it is natural to ask the following.

Problem 1. *Is there a polynomial-time algorithm for calculating* $\mathrm{col}_2(G)$*?*

In [14] it is shown that $\mathrm{gcol}(G)$, and thus also $\mathrm{col}_2(G)$, cannot be bounded in terms of $\chi_a(G)$. A positive answer to the next problem would provide another application of game colouring number to a non-game problem.

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Problem 2. *Can* $\mathrm{col}_2(G)$ *and/or* $\chi_a(G)$ *be bounded in terms of* $\mathrm{gcol}(G)$*?*

In light of Theorem 4.2 it would be interesting to find a good bound on $\mathrm{gcol}(G)$ in terms of $\theta(G)$. Here is a start. If G is regular then $\mathrm{gcol}(G) = \frac{1}{2}\theta(G) + 1$. So suppose G is not regular. The *high*-degree vertices v satisfying $d(v) > \frac{1}{2}\theta(G)$ form an independent set. Only these vertices can be adjacent to more than $\frac{1}{2}\theta(G)$ previously chosen vertices. So Alice will always choose high-degree vertices, if possible. Similarly, Bob will only choose *low*-degree vertices v satisfying $d(v) \leqslant \frac{1}{2}\theta(G)$ which threaten unchosen high-degree vertices. We may assume that the low-degree vertices also form an independent set.

Proposition 6.1. *For every positive integer* $k \geqslant 3$ *there exists a graph* G *with*

$$k + 1 = \Delta(G) + 1 = \mathrm{gcol}(G) = \frac{\theta(G) + 3}{2}.$$

Proof. Fix k. Let G be an X, Y-bigraph, where

$$X := \{x_{i,j} : i \in [k-1], j \in [k]\} \quad \text{and} \quad Y := \{y_j : j \in [k]\} \cup \{z_{i,j} : i \in [k-1], j \in [k]\}.$$

Define $E := E(G)$ by

$$E := \{y_j x_{i,j} : i \in [k-1], j \in [k]\} \cup (z_{i,j} x_{i,h} : i \in [k-1], j, h \in [k], j \neq h\}.$$

Then every vertex in X has degree k and every vertex in Y has degree $k-1$. So $\mathrm{gcol}(G) \leqslant k+1$ and $\theta(G) = 2k-1$. Thus it suffices to show that Bob has a strategy for obtaining a score of at least $k+1$.

While possible, Bob will always choose vertices from Y, and as above, we may assume that Alice always tries to choose a vertex from X. Bob starts and begins by choosing $y_1, \ldots, y_k$. After Alice's first k moves, there exists $i \in [k-1]$ such that at most one vertex of $X_i := x_{i,1}, \ldots, x_{i,k}$ has been chosen. Without loss of generality, no vertex of X_i, other than possibly $x_{i,k}$, has been chosen. Bob chooses $z_{i,k}$. From now on, whenever Alice chooses a vertex $x_{i,j}$ Bob chooses a vertex $z_{i,h}$ with $h = j$ if possible. When Alice chooses the last vertex of X_i, say $x_{i,n}$, each of its neighbours in the set $\{z_{i,j} : j \in [k]\} + y_j - z_{i,n}$ will already be chosen. □

Proposition 6.2. *Suppose that* G *is an* X, Y*-bigraph such that*

$$d(y) = d \leqslant D = d(x) \quad \textit{for all } y \in Y, x \in X.$$

Then $\mathrm{gcol}(G) \leqslant D + 2 - \lfloor \frac{D}{d} \rfloor$.

Proof. Let $k = \lfloor \frac{D}{d} \rfloor$. Obtain an X', Y-bigraph G' from G by splitting each vertex $x \in X$ into k vertices $x_1, \ldots, x_k$ of degree d or $d+1$ so that $\{N(x_i) : i \in [k]\}$ partitions $N(x)$. By Hall's Theorem, G' has a matching that saturates X'. Recombining the vertices $x_1, \ldots, x_k$ into x yields a set $\{S_x : x \in X\}$ of disjoint stars, where S_x has centre x and k edges.

Whenever Bob chooses a vertex y in a star S_x, Alice responds by choosing x if it has not already been chosen. This ensures that when x is finally chosen it will have at most $D - k + 1$ previously chosen neighbours. □

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Problem 3. *Does every graph G satisfy* $\mathrm{gcol}(G) \leqslant \frac{\theta(G)+3}{2}$ *?*

Dedication

Professor Trotter has had a profound influence on our careers as well as the careers of many other mathematicians. It is a pleasure to honour him on the occasion of his 65th birthday and wish him many happy returns of the day.

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