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Polynomial modules and general polynomials

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0. *Introduction.* Throughout k denotes a field and $k[X] = k[X_1, \ldots, X_n]$ the polynomial ring in n indeterminates over k.

Let $u, u_1, \ldots, u_n$ be elements of an extension field of k which are algebraically independent over k, and $L = u + u_1 X_1 + \ldots + u_n X_n$. If $\mathfrak{a}$ is an ideal of $k[X]$, then we set $\mathfrak{a}^e = \mathfrak{a} \cdot k(u, u_1, \ldots, u_n)[X]$. The aim of this paper is to prove and to generalize the following theorem

Theorem. *Let $\mathfrak{a}$ be an ideal of $k[X]$ and $\mathfrak{a} = q_1 \cap \ldots \cap q_t$ a normal decomposition of $\mathfrak{a}$. Suppose that, for $1 \leqslant i \leqslant s$, we have $\dim q_i \geqslant 1$, and that, for $s+1 \leqslant j \leqslant t$, we have $\dim q_j = 0$. Then $\mathfrak{a}^e + (L) = \bigcap_{\sigma=1}^{s} (q_\sigma^e + (L))$ is a normal decomposition of $\mathfrak{a}^e + (L)$, and we have $(\mathfrak{a}^e + (L)) \cap k[X] = q_1 \cap \ldots \cap q_s$.*

The following proofs are very simple and are based on the theory of modules of fractions ((1), Appendix) and on a suitably modified argument used by van der Waerden (see (3), page 116) to prove that, if p is a prime ideal of $k[X]$ and $f_0 \in k[X]$, $p \cdot k(u)[X] + (u - f_0)$ is also a prime ideal.

1. *Prime ideals and general polynomials.* Let u be an element of an extension field of k and transcendental over k, furthermore let $f_0 \in k[X]$ and $f = u - f_0$. If $\mathfrak{a}$ is an ideal of $k[X]$, then we set $\mathfrak{a}^* = \mathfrak{a} \cdot k[u, X]$ and $\mathfrak{a}^e = \mathfrak{a} k(u)[X]$.

Definition. *f_0 is said to be transcendental with respect to an ideal $\mathfrak{a}$ of $k[X]$, if for all $g(u) \in k[u] - \{0\}$, $g(f_0) \notin \mathfrak{a}$.*

Let $\sigma : k[u, X] \to k[X]$ be the $k[X]$-homomorphism of $k[u, X]$ which maps u onto f_0. $(f)\, k[u, X]$ is the kernel of σ.

Proposition 1. *Let $\mathfrak{a}$ be an ideal of $k[X]$. Then the following statements are equivalent:*

(1) *f_0 is transcendental with respect to $\mathfrak{a}$.*

(2) $(\mathfrak{a}^* + (f)) \cap k[u] = (0)$.

(3) $\mathfrak{a}^e + (f) \neq k(u)[X]$.

(4) *$\mathfrak{a}$ possesses a zero (y) such that $f_0(y)$ is transcendental over k.*

Proof. We have $\sigma^{-1}(\mathfrak{a}) = \mathfrak{a}^* + (f)$, thus (1) and (2) are equivalent. $k(u)[X]$ is the quotient ring of $k[u, X]$ with respect to $k[u] - \{0\}$. This shows the equivalence of (2) and (3). If $\mathfrak{a}^e + (f) \neq k(u)[X]$, then $\mathfrak{a}^e + (f)$ possesses a zero (y), hence $f_0(y) = u$ is transcendental over k. Consequently (3) implies (4). (4) implies (2) (if $(\mathfrak{a}^* + (f)) \cap k[u]$ contains a $g(u) \neq 0$, then $g(f_0(y)) = 0$ for every zero (y) of $\mathfrak{a}$).

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Consider for a moment, the case in which $k = k'(u_1, \ldots, u_\mu)$ $(\mu \geqslant 0)$, where k' is a subfield of k and the u_i are algebraically independent over k'. In this situation, let $f_0 = g_0 + u_1 g_1 + \ldots + u_\mu g_\mu$ $(g_0, \ldots, g_\mu \in k'[X])$.

Let $\mathfrak{a}'$ be an ideal in $k'[X]$ and $\mathfrak{a} = \mathfrak{a}' . k[X]$.

PROPOSITION 2. *With the notations explained above, f_0 is transcendental with respect to $\mathfrak{a}$ if and only if there exists an i, $0 \leqslant i \leqslant \mu$, such that g_i is transcendental with respect to $\mathfrak{a}'$.*

Remark. Let $\omega_0, \ldots, \omega_\mu$, $u_{i\rho}$ $(i = 0, \ldots, \mu, \rho = 1, \ldots, \mu)$ be elements of an extension field of k' such that the $u_{i\rho}$ are algebraically independent over $k'(\omega) = k'(\omega_0, \ldots, \omega_\mu)$; furthermore, let $\overline{\omega}_i = \omega_0 + u_{i_1}\omega_1 + \ldots + u_{i\mu}\omega_\mu$ $(i = 0, \ldots, \mu)$. Then we have

$$k'(u_{i\rho}, \omega) = k'(u_{i\rho}, \overline{\omega}) \quad \text{and} \quad [k'(u_{i\rho}, \omega) : k'(u_{i\rho})]_{\mathrm{tr}} = [k'(\omega) : k']_{\mathrm{tr}}$$

(if K_1 is a subfield of a field K_2, then $[K_2 : K_1]_{\mathrm{tr}}$ denotes the transcendence degree of K_2 over K_1). If $\beta = [k'(\omega) : k']_{\mathrm{tr}} - 1$, then $\overline{\omega}_0, \ldots, \overline{\omega}_\beta$ form a transcendence base of $k'(u_{i\rho}, \omega)$ over $k'(u_{i\rho})$. This shows that, if $0 \leqslant s \leqslant \mu$,

$$[k'(u_{0\rho}, \ldots, u_{s\rho}, \overline{\omega}_0, \ldots, \overline{\omega}_s) : k'(u_{0\rho}, \ldots, u_{s\rho})]_{\mathrm{tr}} = \min(s+1, [k'(\omega) : k']_{\mathrm{tr}}).$$

Proof of Proposition 2. The statement is trivial if $\mathfrak{a}' = k'[X]$. Let now $\mathfrak{a}' \neq k'[X]$ and (y) be a zero of $\mathfrak{a}'$. We may assume (without loss of generality) that $u_1, \ldots, u_\mu$ are algebraically independent over $k'(y)$. Then we have

$$[k(f_0(y)) : k]_{\mathrm{tr}} = \min(1, [k'(g_0(y), \ldots, g_\mu(y)) : k']_{\mathrm{tr}}).$$

From this follows the assertion (see condition (4) in Proposition 1).

The degree of a polynomial $h \in k[X]$ is denoted by $[h]$.

COROLLARY 1. *With the notations explained above, let $\mu \geqslant n$ and $g_i \in k'[X_i] - \{0\}$ $(i = 1, \ldots, n)$ such that $[g_i] > 0$. Then f_0 is transcendental with repect to $\mathfrak{a}$ if and only if* $\dim \mathfrak{a}' \geqslant 1$.

The following result is well known (see Krull (2), Hilfssatz 1).

COROLLARY 2. *With the notations explained above, let $\mu \geqslant n$ and $g_i = X_i$ $(i = 1, \ldots, n)$. Then f_0 is transcendental with respect to $\mathfrak{a}$ if and only if* $\dim \mathfrak{a}' \geqslant 1$.

2. *The main results.* Now we consider again the general case.

We set $E = k[X] \oplus \ldots \oplus k[X]$ (s terms), $E^* = k[u, X] \oplus \ldots \oplus k[u, X]$ (s terms) and $E^e = k(u)[X] \oplus \ldots \oplus k(u)[X]$ (s terms). E and E^* are naturally embedded in E^* and E^e, respectively.

If N is a submodule of E, then N^* (respectively N^e) denotes the submodule of E^* (respectively E^e) generated by N.

The mapping $\hat{\sigma} : E^* \to E$ which maps $w = (h_1, \ldots, h_s)$ $(h_\rho \in k[u, X])$ onto

$$\hat{\sigma}(w) = (\sigma(h_1), \ldots, \sigma(h_s))$$

is a group homomorphism with kernel $(f)E^*$. $\hat{\sigma}$ has the following properties:

(1) If $g \in k[u, X]$ and $w \in E^*$, then $\hat{\sigma}(gw) = \sigma(g) . \hat{\sigma}(w)$.

(2) $\hat{\sigma}_{|E}$ is the identity mapping of E.

If Q is a p-primary submodule of a finitely generated module M over a Noetherian ring R, then $\lambda(Q)$ denotes the greatest number r such that there exists a strictly descending chain $Q_1 \supset Q_2 \supset \ldots \supset Q_r = Q$ of p-primary submodules Q_i of M.

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THEOREM 1. *Let Q be a p-primary submodule of E. If f_0 is transcendental with respect to p, then $Q^e+(f)E^e$ is a $(p^e+(f))$-primary submodule of E^e, $(Q^e+(f)E^e)\cap E = Q$, $\lambda(Q) = \lambda(Q^e+(f)E^e)$ and $(Q^e+(f)E^e):E^e = (Q:E)^e+(f)$. If f_0 is not transcendental with respect to p, then $Q^e+(f)E^e = E^e$.*

Proof. We have $\hat{\sigma}^{-1}(Q) = Q^*+(f)E^*$ and $\sigma^{-1}(p) = p^*+(f)$. Consequently $Q^*+(f)E^*$ is a $(p^*+(f))$-primary submodule of E^*, $\lambda(Q) = \lambda(Q^*+(f)E^*)$, $(Q^*+(f)E^*)\cap E = Q$ (by property 2 of $\hat{\sigma}$) and $(Q^*+(f)E^*):E^* = (Q:E)^*+(f)$. $Q^e+(f)E^e$ and E^e are the modules of fractions of $Q^*+(f)E^*$ and E^*, respectively, with respect to $k[u]-\{0\}$. So the theorem follows (see Auslander and Buchsbaum (1)).

In the case of a primary ideal, the first two assertions of Theorem 1 are to be found in Krull ((2), section 2).

THEOREM 2. *Let N be a submodule of E, $N = Q_1\cap\ldots\cap Q_t$ a normal decomposition of N, and p_i the prime ideal of Q_i $(i = 1,\ldots,t)$. Suppose that f_0 is transcendental with respect to $p_1,\ldots,p_\mu$, but not with respect to $p_{\mu+1},\ldots,p_t$. Then $N^e+(f)E^e = \bigcap_{\rho=1}^{\mu}(Q_\rho^e+(f)E^e)$ is a normal decomposition of $N^e+(f)E^e$ and we have $(N^e+(f)E^e)\cap E = Q_1\cap\ldots\cap Q_\mu$. Furthermore, $\mathfrak{r}(N^e+(f)E^e) = \mathfrak{r}(N)^e+(f)$ and $(N^e+(f)E^e):E^e = (N:E)^e+(f)$.*

Remark. $\mathfrak{r}(N)$ and $\mathfrak{r}(N^e+(f)E^e)$ denote the radical of N and $N^e+(f)E^e$, respectively.

Proof. Obviously $N^*+(f)E^* = \bigcap_{\rho=1}^{t}(Q_\rho^*+(f)E^*)$ is a normal decomposition of $N^*+(f)E^* = \hat{\sigma}^{-1}(N)$. Forming fractions with respect to $k[u]-\{0\}$ we get the result that $N^e+(f)E^e = \bigcap_{\rho=1}^{\mu}(Q_\rho^e+(f)E^e)$ is a normal decomposition of $N^e+(f)E^e$, and from this it follows that $(N^e+(f)E^e)\cap E = Q_1\cap\ldots\cap Q_\mu$ (by Theorem 1). Applying this result we get

$$(N^e+(f)E^e):E^e = \bigcap_{\rho=1}^{\mu}((Q_\rho^e+(f)E^e):E^e)$$

$$= \bigcap_{\rho=1}^{\mu}((Q_\rho:E)^e+(f)) = \left(\bigcap_{\rho=1}^{t}(Q_\rho:E)\right)^e+(f) = (N:E)^e+(f)$$

and

$$\mathfrak{r}(N^e+(f)E^e) = \bigcap_{\rho=1}^{\mu}(p_\rho^e+(f)) = \left(\bigcap_{\rho=1}^{t}p_\rho\right)^e+(f) = \mathfrak{r}(N)^e+(f).$$

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