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THE DIFFUSION OF ECONOMIC ACTIVITY ACROSS SPACE: A NEW APPROACH

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Dynamic spatial theory has been a fruitful approach to understanding economic phenomena involving time and space. However, several central questions still remain unresolved in this field. The identification of the social optimal allocation of economic activity across time and space is particularly problematic, not been ensured yet in economic growth. Developing a monotone method, we study the optimal solution of the spatial Ramsey model. Under fairly general assumptions, we prove the existence of unique social optimum. Considering a numerical implementation of our algorithm, we study the role played by capital mobility in the neoclassical growth environment. With capital irreversibility and economic openness, space allows for transitional dynamics. Moreover, in this context, capital mobility is beneficial as well in terms of social welfare and inequality. We also consider an application of our method to an extension of the spatial Ramsey model for optimal land-use planning.

Keywords: Control, Spatial Dynamics, Ramsey Model, Partial Differential Equations

1. INTRODUCTION

A recent strand of the literature on spatial economics has introduced diffusion mechanisms in order to better understand the allocation of economic activity across time and space. This new approach, dynamic spatial theory, turned out to be useful to explain many spatial-dynamic phenomena, such as the mobility of production factors (Boucekkine et al. (2013) and Fabbri (2016)), the dissemination of technology (Desmet and Rossi-Hansberg (2009), Desmet and Rossi-Hansberg

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(2014) and Desmet and Rossi-Hansberg (2015)), the spatial allocation of natural resources (Costello and Polasky (2008) and Brock and Xepapadeas (2010)), or the spread of pollutants due to the economic activity (Camacho and Pérez-Barahona (2015) and La Torre et al. (2015)). It also offers a complement analysis to the new economic geography (NEG) by looking at key additional features. For instance, in spatial economics many authors point out the importance of considering continuous space (Fujita and Mori (2005), Desmet and Rossi-Hansberg (2009) and Desmet and Rossi-Hansberg (2010), and Thisse (2010)), the role played by the dynamics and, in particular, economic growth and the spatial development implications (Fujita and Mori (2005), Gori et al. (2016), Boucekkine et al. (2018), Radi et al. (2019)). Other central elements are the heterogeneity of space, that is, first nature causes (Fujita and Mori (2005) and Boucekkine et al. (2018)), and the spatial mobility of supply (Breinlich (2006)) and, more specifically, capital diffusion (Boucekkine et al. (2018)). However the dynamic spatial theory has come across serious methodological issues, without conclusive answer from either the economic or the mathematical literature. The aim of our paper is to contribute in this regard, providing a general tool to study the solution of the main analytical framework of this literature, namely the spatial Ramsey model.

The spatial Ramsey model is based on the spatial generalization of the Ramsey–Cass–Koopmans model (hereafter the standard Ramsey model), which was first presented in Brito (2004) and Boucekkine et al. (2009). From a general perspective, this framework considers a forward-looking decision maker that chooses the optimal levels of control variables for a given spatial region and time span. In this problem, the social planner additionally takes into account how the state variables spread geographically. The local level of consumption is a usual example of a control, while capital or pollution is often considered as state variables. All this sets an optimal control problem, where the law of motion of the state variables is represented by partial differential equations (PDEs). The partial derivatives included in each PDE specify the spatio-temporal evolution of the corresponding state variable.

More specifically, in the spatial Ramsey model the policy maker takes into account the intertemporal utility (i.e., welfare) of all the individuals distributed in space. Capital is assumed to be mobile across locations, where the corresponding spatio-temporal law of motion is a non-linear parabolic PDE. Considering this model, Boucekkine et al. (2013) and Fabbri (2016) illustrate the role played by the space in economic growth, which turns out to significantly change the dynamical properties of the economy. Assuming a linear technology (i.e., an AK model), Boucekkine et al. (2013) particularly show that transitional dynamics can arise when one adds spatial structure and this despite of constant returns to capital.

The spatial Ramsey model has been useful to describe the dynamic interaction of economic agents that are spatially distributed. Nevertheless, this framework has opened serious methodological concerns. Within the context of social interactions, Mossay and Picard (2011) and Blanchet et al. (2016) identify important difficulties to ensure the existence and uniqueness of equilibrium when we explicitly incorporate space into the economic problem. Considering the spatial Ramsey

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framework, papers such as Boucekkine et al. (2009), Boucekkine et al. (2013), Boucekkine et al. (2013) and Camacho and Pérez-Barahona (2015) encountered a similar trouble. Although a set of necessary conditions can be provided for a general problem, due to including both time and space dimensions, neither the existence nor the uniqueness of solution has been proven so far. This literature then say that the problem is *ill-posed* in the sense of Hadamard (1923). Not being able to ensure the existence of solution is obviously problematic. Moreover, the concern about the non-uniqueness of equilibrium is not less important because it frequently implies indeterminacy, that is, infinite optimal trajectories, in dynamic settings.[1]

In this paper, we generalize the spatial Ramsey model and introduce a monotone method, based on Pao (1992), to analyze the solution of this type of setup. In the literature, this problem has been approached along two different lines of attack.[2] First, using dynamic programming and facing the problem directly. This is the direction taken in Boucekkine et al. (2013, 2018). Not only do the authors prove the existence of a unique solution, even for infinite time horizon, but they also obtain its analytical form. Note, however, that this far, only problems with linear constraints have been solved. Indeed, they specifically concentrate on the case of linear production function, that is, the spatial AK model.

The second line is the most prolific. As in Lions (1966a,b), most of the literature has faced the problem associating a Hamiltonian function and obtaining a set of Pontryagin conditions. Whenever the time space is compact, the existence of a solution is guaranteed (see Raymond and Zidani (1999)). Nevertheless uniqueness, even if the time horizon is finite, has not been proven this far. In the present paper, we deepen into the second approach, working with the set of Pontryagin conditions. This set of conditions is made of a parabolic differential equation, a reverse time parabolic equation plus boundary, initial, and terminal conditions. Although at first glance it might seem trivial to prove uniqueness,[3] notice that it is not a pure parabolic system due to reversed time in one equation. It is neither a pure nor a standard delayed system, since the delay only appears in one equation. Indeed, time evolves forward in one equation and backwards in the other.

Under fairly general assumptions, we prove the existence of unique optimal solution in the spatial Ramsey model when the time horizon is finite. To the best of our knowledge, this is the first paper dealing with the existence and uniqueness of the solution to this framework with a neoclassical technology and non-linear utility.[4] The applicability of our method mainly reduces to ensure the existence of *upper* and *lower solutions*, which are a sort of boundaries to the Pontryagin conditions. After identifying those boundaries, we can then conclude that the solution to this problem exists and it is unique. Consequently, the model is considered as *well-posed*. We also apply our method to an extension of the spatial Ramsey model. Camacho and Pérez-Barahona (2015) adapt this framework to study the social optimal use of land when local pollutants spread across space. However, they could only prove the uniqueness of the time-invariant solution.

The iterative nature of our approach also allows to set an algorithm that provides a numerical check of our technique. It simulates as well the optimal

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trajectories of the economy, across time and space. We concentrate on the importance of capital mobility in economic growth. Barro et al. (1995) present an open-economy neoclassical growth model, without explicitly including spatial structure. Under irreversibility and perfect capital mobility, which is defined as the absence of institutional barriers to spatial capital flows, they observe that one should introduce adjustment costs for capital in order to generate transitional dynamics (see Barro and Sala-i-Martin (2004), too). We reassess this question by considering the spatial Ramsey model. This model can be interpreted as an open-economy neoclassical growth model with spatial structure.

We find that, for plausible values of the parameters, space allows the open-economy neoclassical growth model with perfect capital mobility and irreversibility to display transitional dynamics. In terms of social welfare and income inequality across locations, the model shows that capital mobility can be beneficial in this context. When capital is mobile, the economy lessens the negative effect of diminishing returns to capital, increasing social welfare. Moreover, it reduces the importance of the initial conditions in economic growth because capital diffuses to locations with lower initial endowment. This conclusion is consistent with the observations made by Galor (1996) or Quah (1996a,b).

The paper is structured as follows. Section 2 describes a general version of the spatial Ramsey model. Section 3 presents the result of existence and uniqueness of the social optimum. In Section 4, we apply our approach to the spatial Ramsey model and to its extension for optimal land-use. Finally, we include some concluding remarks in Section 5.

2. THE GENERAL PROBLEM

The objective of the policy maker is to maximize the aggregated social welfare in a bounded region of the space $\Omega \subset \mathbb{R}$ over a finite time period $[0, T]$. To do that, she chooses a time trajectory for a variable c in each location, taking into account the dynamics of the state variable u_1. The choice of the control c affects in turn the dynamics of the state variable. More specifically, the policy maker maximizes the welfare provided by c, aggregating felicity over time and space plus a measure of the final spatial situation of the economy, which is represented by a scrap value function Ψ, as follows:

$$\max_{c} \int_0^T \int_\Omega U(c(x,t))g(x,t)dxdt + \int_\Omega \Psi(u_1(x,T),x)dx \qquad (\mathbf{1})$$

subject to:

$$\begin{cases} u_{1,t}(x,t) - u_{1,xx}(x,t) = f_1(u_1, c), \\ \lim_{x \to \delta\Omega} u_{1,x}(x,t) = 0, \\ u_1(x,0) = u_{1,0}(x) \geq 0 \text{ given.} \end{cases} \qquad (\mathbf{2})$$

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We assume that $U \in C^2(\mathbb{R}^+)$ is measurable, increasing, and concave. As observed in the introduction, the level of local consumption $c(x,t)$ is the typical example of control variable in this type of models.[5] The function $g(x,t)$ represents the usual spatial and time discounting. A possible example is $g(x,t) = \xi(x)e^{-\rho t}$ as in Boucekkine et al. (2013). This functional form assumes that time is exponentially discounted as in the standard Ramsey model, where $\rho > 0$ is the time discount rate. Moreover, the function $\xi(x)$ is assumed to "rapidly decrease" in order to discount space.[6] As in Camacho and Pérez-Barahona (2015), the spatial discounting $\xi(x)$ can also represent the importance that the policy maker gives to each location of the economy.

As in Chapter 2 in Arrow and Kurtz (1970), we introduce a scrap function. In a general context, T is not the end of the world and the stock of variable u_1 will have some value or use in the future, which is measured via the scrap function Ψ. The scrap function can also include space and time discounting since the convergence of the corresponding integral term is required as well. We additionally assume that $\Psi \in C^1(D(u_1), \Omega)$, defined on the domain of u_1, is a positive function with bounded derivative $\Psi_1' > 0$.

The first equation in (2) describes the dynamics of the state variable u_1 across time and space. This is a parabolic PDE, where $u_{1,t}$ denotes the partial derivative with respect to time and $u_{1,xx}$ is the second partial derivative with respect to space. As pointed out in the introduction, this expression can represent the spatio-temporal spread of capital in an economic growth framework. Identifying $u_1(x,t)$ with the capital of location x at time t, that is, $k(x,t)$, the law of motion of capital can be written as

$$k_t(x,t) - k_{xx}(x,t) = I(x,t), \tag{3}$$

where $I(x,t)$ represents the net investment of a location x at time t. The main difference with respect to the standard law of motion without space is the term $-k_{xx}(x,t)$. As in Boucekkine et al. (2009, 2013), it reflects that capital flows from regions with abundant capital to the ones with relatively less capital. They actually show that equation (3) can be economically justified as a balance trade equilibrium in a bounded region.[7]

The second expression in (2) is known in the literature as the Neumann boundary condition. In this framework, we need to describe the behavior of the state variable u_1 in the spatial frontier. This is stablished by the spatial boundary condition. Most of the papers in this literature concentrate on the case of Neumann boundary condition, which assumes that there is no flow of the state variable at locations that are far away from the origin (for further discussion in this respect, see Desmet and Rossi-Hansberg (2010) or Boucekkine et al. (2013), among others).[8] Finally, the last equation is the initial spatial distribution of u_1, which is assumed to be a known positive and continuous function $u_{1,0}(x) \in C(\bar{\Omega})$. In line with the standard Ramsey model, we need to set the initial amount of u_1 at every location. Then, the initial spatial distribution of u_1 represents the initial conditions of our economy.

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In this literature, other papers have considered finite time horizon with end-of-horizon scrap function, together with bounded spatial domain and Neumann boundary conditions. La Torre et al. (2015) introduce a Ramsey-type problem with diffusion of capital and pollution. However, as in Boucekkine et al. (2009), they assume a linear utility function. Notice as well that, for the sake of exposition, we focus on a single state variable. Our analysis can be extended though to the diffusion of multiple states. La Torre et al. (2019) use a parabolic PDE to model migration as a diffusion process. In their model, utility is assumed to be concave, but there is no capital and the production technology is a linear function of labor. Finally, as in Brock and Xepapadeas (2008) and Brock and Xepapadeas (2010), La Torre et al. (2019) mainly concentrate on linear-quadratic objective functions.[9]

In the optimal control literature, only few papers deal with the existence of unique solution in this type of problems. A unique solution exists in some particular cases, namely assuming a linear function $f_1(u_1, c)$ (Lions (1966a,b) and Barbu and Precupanu (2012)) or under specific non-linearities allowing for very limited interactions between the state and the control variables (Lions (1966a,b, 1972) and Ahmed (1977)). Yet, using an appropriated maximum principle, Pontryagin conditions can be associated with general non-linear problems. Fattorini (1990) and Raymond and Zidani (1999) prove that the set of Pontryagin conditions is necessary and sufficient to characterize the solutions of this problem. However, to the best of our knowledge, there is no existence result for this type of dynamical systems.

Using the Ekeland (1974) variational principle, we obtain the associated set of Pontryagin necessary conditions as:[10]

$$(O)\begin{cases} u_{1,t}(x,t) - u_{1,xx}(x,t) = f_1(u_1, c), \\ u_{2,t}(x,t) + u_{2,xx}(x,t) = f_2(u_1, u_2, c), \\ u_2 \frac{\partial f_1}{\partial c} = -U'(c) g(x,t), \\ \lim_{x \to \delta\Omega} u_{i,x}(x,t) = 0, \quad i = 1, 2, \\ u_1(x,0) = u_{1,0}(x) \geq 0 \ \text{ given}, \\ u_2(x,T) = \Psi_1'(u_1(x,T), x), \end{cases}$$

where $\delta\Omega = \bar{\Omega} \backslash \mathring{\Omega}$ is Ω's frontier and u_2 is the corresponding co-state variable (shadow price). As we will see in the next section, ensuring the existence of the unique solution to system (O) is far from being obvious due to the presence of two coupled non-linear parabolic PDE, where one of them is time-reversed.

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION

We will prove in this section the existence of the unique solution to our system of non-linear PDE of real functions on $\Omega \times [0, T]$. For the ease of

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presentation, since c can be expressed as a function of u_1 and u_2, let us rewrite the system (O) as:

$$(I)\begin{cases} u_{1,t}(x,t) - u_{1,xx}(x,t) = f_1(u_1,u_2), \\ u_{2,t}(x,t) + u_{2,xx}(x,t) = f_2(u_1,u_2), \\ \lim_{x\to\delta\Omega} u_{i,x}(x,t) = 0, \;\; i=1,2, \\ u_1(x,0) = u_{1,0}(x) \geq 0, \;\; \text{given}, \\ u_2(x,T) = \Psi_1'(u_1(x,T),x). \end{cases}$$

System (I) is a generalization of the set of necessary and sufficient conditions identified in Boucekkine et al. (2009). One should observe that this system is composed of two PDE. The one corresponding to u_1 is parabolic. However, as we will show later, the second PDE is parabolic only after reversing time. The system also includes conditions on the partial derivatives of u_1 and u_2 at the spatial borders, together with the given initial distribution for u_1. Moreover, $\Psi_1'(u_1(x,T),x)$ determines the final distribution for u_2 that is a function of the final distribution for u_1. This last condition ties the final distribution of the state variable to the final distribution of the associated co-state. As in Camacho et al., (2008), it reveals that the shadow price of the state variable equals the marginal social scrap value of the stock at the end of the planning horizon.

In this paper, we introduce a monotone method based on Pao (1992). These methods were widely used to prove the existence of solution for systems of PDEs, of parabolic and hyperbolic type, with both boundary and initial conditions (see, among others, Sattinger (1972), and Bebernes and Schimitt (1979)). Before going into the details of the proof, let us briefly describe the idea of our method. In a nutshell, we build two converging sequences of functions that unambiguously encircle the unique solution to the system. First, we define and identify appropriate *upper* and *lower* solutions of the original problem (I), which are typically easier to ensure than the solution itself. Starting then from these solutions, we find sequences of associated linear problems, which have a unique solution. We finally show that these associated problems converge to problem (I) from above (below), starting with the upper (lower) solution.

More formally, let us begin by describing the norms that we will use to size functions and variables in our method. We will consider two norms along the proof, the usual norm in $\mathbb{R}^2$ and the supremum norm. For every $u(x,t) \equiv (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$ and at every $(x,t) \in \Omega \times [0,T]$, the Euclidean norm in $\mathbb{R}^2$ is defined as

$$|| u(x,t) || \equiv | u_1(x,t) | + | u_2(x,t) | .$$

We will also use the supremum norm $|| \cdot ||_0$, defined for any $u \in \mathbb{R}^2$ as

$$|| u ||_0 \equiv \sup_{(x,t)\in\Omega\times[0,T]} || u || .$$

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As pointed out before, the dynamics of u_2 is described by a reversed-time parabolic PDE. Consequently, one of the key ingredients of the proof is to reverse time in the second PDE of the system (I). We define in this respect the variables w_1 and w_2 as

$$w_1(x,t) \equiv u_1(x, T-t),$$

$$w_2(x,t) \equiv u_2(x, T-t).$$

Then, the dynamics of w_2 is described by a parabolic PDE with known initial distribution $\Psi_1'(w_1(x,0),x)$:

$$\begin{cases} w_{2,t}(x,t) - w_{2,xx}(x,t) = -f_2(w_1, w_2), \\ w_2(x,0) = \Psi_1'(w_1(x,0),x). \end{cases} \tag{4}$$

The system (I) mixes two types of non-linear parabolic PDE, where one of them is parabolic only after reversing time (as shown in equation (4) above). Indeed, we aim at extending the monotone method of Pao (1992), considering in our paper dynamical systems of mixed types of parabolic PDE. In order to do so, we need to define adequate upper and lower solutions. We say that two functions $\hat{u} = (\hat{u}_1, \hat{u}_2)$ and $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ are upper and lower solutions to (I) if they satisfy for every t:

$$\begin{aligned} &\tilde{u}_{1,t}(x,t) - \tilde{u}_{1,xx}(x,t) - f_1(\tilde{u}_1, \tilde{u}_2) \le 0 \le \hat{u}_{1,t}(x,t) - \hat{u}_{1,xx}(x,t) - f_1(\hat{u}_1, \hat{u}_2), \\ &\hat{u}_{2,t}(x,t) + \hat{u}_{2,xx}(x,t) - f_2(\hat{u}_1, \hat{u}_2) \le 0 \le \tilde{u}_{2,t}(x,t) + \tilde{u}_{2,xx}(x,t) - f_2(\tilde{u}_1, \tilde{u}_2), \\ &\lim_{x \to \delta\Omega} \hat{u}_{i,x}(x,t) = 0, \ \ \lim_{x \to \delta\Omega} \tilde{u}_{i,x}(x,t) = 0, \ i = 1, 2, \end{aligned} \tag{5}$$

with $\tilde{u}_1 \le \hat{u}_1$, $\tilde{u}_2 \le \hat{u}_2$, and

$$\begin{aligned} \tilde{u}_1(x,0) < u_1(x,0) < \hat{u}_1(x,0), \\ \tilde{u}_2(x,0) < u_2(x,0) < \hat{u}_2(x,0), \end{aligned} \tag{6}$$

for all $x \in \Omega$.[11] Notice that due to the presence of a reversed-time parabolic PDE, considering (4) the PDE condition for $\hat{u}_2$ and $\tilde{u}_2$ in (5) becomes

$$\tilde{w}_{2,t}(x,t) - \tilde{w}_{2,xx}(x,t) + f_2(\tilde{w}_1, \tilde{w}_2) \le 0 \le \hat{w}_{2,t}(x,t) - \hat{w}_{2,xx}(x,t) + f_2(\hat{w}_1, \hat{w}_2). \tag{7}$$

That is to say, $\hat{w}_2$ and $\tilde{w}_2$ are, respectively, the upper and lower solution of the corresponding parabolic PDE, as it is usually defined in the literature. Moreover,

$$\tilde{w}_2(x,0) < \Psi_1'(u_1(x,T),x) < \hat{w}_2(x,0).$$

For $i = 1, 2$, $\tilde{u}_i$, $\hat{u}_i$ are required to be $C(\bar{\Omega} \times [0,T]) \cap C^{1,2}(\Omega \times [0,T])$, that is, they are continuous in $\bar{\Omega} \times [0,T]$, continuously differentiable in t, and twice continuously differentiable in x for all $(x,t) \in \Omega \times [0,T]$.

Having established the appropriate definition of lower and upper solutions, let us also define the set of continuous functions between $\tilde{u}$ and $\hat{u}$ as

$$< \tilde{u}, \hat{u} > \equiv \left\{ (u_1, u_2) \in C(\Omega \times [0,T]) : (\tilde{u}_1, \tilde{u}_2) \le (u_1, u_2) \le (\hat{u}_1, \hat{u}_2) \right\}.$$

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The interest of $< \tilde{u}, \hat{u} >$ is that it defines the precise space where the solutions will be confined, that is, all solutions are limited from below by $\tilde{u}$ and from above by $\hat{u}$ (see Theorem 2 at the end of this section).

As usual in this literature, we need to impose boundedness conditions on initial distributions and functions f_1 and f_2 in order to avoid explosive solutions. Assumption 1 sets the conditions for the initial distributions:

ASSUMPTION 1. *$u_{1,0}(x) \in C(\bar{\Omega})$ is measurable. The initial distributions satisfy that*

$$\begin{cases} \tilde{u}_1(x,0) < h_1 e^{h_2|x|^2} \\ u_1(x,0) < h_1 e^{h_2|x|^2} \\ \hat{u}_1(x,0) < h_1 e^{h_2|x|^2} \end{cases} \text{ and } \begin{cases} \tilde{u}_2(x,0) < h_1 e^{h_2|x|^2} \\ \hat{u}_2(x,0) < h_1 e^{h_2|x|^2} \end{cases}$$

for some positive constants h_1, h_2 with $h_2 < \frac{1}{4T}$.

Next, we introduce the conditions on functions f_1 and f_2. Assumption 2 imposes conditions on the change of f_1 and f_2, which are less restrictive than differentiability:

ASSUMPTION 2. *There exist bounded functions $\underline{a}_i = \underline{a}_i(x,t)$, $\bar{a}_i = \bar{a}_i(x,t)$, $\underline{c}_i = \underline{c}_i(x,t)$ and $\bar{c}_i = \bar{c}_i(x,t)$, with $i = 1, 2$, such that for every $u, v \in < \tilde{u}, \hat{u} >$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$, (f_1, f_2) satisfies*

$$-\underline{a}_1(u_1 - v_1) - \underline{a}_2(u_2 - v_2) \leq f_1(u_1, u_2) - f_1(v_1, v_2) \leq \bar{a}_1(u_1 - v_1) + \bar{a}_2(u_2 - v_2),$$

$$-\underline{c}_1(u_1 - v_1) - \underline{c}_2(u_2 - v_2) \leq f_2(u_1, u_2) - f_2(v_1, v_2) \leq \bar{c}_1(u_1 - v_1) + \bar{c}_2(u_2 - v_2),$$

for $u_1 \geq v_1$ and $u_2 \geq v_2$. Furthermore, functions $\underline{a}_1$ and $\bar{c}_2$ are Hölder continuous in $\Omega \times [0, T]$.

The domain of f_i is restricted to sets in which Assumption 3 below is verified. We denote these reduced domains for f_1 and f_2 by J_1 and J_2, respectively. Our analysis will be limited to functions f_1 and f_2 which satisfy the following assumption:

ASSUMPTION 3. *For every $u, v \in < \tilde{u}, \hat{u} >$, either f_1 is increasing in both u_1 and u_2 and $\underline{a}_1 \geq 0$, or $\underline{a}_2 \leq 0$. Similarly, either f_2 is decreasing in both u_1 and u_2 and $\bar{c}_2 \geq 0$, or $\bar{c}_1 \leq 0$.*

Hence, either f_i is increasing in both u_1 and u_2, either f_i is increasing in one of the variables and decreasing in the other, or even decreasing in each of them, but the change of f_i when both u_1 and u_2 increase needs to be bounded from below. For instance, if f_1 is not increasing in u_1 nor in u_2, then $\underline{a}_2 \leq 0$ means that the following constraint should hold: $f_1(u_1, u_2) - f_1(v_1, v_2) \geq -\underline{a}_1(u_1 - v_1) - \underline{a}_2(u_2 - v_2)$, with $-\underline{a}_2(u_2 - v_2) > 0$. If $\underline{a}_1 > 0$, then f_1 could still decrease when both u_1 and u_2 increase.

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Finally, we include some continuity assumptions as well:

ASSUMPTION 4. *For $i = 1, 2$, functions f_i are Hölder continuous in $J_1 \times J_2$ for some subset $J_1 \times J_2 \subseteq \mathbb{R}^2$.*[12]

Defining functions M_1 and M_2 as

$$M_1 \equiv \max\{|\underline{a}_1|, |\underline{a}_2|, |\overline{a}_1|, |\overline{a}_2|\},$$
$$M_2 \equiv \max\{|\underline{c}_1|, |\underline{c}_2|, |\overline{c}_1|, |\overline{c}_2|\},$$

implies that functions f_1 and f_2 are Lipschitz continuous since $M_i = M_i(x, t)$ are bounded in $\Omega \times [0, T]$. That is to say, one can find bounded functions $M_i = M_i(x, t)$ in $\Omega \times [0, T]$ such that

$$|f_i(u_1, u_2) - f_i(v_1, v_2)| \leq M_i |u - v|,$$

for any u, v in $< \tilde{u}, \hat{u} >$.

With functions $\underline{a}_1$ and $\bar{c}_2$ from Assumption 2, we build functions F_1 and F_2 as

$$\begin{aligned} F_1(x, t, u_1, u_2) &\equiv f_1(u_1, u_2) + \underline{a}_1(x, t)u_1, \\ F_2(x, t, u_1, u_2) &\equiv f_2(u_1, u_2) - \bar{c}_2(x, t)u_2. \end{aligned} \tag{8}$$

By Assumptions 2 and 4, functions F_i are Hölder continuous in $\Omega \times [0, T] \times < \tilde{u}, \hat{u} >$, for $i = 1, 2$. Furthermore, we also define two linear operators $\mathbb{L}_1$ and $\mathbb{L}_2$ of u_1 and u_2 as

$$\mathbb{L}_1 u_1 \equiv u_{1,t} - u_{1,xx} + \underline{a}_1 u_1,$$
$$\mathbb{L}_2 u_2 \equiv u_{2,t} + u_{2,xx} - \bar{c}_2 u_2.$$

Introducing the functions F_i and the operators $\mathbb{L}_i$ defined above, we can rewrite the two PDE in (I) as

$$\mathbb{L}_i u_i = F_i(u_1, u_2),$$

in $\Omega \times [0, T]$ for $i = 1, 2$.

As pointed out before, the PDE corresponding to u_2 is parabolic after reversing time. Therefore, let us define the following alternative operator to $\mathbb{L}_2$ in order to consider only parabolic PDEs:

$$\mathbb{L}_2' w_2 \equiv w_{2,t} - w_{2,xx} + \bar{c}_2 w_2.$$

Taking then the operator $\mathbb{L}_2'$, the PDE in (4) becomes

$$\mathbb{L}_2' w_2 = -F_2(w_1, w_2).$$

This new writing turns out to be extremely useful in the upcoming Theorem 1. It allows to consider sequences of single linear parabolic PDE that will be used to prove the existence of the unique solution of (I). As we announced before, the theorem begins by building two monotonic sequences of functions. On the one hand, we set up a non-increasing sequence of continuous functions of time and space $\{\bar{u}_1^k, \bar{u}_2^k\}_{k \in \mathbb{N}}$ and, on the other hand, a non-decreasing sequence of continuous functions $\{\underline{u}_1^k, \underline{u}_2^k\}_{k \in \mathbb{N}}$. Without loss of generality, we construct $\{\bar{u}_1^k\}_{k \in \mathbb{N}}$

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initiating the sequence with the upper solution $\{\hat{u}_1, \hat{u}_2\}$. At every step, we generate $\bar{u}_1^k$ taking $(\bar{u}_1^{k-1}, \bar{u}_2^{k-1})$. Then, with $\bar{w}_1^k(x,t) = \bar{u}_1^k(x, T-t)$ and $(\bar{w}_1^{k-1}, \bar{w}_2^{k-1})$, we will obtain $\bar{w}_2^k$. The step ends when we reverse time again in order to obtain $\bar{u}_2^k(x,t) = \bar{w}_2^k(x, T-t)$. We can repeat this procedure at infinitum to build our sequence. We follow a similar strategy for the sequences $\{\underline{u}_1^k\}_{k\in\mathbb{N}}$ and $\{\underline{u}_2^k\}_{k\in\mathbb{N}}$. However, in this second case the process initiates with the lower solution $\{\tilde{u}_1, \tilde{u}_2\}$. We then prove that these sequences uniformly converge:

$$\lim_{k\to\infty} \bar{u}_i^k = \bar{u}_i \text{ and } \lim_{k\to\infty} \underline{u}_i^k = \underline{u}_i,\ i = 1, 2.$$

Finally, we show that $\mathbb{L}^{-1}F$ possesses the contraction property, where $\mathbb{L}$ is the diagonal matrix $\mathbb{L} \equiv \text{diag}(\mathbb{L}_1, \mathbb{L}_2)$ and F defines the transpose of the vector of F_i functions, that is, $F \equiv (F_1, F_2)'$. Hence, the limits of the above sequences coincide and define the unique solution to (I).

Developing those steps, we establish the following theorem. It states that problem (I) has a unique solution in $\Omega \times [0, T]$ under the assumptions introduced before:

THEOREM 1. *Under the Assumptions 1–4, the problem*

$$\begin{cases} u_{1,t}(x,t) - u_{1,xx}(x,t) = f_1(u_1, u_2), \\ u_{2,t}(x,t) + u_{2,xx}(x,t) = f_2(u_1, u_2), \\ \lim_{x\to\delta\Omega} u_{i,x}(x,t) = 0,\ \ i = 1, 2, \\ u_1(x,0) = u_{1,0}(x)\ \ \textit{given}, \\ u_2(x,T) = \Psi_1'(u_1(x,T), x), \end{cases} \tag{9}$$

has a unique solution (u_1, u_2) in $\Omega \times [0, T]$. Moreover, $u = (u_1, u_2)$ is the limit of the monotonic sequence $\{u_1^k, u_2^k\}_{k\in\mathbb{N}}$, where $u_2^k(x,t) = w_2^k(x, T-t)$. The sequence $\{u_1^k, w_2^k\}_{k\in\mathbb{N}}$ is generated as

$$\begin{cases} \mathbb{L}_1 u_1^k = F_1\left(u_1^{k-1}, u_2^{k-1}\right), \ \ \lim_{x\to\delta\Omega} u_{1,x}^k(x,t) = 0, \ \ \forall t \geq 0, \\ \mathbb{L}_2' w_2^k = -F_2\left(w_1^{k-1}, w_2^{k-1}\right), \ \ \lim_{x\to\delta\Omega} w_{2,x}^k(x,t) = 0, \ \ \forall t \geq 0, \end{cases} \tag{10}$$

for all $k = 1, 2, \ldots$ We can choose between two sets of initial conditions and initial distributions. The first set $(\hat{S})$ of conditions below generates a non-increasing sequence, whereas the second set $(\tilde{S})$ generates a non-decreasing sequence:

$$(\hat{S}) \quad \begin{cases} \left(u_1^0, u_2^0\right) = (\hat{u}_1, \hat{u}_2), \\ u_1^k(x,0) = \frac{\Upsilon}{k}\hat{u}_1(x,0) + \left(1 - \frac{\Upsilon}{k}\right) u_1(x,0), \\ w_2^k(x,0) = \frac{\psi}{k}\hat{w}_2(x,0) + \left(1 - \frac{\psi}{k}\right) \Psi_1'(w_1^{k-1}(x,0), x); \end{cases}$$

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$$(\tilde{S}) \quad \begin{cases} \left(u_1^0, u_2^0\right) = (\tilde{u}_1, \tilde{u}_2), \\ u_1^k(x,0) = \frac{\zeta}{k}\tilde{u}_1(x,0) + \left(1 - \frac{\zeta}{k}\right) u_1(x,0), \\ w_2^k(x,0) = \frac{\phi}{k}\tilde{w}_2(x,0) + \left(1 - \frac{\phi}{k}\right) \Psi_1'(w_1^{k-1}(x,0), x), \end{cases}$$

with $0 < \zeta < \Upsilon < 1$ *and* $0 < \phi < \psi < 1$.

Proof. See Appendix B. ■

We have proved this far that, once a pair of lower and upper solutions is identified, there exists a unique solution u to our original problem between those boundaries. A natural question arises, wondering about how many solutions could be missing below the specific lower solution or above the chosen upper solution. The answer is that, under the assumptions of the model, we will not find any. Then, the unique solution of the problem will be unambiguously located between the pair of lower and upper solutions, that is, $u \in < \tilde{u}, \hat{u} >$. We state this final result in Theorem 2 below:

THEOREM 2. *Let Assumptions 1–4 hold, and* $\tilde{u}$ *and* $\hat{u}$ *be lower and upper solutions to problem* (I). *Let us further assume that:*

(i) either f_1 *is increasing in* u_1 *and* u_2, *with* $\underline{a}_1 \geq 0$; *or* $\underline{a}_2 \leq 0$,
(ii) either f_2 *is decreasing in* u_1 *and* u_2, *with* $\bar{c}_2 \geq 0$; *or* $\bar{c}_1 \leq 0$.

If u *is a solution to* (I), *then* $u \in < \tilde{u}, \hat{u} >$.

Proof. See Appendix C. ■

4. APPLICATIONS

Let us apply our method to the spatial Ramsey model, where ill-posedness had not been solved this far even for a finite decision horizon. As pointed out in Section 1, Boucekkine et al. (2013, 2018) are exceptions in this regard. They focus, however, on the case of a linear production function (i.e., the spatial AK model).[13] Using Theorems 1 and 2, we will conclude that the social optimum exists and is unique. Hence, when the time horizon is finite, we can confirm that the spatial Ramsey model is well-posed. As a numerical check of our result, we will implement the iterative method presented in Theorem 1. We will specifically use this algorithm to study the role played by capital mobility in economic growth and spatial development. Finally, as a second application, we will consider the framework introduced by Camacho and Pérez-Barahona (2015). This is an extension of the spatial Ramsey model for optimal land-use planning, where they could only prove the uniqueness of time-invariant solution.

4.1. The Spatial Ramsey Model

The spatial Ramsey model considers a policy maker that maximizes utility in a region Ω of space from time 0 to T, subject to the dynamic evolution across time and space of the capital stock.[14] The corresponding optimal control problem is

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$$\max_{c} \int_0^T \int_\Omega U(c(x,t))e^{-\rho t}dxdt + \int_\Omega \Psi(k(x,T))e^{-\rho T}dx \tag{11}$$

subject to

$$\begin{cases} k_t(x,t) - k_{xx}(x,t) = Af(k(x,t)) - \delta k(x,t) - c(x,t), \forall (x,t) \in \Omega \times [0,T] \\ \lim_{x \to \delta\Omega} k_x(x,t) = 0 \ \forall t \in [0,T], \\ k(x,0) = k_0(x) > 0 \ \text{given} \ \forall x \in \Omega. \end{cases} \tag{12}$$

The policy maker decides on the consumption $c(x,t)$ of the representative household at every location x and time t. In this respect, she takes into account the spatial-dynamic motion of the stock of capital in each location $k(x,t)$. The first constraint in (12) represents the law of motion of $k(x,t)$. The right-hand side of this equation is the net investment, where $\delta \in (0,1)$ is the rate of capital depreciation and $f(\cdot)$ is the production technology of the economy, with a scale parameter A. As we already observed in Section 2 (see equation (3)), this expression is a parabolic PDE where $k_{xx}(x,t)$ introduces a new feature with respect to the standard law of motion. This term entirely comes from the spatial mobility of capital, which flows across regions depending on their spatial gradient of capital. The second constraint is the usual (Neumann) boundary condition and it assumes that there is no capital flow at locations far away from the origin. Finally, $k_0(x) > 0$ is the initial spatial distribution of capital, which is given in the economy.

$\Psi(\cdot) > 0$ measures how much the policy maker cares about the final state of capital. It is a continuously differentiable function with bounded and positive derivative. We assume that U and f are both positive, continuous, differentiable, and concave functions. Only settings with measurable k_0 and existing integrals are considered. Following the variational principle introduced in Appendix A, the Pontryagin optimal conditions associated with this problem are

$$\begin{cases} k_t(x,t) - k_{xx}(x,t) = Af(k(x,t)) - \delta k(x,t) - c(x,t), \forall (x,t) \in \Omega \times [0,T], \\ \lambda_t(x,t) + \lambda_{xx}(x,t) = \lambda \left(\delta + \rho - Af'(k(x,t))\right), \forall (x,t) \in \Omega \times [0,T], \\ U'(c(x,t)) = \lambda(x,t), \forall (x,t) \in \Omega \times [0,T] \\ \lim_{x \to \delta\Omega} k_x(x,t) = \lim_{x \to \delta\Omega} \lambda_x(x,t) = 0, \ \ \forall t \in [0,T], \\ k(x,0) = k_0(x) \ \text{given}, \forall x \in \Omega, \\ \lambda(x,T) = \Psi'(k(x,T)), \forall x \in \Omega, \end{cases} \tag{13}$$

where λ is the corresponding co-state variable, that is, the shadow price of capital.

4.2. Existence and Uniqueness of Social Optimum

For this illustration, we consider the usual constant relative risk aversion (CRRA) utility function $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$, together with a Cobb–Douglas production technology $f(k) = k^\alpha$ where $\alpha \in (0,1)$. We assume, for the sake of simplicity, a linear scrapping value $\Psi(k(x,T)) = \chi k(x,T)$ with $\chi > 0$. Using our general formulation in (I), $f_1(k,\lambda) = Af(k(x,t)) - \delta k(x,t) - \lambda(x,t)^{-1/\sigma}$ and $f_2(k,\lambda) = \lambda\left(\delta + \rho - Af'(k(x,t))\right)$. Let us apply Theorem 1 in order to prove the

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existence of the unique social optimal solution of the spatial Ramsey model. As we pointed out in the previous section, the main requirement of the theorem is to identify upper and lower solutions to the system (I). This will ensure the existence of unique solution within the space that the upper and lower solutions outline. Taking the definition provided by conditions (5)–(7), we can find the upper and lower solutions required in Theorem 1.[15] One can verify that the following couple is a lower solution to our problem :

$$(\tilde{k}, \tilde{\lambda}) = \left(\left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} + \epsilon, \left(A\tilde{k}^{\alpha} - \delta \tilde{k} \right)^{-\sigma} - \epsilon \right),$$

with $\epsilon > 0$ a sufficiently small constant. Moreover, the couple below provides an upper solution:

$$(\hat{k}, \hat{\lambda}) = (\varphi e^{\phi t}, \psi e^{\zeta t}), \tag{14}$$

with $\zeta \leq \delta + \rho - A\alpha\varphi^{(\alpha-1)\phi}$ and $\varphi(\phi - \delta)A\varphi^{\alpha} > 0$, for positive constants $\varphi, \psi, \phi, \zeta > 0$.[16] Observe that the optimal solution will always remain between this upper and this lower solutions. In particular, the lower solution provides a lower bound for the initial condition. If $k(0) < \tilde{k}$, then the stock of capital is not large enough to ensure the existence of an optimal trajectory for consumption from 0 to T. In the case of the upper bound, it provides us with the maximum growth rate. Indeed, according to (14) capital cannot grow faster than ϕ.

As stated in Theorem 1, we also have to ensure several boundary and continuity requirements (Assumptions 1–4). It is easy to see that Assumption 1 holds if we consider non-explosive initial distributions for the capital and for the upper and lower solutions. Since f_1 and f_2 are continuously differentiable functions, they are Hölder continuous as well. Assumption 4 is then verified. With respect to Assumption 2, this is guaranteed if we focus on interior non-explosive solutions. Lastly, notice that f_1 is increasing in both k and λ.[17] Moreover, f_2 is decreasing in λ but increasing in k (since $\lambda > 0$ and the production function is concave in k). We should then choose $\underline{a}_1 \geq 0$ and $\bar{c}_1 \leq 0$ according to Assumption 3. In addition to that, since f_2 is increasing in k but decreasing in λ, we can set $\bar{c}_1 = 0$ in order to ensure that Assumption 2 holds.

Since Theorem 1 assumptions hold for the functional forms of this illustration, we can then confirm that there exists a unique solution to our problem. Considering Theorem 2, we can guarantee that every solution lies between the upper and lower solutions and, therefore, not missing any additional solution. Hence, we can finally state the following proposition:

PROPOSITION 1. *The spatial Ramsey model (11)–(12) is well-posed.*

4.3. Numerical Check

Let us consider the initial set $(\tilde{S})$ or $(\hat{S})$ in Theorem 1. Each of these sets defines the initialization of an algorithm starting from the lower or the upper solution,

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TABLE 1. Parameter values

A	Scale parameter	1
σ	Utility parameter	1.95
ρ	Time discount rate	0.02
δ	Depreciation rate	0.05
α	Output elasticity	0.75
χ	Scrap parameter	0.1
ϵ	Lower solution constant	1

respectively. Our numerical method iteratively applies (10) in order to generate the sequences stated in the theorem. Notice that, since we have proved that both sequences converge, the algorithm can be initialized either with the lower or with the upper solution. The details of the algorithm are provided in Appendix G.

We assume the same functional forms as in Proposition 1. Since our method involves functions F_1 and F_2 defined in (8), we have to specify $\underline{a}_1$ and $\bar{c}_2$. Taking into account the Assumptions 2 and 3, one can verify that $\underline{a}_1 = \delta$ and $\bar{c}_2 = \delta + \rho$ satisfy all the requirements since we have set $\bar{c}_1 = 0$. All parameter values used in the simulations are displayed in Table 1. For the sake of illustration, we set the scale parameter $A = 1$. Regarding to the CRRA utility, Attanasio and Weber (1993) and Attanasio and Browning (1995) identify an interval for σ between 1.25 and 3.33 using micro-household data. Consistently with this range, Barro et al. (1995) assume $\sigma = 2$. Chetty (2006), however, observes that, even if macroeconomists tend to suggest values of σ greater than in microeconomics, there is an upper bound implying that $\sigma < 2$. Therefore, in line with those studies we choose for our application $\sigma = 1.95$. As in Barro et al. (1995) and Barro and Sala-i-Martin (2004), we consider a time discount rate $\rho = 0.02$ and a depreciation rate $\delta = 0.05$. We set the output elasticity α to 0.75 following the notion of a broad measure of capital that also includes knowledge (see, for instance, Barro and Sala-i-Martin (2004) and Acemoglu (2009)). For the scrapping function, we choose a marginal value $\chi = 0.1$ in order to reduce the direct welfare gains due to the capital left at the end of the decision horizon. Finally, we set the constant ϵ of the lower solution to 1. As observed before, the iterations can start with either the upper or the lower solutions. Since for this model the lower solutions are simpler (see Section 4.2), we decide to initialize the algorithm with $(\tilde{S})$.[18]

We take a decision horizon T of 60 periods. Moreover, the region Ω of space is assumed to be a real line of length 10. We chose these values in order to limit the computing load of the simulations. Since our numerical method requires that both time and space are discretized, we divide them into 100 and 20 points, respectively. Following the notation of Appendix G, this corresponds to $J = 100$ and $N = 20$ with $Dt = 0.1$ and $Dx = 0.5$.

For the numerical check, we eliminate all causes of spatial concentration of economic activity (i.e., first and second nature causes of agglomeration; see, for instance, Boucekkine et al. (2018)). One can reasonable anticipate that capital

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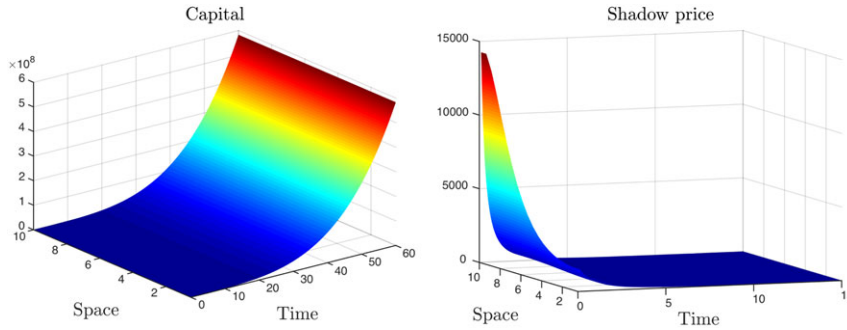


FIGURE 1. Spatial mobility.

mobility across locations will induce spatial homogeneity because only centrifugal forces are put in place. This scenario is particularly useful, on the one hand, to test our analytical method; the algorithm implements the iterative method introduced in Theorem 1. And, on the other hand, this setting allows us to investigate the importance of capital mobility in a neoclassical growth context. Since we will also pay attention to the spatial distributional effects, we assume that locations are spatially heterogenous with respect to their capital endowment. We specifically consider that the initial distribution of capital is given by a logistic function $k(x, 0) = 3000/\{1 + e^{[0.5(x-4)]}\}$ for x in our discretized space $\Omega \simeq \{Dx, 2 * Dx, ..., N * Dx\}$. Consequently, the endowment of capital stock of the different locations would range from 2453 to 1 as we move away from the origin. The initial distribution has been calibrated in order to fit a Gini coefficient of about 0.6, which is consistent with current figures on global inequality (see, for instance, Milanovic (2011) or IBRD (2016)).

Let us start with the case of perfect capital mobility. Our method numerically finds the solution to the problem in 30,762 iterations, with an Euclidean distance between the last two iterations lower than 0.1 for both k and its shadow price λ. We plot their optimal trajectories in Figure 1. We compare this situation with the other extreme case where there is no capital mobility. In the context of our model, this is represented by assuming no spatial diffusion. The policy maker still maximizes the utility in the whole space. However, capital is not allowed to move from one location to another. For this case, the algorithm needs 2264 iterations to converge and, as in the previous scenario, the Euclidean distance between two iterations is lower than 0.1. Figure 2 reproduces the corresponding simulations.[19]

The figures show that the economy displays transitional dynamics even if capital is perfectly mobile. This outcome contrasts with the response of the open-economy neoclassical growth model introduced by Barro et al. (1995). Transitional dynamics is a desirable property for a realistic economic model. However, under the reasonable assumptions of capital irreversibility and economic openness, they observe that one should incorporate capital adjustment costs in order to ensure this property. Yet, an important limitation of their setup is

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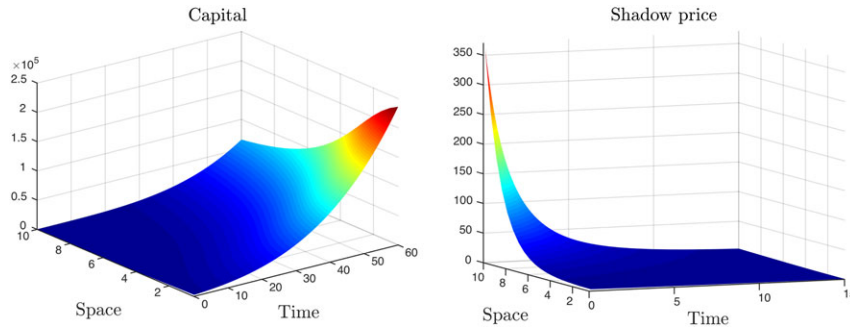


FIGURE 2. No spatial mobility.

that space is not explicitly modeled. We show in this regard that adding spatial structure to the problem, represented in our framework by the diffusion process, induces transitional dynamics even if capital is perfectly mobile.

Another interesting feature that the simulations point out is the effect of capital mobility on social welfare. Capital mobility allows the economy to reduce the negative effect of diminishing returns to capital. When capital is mobile, it flows into locations with lower abundance of capital (and, consequently, with higher shadow prices). Then, the global impact of diminishing returns reduces. As we can see from the figures, capital increases with time in both scenarios due to the linear scrap function. However, at the end of the decision horizon, the minimum and the maximum stock of capital across locations are larger when there is spatial mobility (one can also verify from the simulations that the distribution of capital across locations is monotonic). More specifically, the $\min|_{\Omega}(k(x,T)) = 6.5655 \times 10^4$ and $\max|_{\Omega}(k(x,T)) = 2.3618 \times 10^5$ without mobility, while the $\min|_{\Omega}(k(x,T)) = 5.8301 \times 10^8$ and $\max|_{\Omega}(k(x,T)) = 5.8308 \times 10^8$ with mobility. Knowing the evolution of the shadow price, the condition $U'(c(x,t)) = \lambda(x,t)$ provides the consumption of each location $c(x,t)$. Considering $k(x,T)$ and the corresponding scrap value, one can then compare the levels of social welfare in each scenario. We numerically find that capital mobility results in greater social welfare for the economy. In this scenario, capital is spatially reallocated (diffusion), inducing efficiency gains. This result is consistent with the importance that economic integration gives to capital mobility (see Kose et al. (2010), for a survey of the literature).

We finally consider the effect of capital mobility on the evolution of spatial heterogeneity. As we have anticipated, in this model capital mobility tends to reduce the relative differences with respect to the capital endowment of each location. Indeed, when capital is spatially immobile, at the end of the decision horizon the lowest capital accumulated among all locations represents about 28% of the richest one. This contrasts with the case of mobility, where the stock of capital becomes almost equal across locations. Regarding to income inequality

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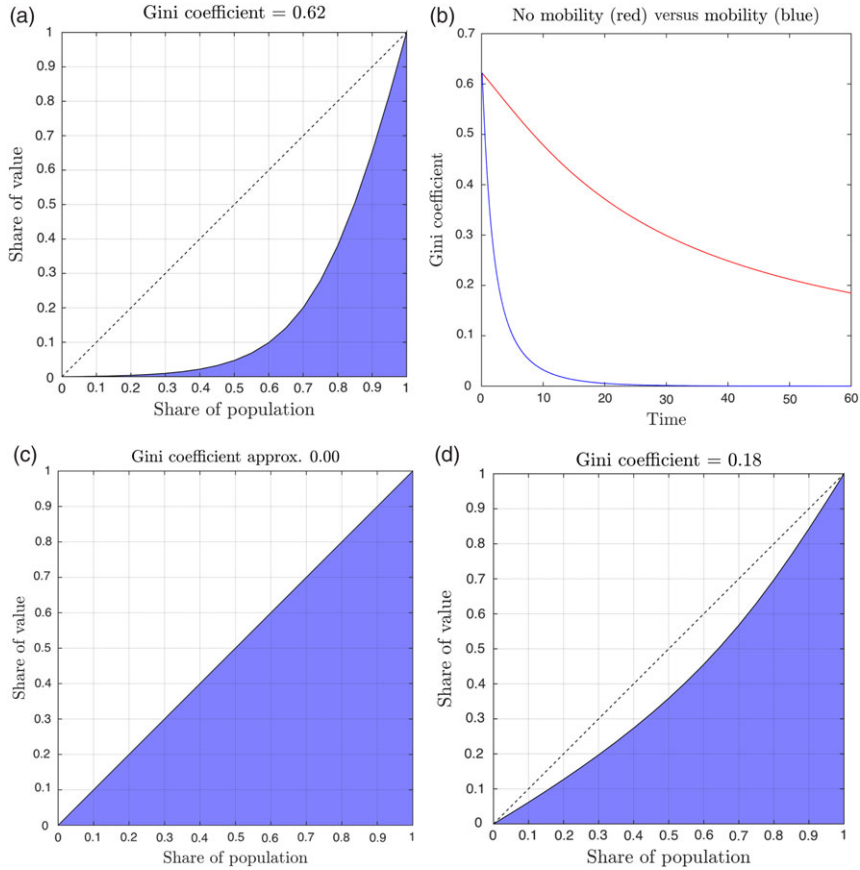


FIGURE 3. Income distribution.

$y(x,t) = Af(k(x,t))$ defines the income of each location at time t. Taking the simulated trajectories for $k(x,t)$, we can analyze the dynamics of the spatial income distribution by means of computing the Lorentz curve and the Gini coefficient of the economy in each time t. Figure 3 represents the evolution of spatial inequality, with and without capital mobility.

Figure 3(a) depicts the initial income inequality due to the spatially heterogeneous distribution of capital endowments. This corresponds to a Gini coefficient of 0.62, as calibrated from world data. In both cases, inequality reduces as capital accumulates (see Figure 3(b)) because of diminishing returns to capital. However, spatial mobility of capital allows the economy to achieve lower inequality in every period (blue line). This is particularly clear if we evaluate the level of inequality in the economy at the end of the decision horizon. Figure 3(c) and (d) shows that the area between the line of equality and the Lorentz curve is significantly smaller when capital is mobile. The Gini coefficient is close to zero when the capital is

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mobile, while it is significantly higher (around 0.18) when spatial mobility is not allowed.

As already observed in Section 1, in line with papers such as Galor (1996) and Quah (1996a,b), this numerical result reveals that improving capital mobility reduces the importance of the initial differences among locations. Moreover, in the spatial Ramsey model capital mobility can be interpreted as economic integration. In this respect, Forslid et al. (2002) and Damijan and Kostevc (2011) identified for Europe that the initially increasing economic integration has raised regional divergence due to strong agglomeration effects. However, as integration intensifies, the growing inflow of capital fosters convergence. While the standard NEG framework is consistent with the former phase, our simulations show that the spatial Ramsey model can contribute to the latter. This feature clearly illustrates the complementarity of both approaches for spatial economics.

Bourguignon (2015) also notices that the historical evolution of the global Gini coefficient is non-monotonic, which contrasts with the transition depicted in Figure 3(b). Although the spatial Ramsey model is a highly stylized framework, it can be useful to identify other key elements behind inequality. Indeed, let us exploit the explanatory power of the spatial Ramsey model extending our application in order to improve the dynamics of the Gini coefficient. We introduce in this regard spatial differences in total factor productivity (TFP), that is, $A(x)$. This scenario can be associated with technology and/or knowledge heterogeneity across locations.

Building on the previous scenario, we underline the role played by TFP heterogeneity assuming this time that $A(x) = 1 + 1/\{1 + e^{[-0.5(x-15)]}\}$. In sharp contrast with $k(x, 0)$, TFP increases (almost doubles) as we move away from the origin. Here, A could be understood as the savoir-faire of a location or as a proxy for locations' knowledge. This spatial heterogeneity could also be related, for instance, to events implying a massive destruction of physical capital. Since in this scenario $k(x, t)$ can only represent physical capital, we set $\alpha = 1/3$ consistently with the restricted definition of capital.[20] Notice that the corresponding Pontryagin optimal conditions are similar to (13), considering $A(x)$ instead of A. We also have to find appropriate upper and lower solutions for this problem (see Appendix F for the details). Figure 4 shows the evolution of the Gini coefficient.[21]

The dynamics of the Gini coefficient is non-monotonic. As in the previous scenario, capital mobility allows for a significant reduction of inequality. However, the higher TFP of locations with incoming capital relaxes this effect. It decreases its strength so much that there is an initial increase in inequality. This transitory increase of inequality is finally compensated by mobility and diminishing returns as before, which slowly reduce the Gini coefficient. Note that this evolution is also consistent with Milanovic (2011). Still, due to the simplified nature of our analysis, further investigations should be done about the origins of the spatial heterogeneity in TFP. A spatial Ramsey model with human capital and endogenous TFP could be a research direction to follow in this respect.

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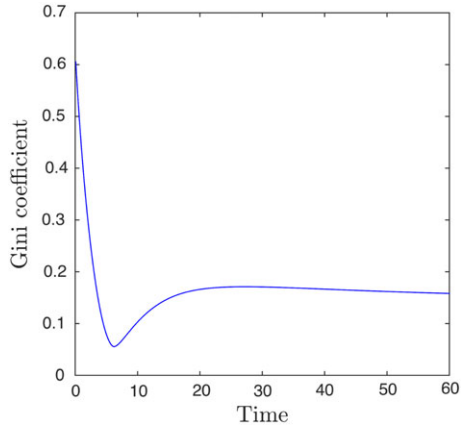


FIGURE 4. TFP heterogeneity.

4.4. Land Use Dynamics and the Environment

Our last application is based on the spatial-dynamic model introduced by Camacho and Pérez-Barahona (2015). This setup extends the spatial Ramsey model in order to study the social optimal use of land. The most distinguishing feature of the model is that the policy maker explicitly takes into account how local pollutants spread across space.

We assume an economy where each location has a unit of land. This can be devoted to three different activities: production, housing, and pollution abatement. For simplicity, the space required for housing is equal to the population density at each location $f(x)$. We also consider no population growth. In this economy, there exists a unique consumption good, which only requires land as a production input. We denote the production by $F(l)$. Production generates pollution. However, the remainder of the land can be used as pollution abatement with a technology $G(1-l-f(x))$. The policy maker chooses the level of consumption *per capita* and the use of land at each location that maximize the discounted social welfare of the entire population. An important characteristic of this problem is that, even if land is immobile by nature, local actions affect the whole space because pollution flows across locations. We describe the spatial dynamics of pollution $p(x,t)$ by means of the Gaussian plume model presented in Section 2 (see footnote 7).

The policy maker maximizes the aggregate welfare in a bounded region of space R over a finite time period $[0,T]$, where $c(x,t)$ denotes the consumption *per capita* at location x and time t:

$$\max_{\{c,l\}} \int_0^T \int_R u(c(x,t))f(x)e^{-\rho t}dxdt + \int_R \psi(p)(x,T)e^{-\rho T}dx \qquad (15)$$

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subject to

$$\begin{cases} p_t(x,t) - p_{xx}(x,t) = P(p,x)A(x,t)F(l(x,t)) - G(1-l-f(x)), \\ \int_R c(x,t)f(x)dx = \int_R P(p,x)A(x,t)F(l)dx, \\ \lim_{x\to\delta R} p_x(x,t) = 0, \\ p(x,0) = p_0(x) \geq 0. \end{cases} \tag{16}$$

Following Camacho and Pérez-Barahona (2015), we introduce two discount factors in the objective function (15). The term $e^{-\rho t}$ is the usual temporal discount with $\rho > 0$. The spatial discount can represent, as observed in Section 2, the weight that the policy maker gives to each location. In order to avoid any subjective spatial preferences this can be identified, for instance, as the population density $f(x)$. Differing from the application considered in the previous section, the scrap value represents in this example the policy maker's concern about the pollution left at the end of the planning horizon. Since pollution is assumed to reduce social welfare, the scrap function will be considered as negative.

In this economy, pollution damages production. The units of final good produced at location x at time t are $P(p,x)A(x,t)F(l)$, where $A(x,t)$ denotes the corresponding total factor productivity. $P(p,x)$ is a decreasing function in pollution, where $1-P$ represents the share of foregone production due to pollution. The first equation in (16) is the Gaussian plume, which is a PDE. The term $PAF(l) - G$ of this equation represents the instantaneous net emissions of each location (i.e., $E(x,t)$ in footnote 7). As in Camacho and Pérez-Barahona (2015), the second constraint in (16) allows for the possibility of spatial reallocation of production. Hence, the policy maker collects all the production and reallocates it across locations at no cost. The limit constraint of the problem is the usual boundary condition. Finally, $p(0,x)$ represents the initial spatial distribution of pollution.[22]

The Pontryagin conditions of the problem are

$$\begin{cases} p_t(x,t) - p_{xx}(x,t) = P(p,x)A(x,t)F(l(x,t)) - G(1-l-f(x)), \\ q_t(x,t) + q_{xx}(x,t) + P'(p,x)A(x,t)F(l)\left[u'(c(x,t)) + q(x,t)\right] + \rho q = 0, \\ \left[u'\left(c(x,t)\right) + q(x,t)\right]P(p,x)A(x,t)F'(l) + q(x,t)G'(1-l-f(x)) = 0, \\ \int_R c(x,t)f(x)dx = \int_R P(p,x)A(x,t)F(l)dx, \\ p(x,0) = p_0(x) \geq 0, \\ \lim_{x\to\delta R}\ p_x(x,t) = 0, \quad \lim_{x\to\delta R}\ p(x,t)q_x(x,t) = 0, \\ q(x,T) = \psi'(x,T), \end{cases} \tag{17}$$

for $(x,t) \in R \times [0,T]$.

As in Boucekkine et al. (2009), one should observe that the boundary condition for the shadow price $q(x,t)$ becomes the standard $\lim_{x\to\delta R} q_x(x,t) = 0$ if we focus

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on interior solutions. Moreover, the last expression in (17) is the terminal condition of the problem. It states that, at the end of the planning horizon, the shadow price of pollution is equal to the marginal concern of the policy maker about the pollution left behind.

Camacho and Pérez-Barahona (2015) show that, due to the possibility of "imports/exports" of consumption goods (represented in this stylized framework by the consumption reallocation), consumption is spatially homogenous. It is then possible to prove that $c(x,t) = c(t) = \int_R P(p,x)A(x,t)F(l)dx / \int_R f(x)dx$. Replacing this expression in the equation describing the trade-off between consumption and pollution (third equation in (17)), one can prove that the optimal land $l(x,t)$ is a unique function of $p(x,t)$ and $q(x,t)$. Consequently, the spatial-dynamic behavior of this economy is described by a set of equations that correspond to our general system (I) in Section 3. Notice though that our general formulation assumes that both state and co-state variables are positive. Therefore, we do the change of variable $\tilde{q}(x,t) \equiv -q(x,t)$ since the shadow price of pollution is negative.

Let us use Theorems 1 and 2 to study the existence of a unique solution to this economy. One can verify that the assumptions of both theorems hold for the functional forms considered in Camacho and Pérez-Barahona (2015). We should also observe that f_1 and f_2 are decreasing functions in both arguments p and $\tilde{q}$. Then, we have to set $\underline{a}_2 \leq 0$ and $\bar{c}_2 \geq 0$. The last condition that we have to ensure is the existence of an upper and a lower solution to the problem. Camacho and Pérez-Barahona (2015) adapt Pao's stationary method (Theorem 3.4 in Pao (1992)) to study the existence of a unique time-invariant solution. In order to apply their result, they identify time-invariant upper and lower solutions. These stationary solutions are particular cases of upper and lower solutions, which can be applied to the case of local pollutants as well. Hence, by applying directly our two theorems, we can conclude that the problem has a unique solution:

PROPOSITION 2. *The land use dynamic problem (15)–(16) is well-posed.*

5. CONCLUDING REMARKS

We have proposed an analysis to study the optimal solution of dynamic models based on the spatial Ramsey framework. As an application, we consider the existence of unique optimal solution to the spatial Ramsey model. The iterative nature of our approach allows us as well to include a numerical check. We specifically use this algorithm to explore the role played by capital mobility in a neoclassical growth environment. Under capital irreversibility and economic openness, space allows for transitional dynamics. In this context, capital mobility is also beneficial in terms of social welfare and inequality. We also apply our method to an extension of the benchmark model for optimal land-use planning, introduced by Camacho and Pérez-Barahona (2015).

As in the existing literature on optimal control with highly non-linear constraints in partial derivatives, we focused on finite decision horizons. Considering

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a finite time span is necessary to reverse time, which is an important step in our method in order to combine parabolic and non-parabolic PDEs. Even if the finite horizon can be as large as one wishes, the problem of infinite horizon is central to study the asymptotic behavior of the economy. Nevertheless, this problem has been handled by the current dynamic spatial literature considering very restrictive assumptions. For instance, Boucekkine et al. (2013) and Boucekkine et al. (2018) have concentrated on the case of linear production function, while Brock and Xepapadeas (2008) propose a linear-quadratic approximation of the original problem. Indeed, Boucekkine et al. (2013) observe in this regard that dealing with an infinite time horizon is still an open question even in the mathematical literature. Compactization techniques or alternative analyses of non-parabolic PDE might be possible directions to explore in order to extend our method to the problem of infinite decision horizon.

We have presented our approach in a fairly general manner, taking the spatial Ramsey model and its optimal land-use extension as examples. However, in Camacho and Pérez-Barahona (2015), the latter also allows for the effect of pollution as a whole. This is represented by an integral across locations. Including spatial integral constraints to the optimal control problem (1)–(2) could be an extension of our method to cope with global pollutants such as anthropogenic greenhouse gases (GHGs). Another question to investigate, not studied enough in the literature, is the decentralization of the social optimum in the spatial Ramsey framework. This would introduce the possibility of implementing public policy schemes that take into account the geographical characteristics of the economy (for some recent studies in this direction, see Brock and Xepapadeas (2017) and La Torre et al. (2019).[23] In line with papers considering the interaction between capital mobility and tax coordination (e.g., Bental (1985) and Rodrik and van Ypersele (2001), and Baldwin and Krugman (2004)), our approach would be also useful to study the optimal degree of capital mobility in the economy. Indeed, note that our setup can be adapted to allow for a heterogeneous diffusion coefficient. This would allow to include frontiers between countries if thought as a step function, or to take into account various spatially continuous diffusion rates depending on regional, national, or supranational investment conditions and capital mobility laws. More challenging would be the case of a targeted or controlled diffusion, which we also include it in our research agenda.

NOTES

1. Many economists consider that models with indeterminacy are inappropriate for economic dynamic analysis (among others, Kehoe et al. (1990) and Acemoglu (2009)). It is desirable that there is a unique optimal trajectory for each initial condition (notice that this feature does not rule out the possibility of multiple long-run equilibria). This is typically the case of saddle-path stability.

2. Other papers follow "pragmatic" ways to deal with this problem. Brito (2004) or Brock and Xepapadeas (2008) focus on particular solutions (spatially periodic or time-constant solutions) to the spatial Ramsey model. Assuming linearity has also allowed to bypass ill-posedness. For instance, Boucekkine et al. (2009) and La Torre et al. (2015) use a linear utility function, while Boucekkine et al.

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(2013) and La Torre et al. (2019) opt for a linear technology. Unfortunately, these sorts of linearity are in general strong assumptions for macroeconomics and economic growth analysis.

3. Boucekkine et al. (2013, p. 338) claim that the case of finite-horizon is "manageable from the optimal control point of view."

4. Notice that Santambrogio et al. (forthcoming) study ill-posedness in a version of the spatial Ramsey model without diffusion. In their framework, capital does not move across locations, but it creates a spatial externality that can be interpreted as a knowledge spillover. The effect of this externality in a location is summarized by means of a spatial aggregator (a kernel function). Moreover, they focus on the decentralized solution considering an economy with "local" social planners.

5. For an extensive description of the economic assumptions behind the spatial Ramsey model see, for instance, Desmet and Rossi-Hansberg (2010) and Boucekkine et al. (2013).

6. Notice that both time and space discounting can be dropped in our setup since the time horizon and space are assumed to be bounded. Section 5 provides further discussion about the implications behind the decision horizon.

7. Camacho and Pérez-Barahona (2015) provide another example of this type of law of motion in an optimal land-use framework, where the dynamics of pollution in each location $p(x,t)$ follows $p_t(x,t) - p_{xx}(x,t) = E(x,t)$. The variable $E(x,t)$ summarizes the emissions in time t of a single source located at x. The term $-p_{xx}(x,t)$ above states that a pollutant diffuses from regions where its concentration is high to regions with a lower concentration (Fick's law of diffusion). This PDE is, indeed, the well-known model in physics called the Gaussian plume, which is frequently used to describe the spread of pollution across locations.

8. In spatial economic analysis, it is common to assume a bounded region Ω (e.g., Desmet and Rossi-Hansberg (2010, 2014)). If we consider the social optimum of the world economy, one can model continuous space as an sphere or, if $\Omega \subset \mathbb{R}$, as a circle. Then, as in Brock and Xepapadeas (2008) or Boucekkine et al. (2013), one can relax the Neumann boundary condition by setting $k(0,t) = k(2\pi,t)$. Still, for the case of unbounded space the corresponding Neumann boundary condition becomes $\lim_{x\to\pm\infty} u_{1,x}(x,t) = 0$ (La Torre et al. (2019), provide an example of unbounded space).

9. La Torre et al. (2019) also outline a spatial Ramsey model. However, they do not study ill-posedness.

10. See Appendix A for details.

11. We say that a function m is larger than another function m' in $\Omega \times [0,T]$ if and only if $m(x,t) \geq m'(x,t)$ for all $(x,t) \in \Omega \times [0,T]$. We denote this property as $m \geq m'$.

12. Our results can be easily extended to functions f_i that depend on x and t as well, that is, $f_i(x,t,u_1,u_2)$. In this case, f_i would be Hölder continuous in $\Omega \times [0,T] \times J_1 \times J_2$.

13. In Appendix D, we also apply our method to the spatial AK model, proving existence and uniqueness of solution.

14. Spatial economics usually considers space as a physical characteristic of the problem. However, it is possible to think about alternative interpretations such as the idea of "economic distance" suggested by Patel (1964) or Quah (1996a).

15. See Appendix E for the computational details.

16. Notice that these constants can be conveniently set to satisfy the conditions $\tilde{k}(x,0) < k(x,0) < \hat{k}(x,0)$ and $\tilde{\lambda}(x,T) < \Psi_1'(k(x,T),x) < \hat{\lambda}(x,T)$.

17. $\partial f_1/\partial\lambda > 0$ because $\lambda(x,t) = U'(c(x,t)) > 0$. For $\partial f_1/\partial k > 0$, the scale parameter A should be high enough, that is, $A > \delta(\varphi e^{\phi T})^{1-\alpha}/\alpha$.

18. We did a number of robustness checks. We repeated simulations for $\alpha = 1/3$, which is a value frequently assumed for a restricted definition of capital; alternative marginal scrap values $\chi = \{0.001, 0.01, 0.5, 1\}$ were considered as well as different time horizons too, considering $T = \{10, 20, 30, 40, 50, 60\}$. All these inspections gave similar qualitative results. We additionally verified that the simulated trajectories remain identical for different lower solutions, with $\epsilon = \{0.1, 0.5, 1.5, 2\}$.

19. For the sake of presentation, we have shortened the time axis of the plot for the shadow price in both scenarios.

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20. We repeated simulations with $\alpha = 0.75$, getting similar qualitative results.

21. The dynamics of capital and its shadow price are almost identical to the previous scenario in Figure 1.

22. Notice that our example focuses on local pollutants: the production in (x, t) is negatively affected by $p(x, t)$. Camacho and Pérez-Barahona (2015) provide a more general framework, allowing for both local and global ($\int_R p(x, t)dx$) effects of pollution. The later would add an integral equation to the general problem (I).

23. Notice that Brock and Xepapadeas (2017) concentrate on the heat transport between two regions, where the dynamics of the temperature is not described by means of PDE.

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APPENDICES

APPENDIX A: PONTRYAGIN OPTIMAL CONDITIONS

We define the value function associated with our problem as

$$V(c,u_1,u_2)\equiv \int_0^T \int_\Omega U\left(c(x,t)\right) g(x,t)dxdt + \int_\Omega \Psi(u_1)(x,T)dx \\ - \int_0^T \int_\Omega u_2(x,t)\left[u_{1,t}(x,t) - u_{1,xx}(x,t) - f_1(u_1,c)\right]dxdt, \tag{A1}$$

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where u_2 is an auxiliary function. V is a function of c, u_1, and u_2. If there exists an optimal solution (c^*, u_1^*, u_2^*), then any other solution to the problem (1)–(2) can be written as a deviation from the optimal solution as

$$\begin{aligned} c(x,t) &= c^*(x,t) + \epsilon\kappa(x,t), \\ u_1(x,t) &= u_1^*(x,t) + \epsilon v_1(x,t), \\ u_2(x,t) &= u_2^*(x,t) + \epsilon v_2(x,t), \end{aligned} \tag{A2}$$

for $\epsilon \in \mathbb{R}$. We can take the first-order derivative of the value function V with respect to ϵ in order to minimize the deviation of the trajectory from the optimal. Beforehand and using integration by parts, we rearrange some integral terms in V:

$$\begin{aligned} \int_0^T \int_\Omega u_2(x,t)u_{1,xx}(x,t)dxdt = \int_0^T u_2(x,t)u_{1,x}(x,t)|_{\delta\Omega}dt - \int_0^T u_{2,x}(x,t)u_1(x,t)|_{\delta\Omega}dt \\ + \int_0^T \int_\Omega u_{2,xx}(x,t)u_1(x,t)dxdt, \end{aligned} \tag{A3}$$

and as usual

$$\begin{aligned} \int_0^T \int_\Omega u_2(x,t)u_{1,t}(x,t)dxdt &= \int_\Omega u_1(x,t)u_2(x,t)|_0^T dx - \int_0^T \int_\Omega u_1(x,t)u_{2,t}(x,t)dxdt \\ &= \int_\Omega u_1(x,T)u_2(x,T)dx - \int_\Omega u_1(x,0)u_2(x,0)dx \\ &\quad - \int_0^T \int_\Omega u_2(x,t)u_{2,t}(x,t)dxdt. \end{aligned} \tag{A4}$$

We then obtain

$$\begin{aligned} \frac{\partial V(c,u_1,u_2)}{\partial \epsilon} &= \int_0^T \int_\Omega U'(c(x,t))\kappa(x,t)g(x,t)dxdt + \int_\Omega \Psi_1'(u_1(x,T))v_1(x,T)dx \\ &\quad + \int_0^T \int_\Omega v_1(x,t)\left[u_{2,t}(x,t) + u_{2,xx}(x,t)\right]dxdt \\ &\quad - \int_\Omega u_2(x,T)v_1(x,T)dx - \int_0^T v_1(x,t)u_{2,x}(x,t)|_{\delta\Omega}dt \\ &\quad + \int_0^T \int_\Omega u_2(x,t)\left(\frac{\partial f_1}{\partial c}\kappa(x,t) + \frac{\partial f_1}{\partial u_1}v_1\right)dxdt. \end{aligned}$$

In order to get the necessary conditions, we can group the elements multiplying κ and v_1, and equate them to zero. If all factors multiplying deviations from optimal values for c and u_1 are equal to zero, then $\frac{\partial V}{\partial \epsilon} = 0$. We would hence need

$$\begin{cases} \kappa: \ U'(c)g(x,t) + u_2\dfrac{\partial f_1}{\partial c} = 0, \\ v_1: u_{2,t} + u_{2,xx} + \dfrac{\partial f_1}{\partial u_1}u_2 = 0. \end{cases} \tag{A5}$$

The first equation in (A5) shows that c is an implicit function of u_1 and u_2. Moreover, we additionally require the following spatial boundary and transversality conditions:

$$\begin{cases} \lim_{x\to\delta\Omega} \ u_{2,x}(x,t) = 0, \\ u_2(x,T) = \Psi_1'(u_1(x,T)). \end{cases}$$

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We then obtain the Pontryagin conditions (O) by defining $f_2(u_1, u_2) \equiv -u_2 \frac{\partial f_1}{\partial u_1}$. Finally, taking into account that c can be expressed as an implicit function of u_1 and u_2, and with an abuse of notation, we have denoted $f_1(u_1, u_2)$ when it should have been $f_1(u_1, c(u_1, u_2))$. We can rewrite the set of necessary conditions as

$$\begin{cases} u_{1,t}(x,t) - u_{1,xx}(x,t) = f_1(u_1, u_2), \\ u_{2,t}(x,t) + u_{2,xx}(x,t) = f_2(u_1, u_2), \\ \lim_{x \to \delta\Omega} u_{1,x}(x,t) = 0, \\ \lim_{x \to \delta\Omega} u_{2,x}(x,t) = 0, \\ u_1(x,0) = u_{1,0}(x), \text{ given}, \\ u_2(x,T) = \Psi_1'(u_1(x,T)). \end{cases}$$

For the sake of exposition, we will use this alternative formulation of (O) in Section 3.

APPENDIX B: THEOREM 1 PROOF

The final objective of this proof is to demonstrate the contraction property of $\mathbb{L}^{-1}F$. Then, with this result in hand, we can directly conclude that the sequence

$$\begin{pmatrix} u_1^k \\ u_2^k \end{pmatrix} = \mathbb{L}^{-1} F(u^{k-1})$$

converges to the unique solution $(u_1, u_2)'$. We will require some intermediate results in order to prove the contraction property. Let us first summarize our method to build the sequences of the theorem. It mainly consists of four steps, which will be developed in detail in the subsequent Proposition B.1:

Step 1. We build a sequence of functions $\{\bar{u}_1^k\}_{k \in \mathbb{N}}$ starting from $(\bar{u}_1^0, \bar{u}_2^0) = (\hat{u}_1, \hat{u}_2)$. $\bar{u}_1^1$ is the solution to

$$\begin{cases} \mathbb{L}_1 \bar{u}_1^1 = F_1(\bar{u}_1^0, \bar{u}_2^0) = F_1(\hat{u}_1, \hat{u}_2), \ \forall (x,t) \in \Omega \times [0,T], \\ \lim_{x \to \delta\Omega} \bar{u}_{1,x}^1(x,t) = 0, \ \forall t \geq 0, \end{cases} \tag{B1}$$

with a suitable initial condition for $\bar{u}_1^1$ satisfying that $u_1(x,0) \leq \bar{u}_1^1(x,0) \leq \hat{u}_1(x,0)$. Equation (B1) together with the initial condition for $\bar{u}_1^1$ set up a linear parabolic PDE. Given the structure of the left-hand side of the PDE in (B1), and since $F_1(\hat{u}_1, \hat{u}_2)$ is a known bounded function of time and space, there exists a unique solution to (B1) from $\bar{u}_1^1(x,0)$ and subject to the boundary condition, as proven in Theorem 1.7 in Friedman (1959). Furthermore, this solution verifies that

$$\int_0^T \int_\Omega | \bar{u}_1^1(x,t) | \, e^{-\varrho |x|^2} dx dt < \infty,$$

for some positive constant ϱ.

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Step 2. We initiate the sequence $\{\bar{w}_2^k\}_{k\in\mathbb{N}}$ as $\bar{w}_2^0(x,t)=\hat{u}_2(x,T-t)$. Using $\bar{u}_1^0$ as $\bar{w}_1^0(x,t)=\hat{u}_1^1(x,T-t)$, $\bar{w}_2^1$ is the solution to

$$\begin{cases}\mathbb{L}_2'\bar{w}_2^1=-F_2(\bar{w}_1^0,\bar{w}_2^0),\\ \lim_{x\to\delta\Omega}\bar{w}_2^1(x,t)=0,\end{cases}\tag{B2}$$

and a suitable initial condition for $\bar{w}_2^1$ satisfying $w_2(x,0)\le\bar{w}_2^1(x,0)\le\hat{w}_2(x,0)$. As in Step 1, (B2) is a linear parabolic PDE, which has a unique positive solution under the boundedness assumption on Ψ and Assumption 1. Furthermore, the solution is bounded as before:

$$\int_0^T\int_\Omega|\,\bar{w}_2^1(x,t)\,|\,e^{-\varrho|x|^2}dxdt<\infty,$$

for some positive constant ϱ. We proceed to the next iteration by undoing the change of variable $\bar{u}_2^1(x,t)=\bar{w}_2^1(x,T-t)$. Likewise, building on $(\bar{u}_1^{k-1},\bar{u}_2^{k-1})$, we compute $(\bar{u}_1^k,\bar{u}_2^k)$ following the procedure just described for $k=2,3,\ldots$ Moreover, for the successive iterations, we also need to state appropriate initial conditions $\bar{u}_1^k(x,0)$ and $\bar{w}_2^k(x,0)$ in order to ensure that the initial and terminal conditions in (I) hold as $k\to\infty$ (notice that the terminal condition for u_2 is equivalent to the initial condition for w_2 after reversing time).

Step 3. In a similar manner, we can build a sequence $\{\underline{u}_1^k,\underline{u}_2^k\}_{k\in\mathbb{N}}$ and $\{\underline{w}_1^k,\underline{w}_2^k\}_{k\in\mathbb{N}}$ considering the lower solution.

Step 4. We prove that $\{\bar{u}_1^k,\bar{w}_2^k\}_{k\in\mathbb{N}}$ is monotonic non-increasing and that $\{\underline{u}_1^k,\underline{w}_2^k\}_{k\in\mathbb{N}}$ is monotonic non-decreasing. Undoing the change of time variable, we obtain the convergence of sequences $\{\bar{u}_1^k,\bar{u}_2^k\}_{k\in\mathbb{N}}$ and $\{\underline{u}_1^k,\underline{u}_2^k\}_{k\in\mathbb{N}}$ in $\Omega\times[0,T]$. As it will become clear in the proposition, the monotonicity and convergence of the generated sequence hinge on the choice of the initial conditions at each iteration. This choice will depend on the monotonicity characteristics of functions F_i.

We formally enunciate this iterative method as a proposition. This proposition follows the spirit of Lemmas 8.2.1 and 8.2.2 in Pao (1992), which shows the same result for pure parabolic systems of PDEs. The proposition also underlines the monotonicity property of the sequences generated, which will be useful for the upcoming results:

PROPOSITION B.1. *Let us consider the assumptions of the theorem. Let us introduce variables w as the time-reverse of u, that is,* $w_i(x,t)\equiv u_i(x,T-t)$, $\hat{w}_i(x,t)\equiv\hat{u}_i(x,T-t)$, *and* $\tilde{w}_i(x,t)\equiv\tilde{u}_i(x,T-t)$ *for* $i=1,2$. *We build sequences* $\{\bar{u}_1^k,\bar{w}_2^k\}_{k\in\mathbb{N}}$ *and* $\{\underline{u}_1^k,\underline{w}_2^k\}_{k\in\mathbb{N}}$ *following the process*

$$\begin{cases}\mathbb{L}_1\bar{u}_1^k=F_1(\bar{u}_1^{k-1},\bar{u}_2^{k-1}),\\ \bar{u}_1^k(x,0)=\frac{\Upsilon}{k}\hat{u}_1(x,0)+\left(1-\frac{\Upsilon}{k}\right)u_1(x,0),\\ \lim_{x\to\delta\Omega}\bar{u}_{1,x}^k=0,\end{cases}$$

and

$$\begin{cases}\mathbb{L}_1\underline{u}_1^k=F_1(\underline{u}_1^{k-1},\underline{u}_2^{k-1}),\\ \underline{u}_1^k(x,0)=\frac{\varsigma}{k}\tilde{u}_1(x,0)+\left(1-\frac{\varsigma}{k}\right)u_1(x,0),\\ \lim_{x\to\delta\Omega}\underline{u}_{1,x}^k=0,\end{cases}$$

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with $0 < \zeta < \Upsilon < 1$,

$$\begin{cases} \mathbb{L}_2' \bar{w}_2^k = -F_2\left(\bar{w}_1^{k-1}, \bar{w}_2^{k-1}\right), \\ \bar{w}_2^k(x,0) = \frac{\psi}{k}\hat{w}_2(x,0) + \left(1 - \frac{\psi}{k}\right)\Psi_1'\left(\bar{w}_1^{k-1}(x,0), x\right), \\ \lim_{x\to\delta\Omega} \bar{w}_{2,x}^k = 0, \end{cases}$$

and

$$\begin{cases} \mathbb{L}_2' \underline{w}_2^k = -F_2\left(\underline{w}_1^{k-1}, \underline{w}_2^{k-1}\right), \\ \underline{w}_2^k(x,0) = \frac{\phi}{k}\tilde{w}_2(x,0) + \left(1 - \frac{\phi}{k}\right)\Psi_1'(\underline{w}_1^{k-1}(x,0), x), \\ \lim_{x\to\delta\Omega} \underline{w}_{2,x}^k = 0, \end{cases}$$

with $0 < \phi < \psi < 1$. *We initialize the sequences as*

$$\begin{aligned} \left(\bar{u}_1^0, \bar{u}_2^0\right) &= \left(\hat{u}_1, \hat{u}_2\right), \\ \left(\bar{w}_1^0, \bar{w}_2^0\right) &= \left(\hat{w}_1, \hat{w}_2\right), \\ \left(\underline{u}_1^0, \underline{u}_2^0\right) &= \left(\tilde{u}_1, \tilde{u}_2\right), \\ \left(\underline{w}_1^0, \underline{w}_2^0\right) &= \left(\tilde{w}_1, \tilde{w}_2\right). \end{aligned}$$

Then, the two sequences $\{\bar{u}_1^k, \bar{w}_2^k\}_{k\in\mathbb{N}}$ *and* $\{\underline{u}_1^k, \underline{w}_2^k\}_{k\in\mathbb{N}}$ *obtained above have the monotone property*

$$\begin{aligned} \underline{u}_1^{k-1} \le \underline{u}_1^k \le \bar{u}_1^k \le \bar{u}_1^{k-1}, \\ \underline{w}_2^{k-1} \le \underline{w}_2^k \le \bar{w}_2^k \le \bar{w}_2^{k-1}. \end{aligned}$$

Proof. For the ease of presentation, we divide the proof into five parts. It is important to keep in mind along the demonstration that

$$\begin{aligned} \tilde{u}_1(x,t) \le \hat{u}_1(x,t), \\ \tilde{w}_2(x,t) \le \hat{w}_2(x,t), \end{aligned}$$

and

$$\begin{aligned} \tilde{u}_1(x,0) < u_1(x,0) < \hat{u}_1(x,0), \\ \tilde{w}_2(x,0) < \Psi_1'(\tilde{u}_1(x,T), x), \\ \Psi_1'(\hat{u}_1(x,T), x) < \hat{w}_2(x,0). \end{aligned}$$

(*i*) Let us first construct $\{\bar{u}_1^k\}_{k\in\mathbb{N}}$ and $\{\bar{w}_2^k\}_{k\in\mathbb{N}}$. We initialize the sequences with

$$\begin{aligned} (\bar{u}_1^0, \bar{u}_2^0) = (\hat{u}_1, \hat{u}_2), \\ (\bar{w}_1^0, \bar{w}_2^0) = (\hat{w}_1, \hat{w}_2). \end{aligned}$$

As observed in the short description of our iterative method, the first element of the sequence $\{\bar{u}_1^k\}_{k\in\mathbb{N}}$ is the solution to

$$\begin{cases} \mathbb{L}_1 \bar{u}_1^1 = F_1(\bar{u}_1^0, \bar{u}_2^0) = F_1(\hat{u}_1, \hat{u}_2), \\ \bar{u}_1^1(x,0) = \Upsilon\hat{u}_1(x,0) + (1-\Upsilon)u_1(x,0), \\ \lim_{x\to\delta\Omega} \bar{u}_{1,x}^1 = 0. \end{cases} \quad \textbf{(B3)}$$

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The PDE in (B3) has a unique solution $\bar{u}_1^1$ since $F_1(\hat{u}_1, \hat{u}_2)$ is known, $\mathbb{L}_1$ is linear, and $\bar{u}_1^1(x,0)$ is strictly positive. Next, we show that $\bar{u}_1^0 \geq \bar{u}_1^1$. Let us construct in this respect the auxiliary variable v_1^0 on $\Omega \times [0,T]$:[24]

$$v_1^0 \equiv \bar{u}_1^0 - \bar{u}_1^1 = \hat{u}_1 - \bar{u}_1^1.$$

Then,

$$\begin{aligned}\mathbb{L}_1 v_1^0 &= \mathbb{L}_1(\bar{u}_1^0 - \bar{u}_1^1) = \hat{u}_{1,t} - \hat{u}_{1,xx} + \underline{a}_1(x,t)\hat{u}_1 - F_1(\hat{u}_1, \hat{u}_2) \\ &= \hat{u}_{1,t} - \hat{u}_{1,xx} + \underline{a}_1(x,t)\hat{u}_1 - f_1(\hat{u}_1, \hat{u}_2) - \underline{a}_1(x,t)\hat{u}_1 \geq 0.\end{aligned}$$

We know that the expression above is non-negative because $(\hat{u}_1, \hat{u}_2)$ is an upper solution of problem (I). Moreover,

$$\begin{aligned}v_1^0(x,0) &= \bar{u}_1^0(x,0) - \bar{u}_1^1(x,0) = \hat{u}_1(x,0) - \Upsilon\hat{u}_1(x,0) - (1-\Upsilon)u_1(x,0) \\ &= (1-\Upsilon)\left(\hat{u}_1(x,0) - u_1(x,0)\right) > 0.\end{aligned}$$

Then, invoking classical results on positiveness as in Lemma 2.2.1 in Pao (1992), we conclude that $v_1^0(x,t) \geq 0$. This implies that $\bar{u}_1^0(x,t) \geq \bar{u}_1^1(x,t)$ for all $(x,t) \in \Omega \times [0,T]$.

Similarly, let us consider $\bar{w}_2^1$ solution to

$$\begin{cases} \mathbb{L}_2'\bar{w}_2^1 = -F_2(\bar{w}_1^0, \bar{w}_2^0) = -F_2(\hat{w}_1, \hat{w}_2), \\ \bar{w}_2^1(x,0) = \psi\hat{w}_2(x,0) + (1-\psi)\Psi_1'(\bar{w}_1^0(x,0), x), \\ \lim_{x\to\delta\Omega} \bar{w}_{2,x}^1 = 0, \end{cases}$$

and construct $v_2^0 \equiv \bar{w}_2^0 - \bar{w}_2^1 = \hat{w}_2 - \bar{w}_2^1$, where $\hat{w}_2(x,t) = \hat{u}_2(x, T-t)$ is the time reverse of the second coordinate of the upper solution $(\hat{u}_1, \hat{u}_2)$. As before, we can show for v_2^0 that

$$\begin{aligned}\mathbb{L}_2' v_2^0 &= \mathbb{L}_2'(\hat{w}_2 - \bar{w}_2^1) = \hat{w}_{2,t} - \hat{w}_{2,xx} + \bar{c}_2(x,t)\hat{w}_2 - F_2(\hat{w}_1, \hat{w}_2) \\ &= \hat{w}_{2,t} - \hat{w}_{2,xx} + \bar{c}_2(x,t)\hat{w}_2 + f_2(\hat{w}_1, \hat{w}_2) - \bar{c}_2(x,t)\hat{w}_2 \geq 0,\end{aligned}$$

because of (7). We need to check the sign of $v_2^0(x,0)$. Using the definition of the initial conditions, we conclude that

$$\begin{aligned}v_2^0(x,0) &= \hat{w}_2(x,0) - \bar{w}_2^1(x,0) = \hat{w}_2(x,0) - \psi\hat{w}_2(x,0) - (1-\psi)\Psi_1'(\bar{w}_1^0(x,0), x) \\ &= (1-\psi)\left(\hat{w}_2(x,0) - \Psi_1'(\hat{w}_1(x,0), x)\right) > 0.\end{aligned}$$

Then, $v_2^0(x,t) \geq 0$ and, consequently, $\hat{w}_2(x,t) \geq \bar{w}_2^1(x,t)$ for every $(x,t) \in \Omega \times [0,T]$.

(*ii*) We apply the same procedure to the other sequences, starting instead with the lower solutions:

$$(\underline{u}_1^0, \underline{u}_2^0) = (\tilde{u}_1, \tilde{u}_2),$$

$$(\underline{w}_1^0, \underline{w}_2^0) = (\tilde{w}_1, \tilde{w}_2).$$

$\underline{u}_1^1$ is the solution to

$$\begin{cases} \mathbb{L}_1\underline{u}_1^1 = F_1(\underline{u}_1^0, \underline{u}_2^0) = F_1(\tilde{u}_1, \tilde{u}_2), \\ \underline{u}_1^1(x,0) = \zeta\tilde{u}_1(x,0) + (1-\zeta)u_1(x,0), \\ \lim_{x\to\delta\Omega} \underline{u}_{1,x}^1 = 0. \end{cases}$$

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Then, defining v_1^0 and v_2^0 as

$$v_1^0 \equiv \underline{u}_1^1 - \underline{u}_1^0 = \underline{u}_1^1 - \tilde{u}_1,$$

$$v_2^0 \equiv \underline{w}_2^1 - \underline{w}_2^0 = \underline{w}_2^1 - \tilde{w}_2,$$

we will prove that $\underline{u}_1^1 \geq \underline{u}_1^0$ and $\underline{w}_2^1 \geq \underline{w}_2^0$. Let us consider $v_1^0 = \underline{u}_1^1 - \underline{u}_1^0 = \underline{u}_1^1 - \tilde{u}_1$.

Applying the operator $\mathbb{L}_1$,

$$\mathbb{L}_1 v_1^0 = \mathbb{L}_1(\underline{u}_1^1 - \tilde{u}_1) = F_1(\underline{u}_1^0, \underline{u}_2^0) - \left(\tilde{u}_{1,t} - \tilde{u}_{1,xx} + \bar{c}_1 \tilde{u}_1\right)$$

$$= f_1(\tilde{u}_1, \tilde{u}_2) + \bar{c}_1 \tilde{u}_1 - \tilde{u}_{1,t} + \tilde{u}_{1,xx} - \bar{c}_1 \tilde{u}_1 \geq 0$$

because $(\tilde{u}_1, \tilde{u}_2)$ is a lower solution. Regarding the initial condition,

$$v_1^0(x,0) = \underline{u}_1^1(x,0) - \underline{u}_1^0(x,0) = \underline{u}_1^1(x,0) - \tilde{u}_1(x,0)$$

$$= \zeta \tilde{u}_1(x,0) + (1-\zeta)u_1(x,0) - \tilde{u}_1(x,0) = (1-\zeta)(u_1(x,0) - \tilde{u}_1(x,0)) > 0.$$

Then, we conclude that $\underline{u}_1^1 \geq \underline{u}_1^0$.

Similarly, let us prove that $\underline{w}_2^0 \leq \underline{w}_2^1$. Taking the definition of v_2^0 we can show that

$$\mathbb{L}_2' v_2^0 = \mathbb{L}_2'(\underline{w}_2^1 - \underline{w}_2^0) = -F_2(\underline{w}_1^0, \underline{w}_2^0) - \left(\underline{w}_{2,t} - \underline{w}_{2,xx} + \bar{c}_2 \underline{w}_2\right)$$

$$= -f_2(\tilde{w}_1, \tilde{w}_2) + \bar{c}_2 \tilde{w}_2 - \left(\tilde{w}_{2,t} - \tilde{w}_{2,xx} + \bar{c}_2 \tilde{w}_2\right) \geq 0.$$

Moreover,

$$v_2^0(x,0) = \underline{w}_2^1(x,0) - \underline{w}_2^0(x,0)$$

$$= \underline{w}_2^1(x,0) - \tilde{w}_2(x,0) = \phi \tilde{w}_2(x,0) + (1-\phi)\Psi_1'(\underline{w}_1^0(x,0), x) - \tilde{w}_2(x,0)$$

$$= (1-\phi)\left(\Psi_1'(\tilde{w}_1(x,0), x) - \tilde{w}_2(x,0)\right) > 0.$$

We therefore confirm our statement.

(*iii*) We prove next that $\underline{u}_1^1 \leq \bar{u}_1^1$ and $\underline{w}_2^1 \leq \bar{w}_2^1$. We define the auxiliary variables v_1^1 and v_2^1 on $\Omega \times [0,T]$ as

$$v_1^1 \equiv \bar{u}_1^1 - \underline{u}_1^1,$$

$$v_2^1 \equiv \bar{w}_2^1 - \underline{w}_2^1.$$

Let us show first that $\bar{u}_1^1 \leq \underline{u}_1^1$. Applying the operator $\mathbb{L}_1$,

$$\mathbb{L}_1 v_1^1 = \mathbb{L}_1(\bar{u}_1^1 - \underline{u}_1^1) = F_1(\hat{u}_1, \hat{u}_2) - F_1(\tilde{u}_1, \tilde{u}_2)$$

$$= f_1(\hat{u}_1, \hat{u}_2) - f_1(\tilde{u}_1, \tilde{u}_2) + \underline{a}_1(\hat{u}_1 - \tilde{u}_1).$$

Since $\hat{u}_1 \geq \tilde{u}_1$, we can see that $\mathbb{L}_1 v_1^1 \geq 0$ if $\underline{a}_1 \geq 0$ and f_1 increasing in both arguments u_1 and u_2, or if alternatively $\underline{a}_2 \leq 0$. Notice that in the later case we would have

$$\mathbb{L}_1 v_1^1 = f_1(\hat{u}_1, \hat{u}_2) - f_1(\tilde{u}_1, \tilde{u}_2) + \underline{a}_1(\hat{u}_1 - \tilde{u}_1) \geq -\underline{a}_2(\hat{u}_2 - \tilde{u}_2) \geq 0.$$

Taking into account

$$v_1^1(x,0) = \bar{u}_1^1(x,0) - \underline{u}_1^1(x,0) = \Upsilon \hat{u}_1(x,0) + (1-\Upsilon)u_1(x,0) - (\zeta \tilde{u}_1(x,0) + (1-\zeta)u_1(x,0))$$

$$= \Upsilon \hat{u}_1(x,0) - \zeta \tilde{u}_1(x,0) - (\Upsilon - \zeta)u_1(x,0)$$

$$\geq \Upsilon \hat{u}_1(x,0) - \zeta \hat{u}_1(x,0) - (\Upsilon - \zeta)u_1(x,0) = (\Upsilon - \zeta)\left(\hat{u}_1(x,0) - u_1(x,0)\right) > 0,$$

we conclude that $v_1^1(x,t) \geq 0$, $\forall (x,t) \in \Omega \times [0,T]$. Hence, $\underline{u}_1^1 \leq \bar{u}_1^1$.

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(*iv*) Let us consider v_2^1. Applying $\mathbb{L}_2'$,

$$\begin{aligned}\mathbb{L}_2' v_2^1 &= \mathbb{L}_2'(\bar{w}_2^1 - \underline{w}_2^1) = -F_2(\hat{w}_1, \hat{w}_2) + F_2(\tilde{w}_1, \tilde{w}_2)\\ &= -f_2(\hat{w}_1, \hat{w}_2) + f_2(\tilde{w}_1, \tilde{w}_2) + \bar{c}_2(\hat{w}_2 - \tilde{w}_2).\end{aligned}$$

This can be positive if either f_2 decreasing in both w_1 and w_2 and $\bar{c}_2 \geq 0$, or if $\bar{c}_1 \leq 0$ whatever the sign of the partial derivatives of f_2 since

$$\mathbb{L}_2' v_2^1 \geq -\bar{c}_1(\hat{w}_1 - \tilde{w}_1) \geq 0.$$

Moreover, one can verify that

$$\begin{aligned}v_2^1(x,0) &= \bar{w}_2^1(x,0) - \underline{w}_2^1(x,0)\\ &= \psi \hat{w}_2(x,0) + (1-\psi)\Psi_1'(\bar{w}_1^0(x,0), x) - \phi \tilde{w}_2(x,0) - (1-\phi)\Psi_1'(\underline{w}_1^0(x,0), x)\\ &\geq (\psi - \phi)\left(\hat{w}_2(x,0) - \Psi_1'(\hat{w}_1(x,0), x)\right) > 0.\end{aligned}$$

We therefore conclude that $v_2^1 \geq 0$ and $\bar{w}_2^1 \geq \underline{w}_2^1$.

(*v*) We have proven so far that

$$\underline{u}_1^0 \leq \underline{u}_1^1 \leq \bar{u}_1^1 \leq \bar{u}_1^0$$

and

$$\underline{w}_2^0 \leq \underline{w}_2^1 \leq \bar{w}_2^1 \leq \bar{w}_2^0.$$

These two conclusions provide the base case that will allow us to show by mathematical induction that our sequences are ordered. Following this strategy, let us assume

$$\underline{u}_1^{k-1} \leq \underline{u}_1^k \leq \bar{u}_1^k \leq \bar{u}_1^{k-1}$$

and

$$\underline{w}_2^{k-1} \leq \underline{w}_2^k \leq \bar{w}_2^k \leq \bar{w}_2^{k-1}.$$

We have to show, as inductive step, that

$$\underline{u}_1^k \leq \underline{u}_1^{k+1} \leq \bar{u}_1^{k+1} \leq \bar{u}_1^k$$

and

$$\underline{w}_2^k \leq \underline{w}_2^{k+1} \leq \bar{w}_2^{k+1} \leq \bar{w}_2^k.$$

Let us begin by verifying that the inequality $\bar{u}_1^{k+1} \leq \bar{u}_1^k$ holds. We define in this regard the auxiliary variable v_1^k as

$$v_1^k \equiv \bar{u}_1^k - \bar{u}_1^{k+1}.$$

Applying the operator $\mathbb{L}_1$,

$$\begin{aligned}\mathbb{L}_1 v_1^k &= F_1\left(\bar{u}_1^{k-1}, \bar{u}_2^{k-1}\right) - F_1\left(\bar{u}_1^k, \bar{u}_2^k\right)\\ &= f_1\left(\bar{u}_1^{k-1}, \bar{u}_2^{k-1}\right) - f_1\left(\bar{u}_1^k, \bar{u}_2^k\right) + \underline{a}_1\left(\bar{u}^{k-1} - \bar{u}_1^k\right) \geq 0.\end{aligned}$$

Since $u_2^k(x,t) = w_2^k(x, T-t)$, it is also true that $\bar{u}_2^k(x,t) = \bar{w}_2^k(x, T-t)$ and $\underline{u}_2^k(x,t) = \underline{w}_2^k(x, T-t)$. Given that $\bar{w}_2^k(x,t) \leq \bar{w}_2^{k-1}(x,t)$ for all t, then $\bar{w}_2^k(x, T-t) \leq \bar{w}_2^{k-1}(x, T-t)$. This implies in particular that $\bar{u}_2^k(x,t) \leq \bar{u}_2^{k-1}(x,t)$. Since $\bar{u}_1^k \leq \bar{u}_1^{k-1}$ and

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$\bar{u}_2^k \leq \bar{u}_2^{k-1}$, we can see that $\mathbb{L}_1 v_1^k \geq 0$ if $\underline{a}_1 \geq 0$ and f_1 increasing in both arguments, or if alternatively $\underline{a}_1 \leq 0$. In the later case,

$$\mathbb{L}_1 v_1^k = f_1(\bar{u}_1^{k-1}, \bar{u}_2^{k-1}) - f_1(\bar{u}_1^k, \bar{u}_2^k) + \underline{a}_1 \left(\bar{u}^{k-1} - \bar{u}_1^k\right) \geq -\underline{a}_2 \left(\bar{u}_2^{k-1} - \bar{u}_2^k\right) \geq 0.$$

Moreover,

$$\begin{aligned} v_1^k(x,0) &= \bar{u}_1^k(x,0) - \bar{u}_1^{k+1}(x,0) \\ &= \frac{\Upsilon}{k}\hat{u}_1(x,0) + \left(1 - \frac{\Upsilon}{k}\right) u_1(x,0) - \frac{\Upsilon}{k+1}\hat{u}_1(x,0) - \left(1 - \frac{\Upsilon}{k+1}\right) u_1(x,0) \\ &= \Upsilon \hat{u}_1(x,0)\left(\frac{1}{k} - \frac{1}{k+1}\right) - \Upsilon u_1(x,0)\left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \Upsilon \left(\frac{1}{k} - \frac{1}{k+1}\right)\left(\hat{u}_1(x,0) - u_1(x,0)\right) > 0, \end{aligned}$$

since $\hat{u}_1(x,0) > u_1(x,0)$. Hence, $v_1^k \geq 0$ and $\bar{u}_1^k \geq \bar{u}_1^{k+1}$.

Let us similarly prove the inequality $\underline{u}_1^k \leq \underline{u}_1^{k+1}$. For this results, we define the auxiliary variable v_1^k as

$$v_1^k \equiv \underline{u}_1^{k+1} - \underline{u}_1^k.$$

Applying again $\mathbb{L}_1$,

$$\begin{aligned} \mathbb{L}_1 v_1^k &= F_1(\underline{u}_1^k, \underline{u}_2^k) - F_1(\underline{u}_1^{k-1}, \underline{u}_2^{k-1}) \\ &= f_1(\underline{u}_1^k, \underline{u}_2^k) - f_1(\underline{u}_1^{k-1}, \underline{u}_2^{k-1}) + \underline{a}_1(\underline{u}_1^k - \underline{u}_1^{k-1}) \geq 0. \end{aligned}$$

Since $\underline{u}_1^k \geq \underline{u}_1^{k-1}$ and $\underline{u}_2^k \geq \underline{u}_2^{k-1}$, we can see that $\mathbb{L}_1 v_1^1 \geq 0$ if $\underline{a}_1 \geq 0$ and f_1 increasing in both arguments, or if alternatively $\underline{a}_2 \leq 0$. In the later case, we would have

$$\mathbb{L}_1 v_1^k = f_1(\underline{u}_1^k, \underline{u}_2^k) - f_1(\underline{u}_1^{k-1}, \underline{u}_2^{k-1}) + \underline{a}_1(\underline{u}_1^k - \underline{u}_1^{k-1}) \geq -\underline{a}_2(\underline{u}_2^k - \underline{u}_2^{k-1}) \geq 0.$$

One can also verify that

$$\begin{aligned} v_1^k(x,0) &= \underline{u}_1^{k+1}(x,0) - \underline{u}_1^k(x,0) \\ &= \frac{\zeta}{k+1}\tilde{u}_1(x,0) + \left(1 - \frac{\zeta}{k+1}\right) u_1(x,0) - \frac{\zeta}{k}\tilde{u}_1(x,0) + \left(1 - \frac{\zeta}{k}\right) u_1(x,0) \\ &= \zeta \frac{1}{k(k+1)}\tilde{u}_1(x,0) + \zeta \frac{1}{k(k+1)} u_1(x,0) \\ &= \zeta \frac{1}{k(k+1)}\left(u_1(x,0) - \tilde{u}_1(x,0)\right) > 0. \end{aligned}$$

Therefore, $v_1^k \geq 0$ and $\underline{u}_1^k \leq \underline{u}_1^{k+1}$. Next, we need to prove that $\underline{u}_1^{k+1} \leq \bar{u}_1^{k+1}$. Let us define $v_1^{k+1} \equiv \bar{u}_1^{k+1} - \underline{u}_1^{k+1}$. Then,

$$\mathbb{L}_1 v_1^{k+1} = F_1(\bar{u}_1^k, \bar{u}_2^k) - F_1(\underline{u}_1^k, \underline{u}_2^k) = f_1(\bar{u}_1^k, \bar{u}_2^k) - f_1(\underline{u}_1^k, \underline{u}_2^k) + \underline{a}_1(\bar{u}_1^k - \underline{u}_1^k) \geq 0.$$

Since $\bar{u}_1^k \geq \underline{u}_1^k$ and $\bar{u}_2^k \geq \underline{u}_2^k$, we can see that $\mathbb{L}_1 v_1^{k+1} \geq 0$ if $\underline{a}_1 \geq 0$ and f_1 increasing in both arguments, or if alternatively $\underline{a}_2 \leq 0$. In the later case,

$$\mathbb{L}_1 v_1^{k+1} = f_1(\bar{u}_1^k, \bar{u}_2^k) - f_1(\underline{u}_1^k, \underline{u}_2^k) + \underline{a}_1(\bar{u}_1^k - \underline{u}_1^k) \geq -\underline{a}_2(\bar{u}_2^k - \underline{u}_2^k) \geq 0.$$

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We can verify that the initial distribution of v_1^{k+1} is positive:

$$\begin{aligned}
v_1^{k+1}(x,0) &= \bar{u}_1^{k+1}(x,0) - \underline{u}_1^{k+1}(x,0) \\
&= \frac{\Upsilon}{k+1}\hat{u}_1(x,0) + \left(1 - \frac{\Upsilon}{k+1}\right) u_1(x,0) - \frac{\zeta}{k+1}\tilde{u}_1(x,0) - \left(1 - \frac{\zeta}{k+1}\right) u_1(x,0) \\
&= \frac{\Upsilon}{k+1}\hat{u}_1(x,0) - \frac{\zeta}{k+1}\tilde{u}_1(x,0) - \frac{\Upsilon - \zeta}{k+1} u_1(x,0) \\
&\geq \frac{\Upsilon - \zeta}{k+1}\hat{u}_1(x,0) - \frac{\Upsilon - \zeta}{k+1} u_1(x,0) \\
&= \frac{\Upsilon - \zeta}{k+1}\left(\hat{u}_1(x,0) - u_1(x,0)\right) > 0,
\end{aligned}$$

Hence, $v_1^{k+1} \geq 0$ and $\underline{u}_1^{k+1} \leq \bar{u}_1^{k+1}$. Summarizing, we can conclude that

$$\underline{u}_1^k \leq \underline{u}_1^{k+1} \leq \bar{u}_1^{k+1} \leq \bar{u}_1^k.$$

Let us now consider the second set of inequalities. Defining $v_2^k \equiv \bar{w}_2^k - \bar{w}_2^{k+1}$, we apply the operator $\mathbb{L}_2'$:

$$\mathbb{L}_2' v_2^k = F_2(\bar{w}_1^k, \bar{w}_2^k) - F_2(\bar{w}_1^{k-1}, \bar{w}_2^{k-1}) = f_2(\bar{w}_1^k, \bar{w}_2^k) - f_2(\bar{w}_1^{k-1}, \bar{w}_2^{k-1}) + \bar{c}_2(\bar{w}_2^{k-1} - \bar{w}_2^k) \geq 0.$$

Since $\bar{w}_1^k \leq \bar{w}_1^{k-1}$ and $\bar{w}_2^k \leq \underline{w}_2^{k-1}$, we can see that $\mathbb{L}_2' v_2^k \geq 0$ if $\bar{c}_2 \geq 0$ and f_2 decreasing in both arguments, or if alternatively $\bar{c}_1 \leq 0$. In the later case,

$$\begin{aligned}
\mathbb{L}_2' v_2^k &= f_2(\bar{w}_1^k, \bar{w}_2^k) - f_2(\bar{w}_1^{k-1}, \bar{w}_2^{k-1}) + \bar{c}_2(\bar{w}_2^{k-1} - \bar{w}_2^k) \\
&\geq -\bar{c}_1(\bar{w}_1^{k-1} - \bar{w}_1^k) \geq 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
v_2^k(x,0) &= \bar{w}_2^k(x,0) - \bar{w}_2^{k+1}(x,0) \\
&= \psi\left(\frac{1}{k} - \frac{1}{k+1}\right)\hat{w}_2(x,0) + \left(1 - \frac{\psi}{k}\right)\Psi_1'(\bar{w}_1^{k-1}(x,0), x) \\
&\quad - \left(1 - \frac{\psi}{k+1}\right)\Psi_1'(\bar{w}_1^k(x,0), x).
\end{aligned}$$

Since $\bar{w}_1^{k+1}(x,0) \leq \bar{w}_1^k(x,0)$, for a convex function Ψ

$$\begin{aligned}
w_2^k(x,0) &\geq \psi\left(\frac{1}{k} - \frac{1}{k+1}\right)\hat{w}_2(x,0) + \left(1 - \frac{\psi}{k}\right)\Psi_1'(\bar{w}_1^{k-1}(x,0), x) \\
&\quad - \left(1 - \frac{\psi}{k+1}\right)\Psi_1'(\bar{w}_1^{k-1}(x,0), x) \\
&= \psi\left(\frac{1}{k} - \frac{1}{k+1}\right)\left[\hat{w}_2(x,0) - \Psi_1'(\bar{w}_1^{k-1}(x,0), x)\right] \\
&> \psi\left(\frac{1}{k} - \frac{1}{k+1}\right)\left[\hat{w}_2(x,0) - \Psi_1'(\hat{w}_1(x,0), x)\right] > 0.
\end{aligned}$$

Hence, we conclude that $v_2^k(x,0) > 0$ and $\bar{w}_2^{k+1} \leq \bar{w}_2^k$.

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We prove next that $\underline{w}_2^k \le \underline{w}_2^{k+1}$. Let us define $v_2^k \equiv \underline{w}_2^{k+1} - \underline{w}_2^k$. Considering the operator $\mathbb{L}_2'$ we obtain

$$\begin{aligned}\mathbb{L}_2' v_2^k &= F_2(\underline{w}_1^{k-1}, \underline{w}_2^{k-1}) - F_2(\underline{w}_1^k, \underline{w}_2^k)\\ &= f_2(\underline{w}_1^{k-1}, \underline{w}_2^{k-1}) - f_2(\underline{w}_1^k, \underline{w}_2^k) + \bar{c}_2(\underline{w}_2^k - \underline{w}_2^{k-1}) \ge 0.\end{aligned}$$

Since $\underline{w}_1^{k-1} \le \underline{w}_1^k$ and $\underline{w}_2^{k-1} \le \underline{w}_2^k$, we can see that $\mathbb{L}_2' v_2^k \ge 0$ if $\bar{c}_2 \ge 0$ and f_2 decreasing in both arguments, or if alternatively $\bar{c}_1 \le 0$. In the later case,

$$\begin{aligned}\mathbb{L}_2' v_2^k &= f_2(\underline{w}_1^{k-1}, \underline{w}_2^{k-1}) - f_2(\underline{w}_1^k, \underline{w}_2^k) + \bar{c}_2(\underline{w}_2^k - \underline{w}_2^{k-1})\\ &\ge -\bar{c}_1(\underline{w}_1^k - \underline{w}_1^{k-1}) \ge 0.\end{aligned}$$

We check the initial distribution of v_2^k for a convex function Ψ:

$$\begin{aligned}v_2^k(x,0) &= \underline{w}_2^{k+1}(x,0) - \underline{w}_2^k(x,0)\\ &= \frac{\phi}{k+1}\tilde{w}_2(x,0) + \left(1 - \frac{\phi}{k+1}\right)\Psi_1'(\underline{w}_1^k(x,0),x) - \frac{\phi}{k}\tilde{w}_2(x,0)\\ &\quad - \left(1 - \frac{\phi}{k}\right)\Psi_1'(\underline{w}_1^{k-1}(x,0),x) \ge \phi\frac{1}{k(k+1)}\left(\Psi_1'(\underline{w}_1^k(x,0),x) - \tilde{w}_2(x,0)\right)\\ &> \phi\frac{1}{k(k+1)}\left(\Psi_1'(\tilde{w}_1(x,0),x) - \tilde{w}_2(x,0)\right) > 0,\end{aligned}$$

Therefore, $v_2^k(x,0) > 0$ and $\underline{w}_2^k \le \underline{w}_2^{k+1}$.

Let us finally show that $\underline{w}_2^{k+1} \le \bar{w}_2^{k+1}$. This time we define $w_2^{k+1} \equiv \bar{w}_2^{k+1} - \underline{w}_2^{k+1}$. Applying $\mathbb{L}_2'$,

$$\begin{aligned}\mathbb{L}_2' v_2^{k+1} &= -F_2(\bar{w}_1^k, \bar{w}_2^k) + F_2(\underline{w}_1^k, \underline{w}_2^k)\\ &= -f_2(\bar{w}_1^k, \bar{w}_2^k) + f_2(\underline{w}_1^k, \underline{w}_2^k) + \bar{c}_2(\bar{w}_2^k - \underline{w}_2^k) \ge -\bar{c}_1(\bar{w}_1^k - \underline{w}_1^k) \ge 0.\end{aligned}$$

Since $\bar{w}_2^k \ge \underline{w}_2^k$ and $\bar{w}_1^k \ge \underline{w}_1^k$, we can see that $\mathbb{L}_2' v_2^{k+1} \ge 0$ if $\bar{c}_2 \ge 0$ and f_2 decreasing in both arguments, or if alternatively $\bar{c}_1 \le 0$. In the later case,

$$\mathbb{L}_2' v_2^{k+1} = -f_2(\bar{w}_1^k, \bar{w}_2^k) + f_2(\underline{w}_1^k, \underline{w}_2^k) + \bar{c}_2(\bar{w}_2^k - \underline{w}_2^k) \ge -\bar{c}_1(\bar{w}_1^k - \underline{w}_1^k) \ge 0.$$

With respect to the initial condition $v_2^{k+1}(x,0)$:

$$\begin{aligned}w_2^{k+1}(x,0) &= \frac{\psi}{k+1}\hat{w}_2(x,0) + \left(1 - \frac{\psi}{k+1}\right)\Psi_1'(\bar{w}_1^k(x,0),x)\\ &\quad - \frac{\phi}{k+1}\tilde{w}_2(x,0) - \left(1 - \frac{\phi}{k+1}\right)\Psi_1'(\underline{w}_1^k(x,0),x)\\ &\ge \frac{\psi-\phi}{k+1}\hat{w}_2(x,0) - \frac{\psi-\phi}{k+1}\Psi_1'(\bar{w}_1^k(x,0),x) = \frac{\psi-\phi}{k}\left(\hat{w}_2(x,0) - \Psi_1'(\bar{w}_1^k(x,0),x)\right)\\ &\ge \frac{\psi-\phi}{k}\left(\hat{w}_2(x,0) - \Psi_1'(\hat{w}_1(x,0),x)\right) > 0,\end{aligned}$$

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for a convex function Ψ. This implies that $v_2^{k+1} \geq 0$ and, hence, $\underline{w}_2^{k+1} \leq \bar{w}_2^{k+1}$. We can then state that

$$\underline{w}_2^k \leq \underline{w}_2^{k+1} \leq \bar{w}_2^{k+1} \leq \bar{w}_2^k. \qquad \blacksquare$$

Before proceeding to the next intermediate result, it is important to comment on the generality of Proposition B.1:

REMARK 1. The initial conditions for $\bar{w}_2^k$ and $\underline{w}_2^k$ correspond to a function Ψ that is assumed to be convex in w_1. For Ψ_1 concave we should initialize $\bar{w}_2^k$ and $\underline{w}_2^k$ as

$$\bar{w}_2^k(x,0) = \tfrac{\psi}{k}\hat{w}_2(x,0) + \left(1 - \tfrac{\psi}{k}\right)\Psi_1'(\underline{w}_1^{k-1}(x,0),x),$$
$$\underline{w}_2^k(x,0) = \tfrac{\phi}{k}\tilde{w}_2(x,0) + \left(1 - \tfrac{\phi}{k}\right)\Psi_1'(\bar{w}_1^{k-1}(x,0),x).$$

REMARK 2. In the absence of an upper solution, it is still true that $\{\underline{u}_1^k, \underline{u}_2^k\}$ is monotone non-decreasing. Then, as $k \to \infty$, the sequence either converges to some limit or it becomes unbounded. Nevertheless, we cannot prove that the limit is a solution of the original problem. The same holds true for the monotone non-increasing sequence $\{\bar{u}_1^k, \bar{u}_2^k\}$.

REMARK 3. The initial conditions for our sequences have been set in order to ensure that the iterative process is monotone. Alternative forms might be considered. The main requirement is that they should converge towards the initial and terminal conditions of the original problem when $k \to \infty$.

This far we have constructed four sequences of continuous functions $\{\bar{u}_1^k\}_{k\in\mathbb{N}}$, $\{\underline{u}_1^k\}_{k\in\mathbb{N}}$, $\{\bar{w}_2^k\}_{k\in\mathbb{N}}$ and $\{\underline{w}_2^k\}_{k\in\mathbb{N}}$, which are monotone and bounded. One can then conclude that they converge uniformly. Hence, there exist functions $\underline{u}_1$, $\bar{u}_1$, $\underline{w}_2$, and $\bar{w}_2$ defined over $\Omega \times [0,T]$, such that

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} \underline{u}_1^k(x,t) = \underline{u}_1(x,t),$$
$$\lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} \bar{u}_1^k(x,t) = \bar{u}_1(x,t),$$

and

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} \underline{w}_2^k(x,t) = \underline{w}_2(x,t),$$
$$\lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} \bar{w}_2^k(x,t) = \bar{w}_2(x,t)$$

$\forall (x,t) \in \Omega \times [0,T]$. Notice that $\underline{u}_1(x,0) = \bar{u}_1(x,0) = u_1(x,0)$ and $\underline{w}_2(x,0) = \bar{w}_2(x,0) = \Psi_1'(u_1(x,T),x)$ (see Remark 3 above). Undoing the change of variable, we also obtain the limit functions $\bar{u}_2(x,t) = \bar{w}_2(x,T-t)$ and $\underline{u}_2(x,t) = \underline{w}_2(x,T-t)$. The next question is whether the limit functions $\underline{u}_1(x,t)$, $\bar{u}_1(x,t)$, $\underline{u}_2(x,t)$ and $\bar{u}_2(x,t)$ coincide, that is, whether for all $(x,t) \in \Omega \times [0,T]$ it is true that $\underline{u}_1(x,t) = \bar{u}_1(x,t)$ and $\underline{u}_2(x,t) = \bar{u}_2(x,t)$. At this point of the proof, we directly focus on u_1 and u_2. In order to confirm that $\underline{u}_1 = \bar{u}_1$ and $\underline{u}_2 = \bar{u}_2$, we will show that the linear operators $\mathbb{L}_1$ and $\mathbb{L}_2$, and the corresponding diagonal matrix $\mathbb{L} \equiv \mathrm{diag}(\mathbb{L}_1, \mathbb{L}_2)$, are invertible. Then, once we demonstrate that $\mathbb{L}^{-1}F(u)$ is a contraction where $F(u) \equiv (F_1(u), F_2(u))'$, we can conclude the uniqueness of the limits. The proof follows the same steps as Theorem 8.9.1 in Pao (1992) for pure parabolic systems, simpler to deal with. Let us first consider the existence of $\mathbb{L}^{-1}$:

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LEMMA B.1. *There exists a real positive number γ such that the inverse operators of $\mathbb{L}_1$ and $\mathbb{L}_2$, denoted by $\mathbb{L}_1^{-1}$ and $\mathbb{L}_2^{-1}$, respectively, exist and satisfy*

$$| \mathbb{L}_i^{-1} u_i - \mathbb{L}_i^{-1} u_i' |_0 \leq \gamma^{-1} | u_i - u_i' |_0$$

for any $(x,t) \in \Omega \times [0,T]$, where $|\cdot|_0$ is the supremum norm for functions in $R(\mathbb{L}_i)$ and the range of $\mathbb{L}_i$:

$$\mathbb{L}_i : D(\mathbb{L}_i) \longrightarrow C(\Omega \times [0,T])$$

for $i = 1, 2$, where $\mathbb{L}_i$'s domain is

$$D(\mathbb{L}_1) = \{u_1 \in C(\bar{\Omega} \times [0,T]) \cap C^{1,2}(\Omega \times [0,T]),\ \lim_{x \to \delta\Omega} u_{1,x} = 0,\ u_1(x,0) = u_{1,0}(x),\ \textit{given}\},$$
$$D(\mathbb{L}_2) = \{u_2 \in C(\bar{\Omega} \times [0,T]) \cap C^{1,2}(\Omega \times [0,T]),\ \lim_{x \to \delta\Omega} u_{2,x} = 0,\ u_2(x,T) = \Psi_1'(u_1(x,T),x)\}.$$

Proof. We will treat $\mathbb{L}_1$ and $\mathbb{L}_2$ separately. We begin by proving lemma's result for $\mathbb{L}_1$.

I. Let us take any two $u_1, u_1' \in D(\mathbb{L}_1)$ satisfying $u_1 \neq u_1'$, that is, they do not coincide in all $(x,t) \in \Omega \times [0,T]$. We build $v \equiv u_1 - u_1'$, and let $(x_1,t_1) \in \Omega \times [0,T]$ be a point such that

$$| v |_0 = | v(x_1,t_1) | .$$

That is to say, v attains the supremum of its absolute value at (x_1,t_1). We will show that

$$v(x_1,t_1) \left(\mathbb{L}_1 u_1(x_1,t_1) - \mathbb{L}_1 u_1'(x_1,t_1) \right) \geq \gamma_1 | v |_0^2,$$

which would imply, following Pao (1992), the statement of the lemma for operator $\mathbb{L}_1$. The inequality above is obvious if $v(x_1,t_1) = 0$. If $v(x_1,t_1) \neq 0$, then $v(x_1,t_1)$ is a positive maximum or a negative minimum in $\bar{\Omega} \times [0,T]$.

We know that, for $t = 0$ and for any $x \in \Omega$, $v(x,0) = u_{1,0}(x) - u_{1,0}(x) = 0$. Therefore, $t_1 > 0$. Given that t_1 is an extremum,

$$\begin{cases} v_t(x_1,t_1) = 0, \text{ if } t_1 < T, \\ \text{or } v(x_1,t_1) v_t(x_1,t_1) \geq 0, \text{ if } t_1 = T. \end{cases} \tag{B4}$$

Regarding the space we also have two alternatives. Rather x_1 is in the interior of the space or it is located at the frontier, that is, $x_1 \in \mathring{\Omega}$ or $x_1 \in \delta\Omega$, respectively:

I.1. If $x_1 \in \mathring{\Omega}$, we have that

$$v(x_1,t_1) v_{xx}(x_1,t_1) \leq 0 \tag{B5}$$

because (x_1,t_1) is a extremum. Considering together (B4) and (B5),

$$v(x_1,t_1) \left[v_t(x_1,t_1) - v_{xx}(x_1,t_1) \right] \geq 0$$

or, equivalently, rewriting the inequality above in terms of u_1 and taking into account the definition of $\mathbb{L}_1$,

$$v(x_1,t_1) \{ \mathbb{L}_1 u_1(x_1,t_1) - \mathbb{L}_1 u_1'(x_1,t_1) - \underline{a}_1(x_1,t_1) \left[u_1(x_1,t_1) - u_1'(x_1,t_1) \right] \} \geq 0.$$

Therefore, since $\underline{a}_1(x,t)$ is a bounded function (assumption of the theorem),

$$\begin{aligned} v(x_1,t_1) \left[\mathbb{L}_1 u_1(x_1,t_1) - \mathbb{L}_1 u_1'(x_1,t_1) \right] &\geq \underline{a}_1(x_1,t_1) | u_1(x_1,t_1) - u_1'(x_1,t_1) |^2 \\ &\geq \gamma_1 | v(x_1,t_1) |^2 = \gamma_1 | v |_0^2 \end{aligned} \tag{B6}$$

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for any $\gamma_1 \in \mathbb{R}$ such that $0 < \gamma_1 \leq \underline{a}_1(x_1, t_1)$. Hence, since

$$| v |_0 | \mathbb{L}_1 u_1 - \mathbb{L}_1 u_1' |_0 \geq v(x_1, t_1) \left[\mathbb{L}_1 u_1(x_1, t_1) - \mathbb{L}_1 u_1'(x_1, t_1) \right], \tag{B7}$$

using (B6) and (B7) we can conclude as in Pao (1992) that

$$| \mathbb{L}_1 u_1 - \mathbb{L}_1 u_1' |_0 \geq \gamma_1 | v |_0 . \tag{B8}$$

I.2. Let us consider the case where x_1 is at the frontier. If $x_1 \in \delta\Omega$, then, for any $\epsilon > 0$, there exists a point $(x_\epsilon, t_\epsilon) \in \mathring{\Omega} \times [0, T]$ sufficiently close to (x_1, t_1) such that

$$\begin{cases} v(x_1, t_1) v_t(x_\epsilon, t_\epsilon) \geq \frac{-\epsilon}{2} | v |_0^2, \\ v(x_1, t_1) v_{xx}(x_\epsilon, t_\epsilon) \leq \frac{-\epsilon}{2} | v |_0^2, \end{cases}$$

where $t_\epsilon \equiv t_1 \pm \epsilon$ and $x_\epsilon \equiv x_1 \pm \epsilon$. We can state the first inequality because (x_1, t_1) is a maximum of the absolute value. Hence, when we approach to (x_1, t_1), we must observe that $v_t \to 0$ by continuity of the trajectories. The same is true for the derivative in space. As a consequence,

$$v(x_1, t_1) [v_t(x_\epsilon, t_\epsilon) - v_{xx}(x_\epsilon, t_\epsilon)] \geq -\epsilon | v |_0^2$$

and, as before,

$$v(x_1, t_1) \left(\mathbb{L}_1 u_1 - \mathbb{L}_1 u_1' \right) |_{(x_\epsilon, t_\epsilon)} \geq (\gamma_1 - \epsilon) | v |_0^2 .$$

Therefore, the arbitrariness of ϵ ensures that

$$| \mathbb{L}_1 u_1(x_1, t_1) - \mathbb{L}_1 u_1'(x_1, t_1) | \geq \gamma_1 | u_1(x_1, t_1) - u_1'(x_1, t_1) | = \gamma_1 | u_1 - u_1' |_0$$

for $x \in \delta\Omega$ as well. Hence, since

$$| \mathbb{L}_1 u_1 - \mathbb{L}_1 u_1' |_0 \geq | u_1(x_1, t_1) - \mathbb{L}_1 u_1'(x_1, t_1) |,$$

$\mathbb{L}_1^{-1}$ exists on $R(\mathbb{L}_1)$ and satisfies

$$| \mathbb{L}_1^{-1} u_1 - \mathbb{L}_1^{-1} u_1' |_0 \leq \gamma_1^{-1} | u_1 - u_1' |_0 .$$

II. We prove next that $\mathbb{L}_2^{-1}$ exists and there is a $\gamma_2 > 0$ such that

$$| \mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' |_0 \geq \gamma_2 | u_2 - u_2' |_0, \tag{B9}$$

for any two $u_2, u_2' \in D(\mathbb{L}_2)$. We define $v \equiv u_2 - u_2'$, and let (x_2, t_2) be a point in $\Omega \times [0, T]$ such that

$$| v |_0 = | v(x_2, t_2) | .$$

It is important to notice that $v(x, T) = 0$, for any $x \in \Omega$, in order that $t_2 < T$.

Let us show:

$$v(x_2, t_2) \left(\mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' \right) \geq \gamma_2 | v |_0^2 .$$

If $v(x_2, t_2) = 0$, then the inequality above is trivially verified. If $v(x_2, t_2) \neq 0$, $v(x_2, t_2)$ is rather a positive maximum or a negative minimum in $\Omega \times [0, T)$. Hence,

$$v_t(x_2, t_2) = 0$$

whenever $t_2 < T$. As in point I, x_2 lies either in $\mathring{\Omega}$ or in $\delta\Omega$:

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II.1 If $x_2 \in \mathring{\Omega}$,

$$v_x(x_2, t_2) = 0$$

and $v(x_2, t_2)v_{xx}(x_2, t_2) \leq 0$. We can then write

$$\begin{aligned}
&- v(x_2, t_2)\,[v_t(x_2, t_2) + v_{xx}(x_2, t_2)] \geq 0,\\
&- v(x_2, t_2)\left[\mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' + \bar{c}_2(u_2 - u_2')\right] \geq 0,\\
&- v(x_2, t_2)\left[\mathbb{L}_2 u_2 - \mathbb{L}_2 u_2'\right] \geq \bar{c}_2 v(x_2, t_2)(u_2 - u_2') \geq \gamma_2 \mid v \mid_0^2,
\end{aligned}$$

for any $0 < \gamma_2 \leq \bar{c}_2$. On the other hand,

$$\mid -v \mid_0 \mid \mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' \mid_0 \geq -v(x_2, t_2)\left(\mathbb{L}_2 u_2 - \mathbb{L}_2 u_2'\right)$$

and $\mid -v \mid_0 = \mid v \mid_0$. Therefore,

$$\mid \mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' \mid_0 \geq \gamma_2 \mid u_2 - u_2' \mid_0 .$$

Again, as in Pao (1992), this implies that $\mathbb{L}_2^{-1}$ exists and satisfies

$$\mid \mathbb{L}_2^{-1} u_2 - \mathbb{L}_2^{-1} u_2' \mid_0 \leq \gamma_2^{-1} \mid u_2 - u_2' \mid_0 .$$

II.2 If $x_2 \in \delta\Omega$ then, since $\lim_{x \to \delta\Omega} v_t(x, t) = 0$ for all t, there exists $(x_\epsilon, t_\epsilon) \in \mathring{\Omega} \times [0, T)$ and ϵ sufficiently small such that

$$\begin{cases}
v(x_2, t_2)v_t(x_\epsilon, t_\epsilon) \leq \frac{\epsilon}{2} \mid v \mid_0^2,\\
v(x_2, t_2)v_{xx}(x_\epsilon, t_\epsilon) \leq \frac{\epsilon}{2} \mid v \mid_0^2 .
\end{cases}$$

In particular, we will restrict ϵ to $\epsilon \leq \bar{c}_2(x_2, t_2)$. Therefore,

$$-v(x_2, t_2)\,[v_t(x_\epsilon, t_\epsilon) + v_{xx}(x_\epsilon, t_\epsilon)] \geq -\epsilon \mid v \mid_0^2,$$

that is,

$$- v(x_2, t_2)\left[\mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' + \bar{c}_2(u_2 - u_2')\right] \mid_{(x_\epsilon, t_\epsilon)} \geq -\epsilon \mid v \mid_0^2 . \qquad \textbf{(B10)}$$

It is also true that

$$\mid v(x_2, t_2) \mid_0 \mid \mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' \mid_0 \geq -v(x_2, t_2)\left[\mathbb{L}_2 u_2 - \mathbb{L}_2 u_2'\right] \mid_{(x_\epsilon, t_\epsilon)} . \qquad \textbf{(B11)}$$

Using (B10) and (B11), we obtain

$$\mid v(x_2, t_2) \mid_0 \mid \mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' \mid_0 \geq -\epsilon \mid v \mid_0^2 + \bar{c}_2(x_\epsilon, t_\epsilon) \mid v(x_2, t_2) \mid_0 (u_2 - u_2') \mid_{(x_\epsilon, t_\epsilon)} .$$

Therefore,

$$\mid \mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' \mid_0 \geq -\epsilon \mid v \mid_0 + \bar{c}_2(x_\epsilon, t_\epsilon)(u_2 - u_2') \mid_{(x_\epsilon, t_\epsilon)} . \qquad \textbf{(B12)}$$

Moreover, by continuity and since $u_{2x}(x, t) = 0$ for any point in $\delta\Omega$, we also know that

$$\lim_{(x_\epsilon, t_\epsilon) \to (x_2, t_2)} (u_2 - u_2')(x_\epsilon, t_\epsilon) = v(x_2, t_2).$$

We can choose any $0 < \gamma_2 \leq \bar{c}_2(x_\epsilon, t_\epsilon) - \epsilon$, and take the limit when $(x_\epsilon, t_\epsilon) \to (x_2, t_2)$. Consequently,

$$\begin{aligned}
&- \epsilon \mid v \mid_0 + \bar{c}_2(x_\epsilon, t_\epsilon)(u_2 - u_2') \mid_{(x_\epsilon, t_\epsilon)} \to -\epsilon \mid v \mid_0 + \bar{c}_2(x_2, t_2)(u_2 - u_2') \mid_{(x_2, t_2)}\\
&= -\epsilon \mid v \mid_0 + \bar{c}_2(x_2, t_2) \mid v \mid_0 = [-\epsilon + \bar{c}_2(x_2, t_2)] \mid v \mid_0 .
\end{aligned}$$

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Hence, (B12) implies that

$$| \mathbb{L}_2 u_2 - \mathbb{L}_2 u_2' |_0 \geq [-\epsilon + \bar{c}_2(x_2, t_2)] \, | v |_0 \geq \gamma_2 \, | v |_0 \, . \qquad \blacksquare$$

As a direct consequence of Lemma B.1, we can state the following result:

COROLLARY B.1. *Taking* $\gamma = \min\{\gamma_1, \gamma_2\}$*, the operator* $\mathbb{L} \equiv (\mathbb{L}_1, \mathbb{L}_2)'$ *verifies*

$$|| \mathbb{L}^{-1} u - \mathbb{L}^{-1} u' ||_0 \leq \gamma^{-1} \, || u - u' ||_0,$$

for $u, u' \in R(\mathbb{L})$.

Under the assumptions of the theorem, there exist functions $K_i = K_i(x, t)$ in $\Omega \times [0, T]$ with values in $\mathbb{R}$ such that, for any u, v in the domain of f_i,

$$| F_i(u_1, u_2) - F_i(v_1, v_2) | \leq K_i \, | u - v | \, .$$

We define $K \equiv K_1 + K_2$. Let $\bar{K}$ be an upper bound for K. As we will see in the next lemma, one can then conclude that $\mathbb{L}^{-1} F(u)$ is a contraction. Therefore, the limits of our sequences coincide and define the unique solution to (I):

LEMMA B.2. *Let* $\gamma > \bar{K}$*, and consider* $F(u)$ *defined in (8) and* $\mathbb{L}^{-1} F$ *a mapping from* $X \subset C(\Omega \times [0, T])$ *into* $C(\Omega \times [0, T])$*. Then,* $\mathbb{L}^{-1} F$ *possesses the contraction property*

$$|| \mathbb{L}^{-1} F(u) - \mathbb{L}^{-1} F(u') ||_0 \leq \frac{\bar{K}}{\gamma} \, || u - u' ||_0,$$

for all $u, u' \in X$.

Proof. For each $i = 1, 2$ and $u \in X \subset C(\Omega \times [0, T])$, the function $F_i(u)$ is in X. This implies that $F_i(u) \in R(\mathbb{L}_i)$ and, thus, $F(u) \in R(\mathbb{L})$. Hence, by Lemma 1, $\mathbb{L}^{-1}$ is well defined on X. Since F_i verifies

$$| F_i(u) - F_i(u') | \leq K_i \, | u - u' | \leq K_i \, | u - u' |_0,$$

function F satisfies the condition

$$|| F(u) - F(u') ||_0 \leq \bar{K} \, || u - u' ||_0 \qquad \textbf{(B13)}$$

for all $u, u' \in X$. We also know from Corollary B.1 that

$$|| \mathbb{L}^{-1} F(u) - \mathbb{L}^{-1} F(u') ||_0 \leq \gamma^{-1} \, || F(u) - F(u') ||_0 \, . \qquad \textbf{(B14)}$$

Taking together (B13) and (B14), we obtain

$$|| \mathbb{L}^{-1} F(u) - \mathbb{L}^{-1} F(u') ||_0 \leq \gamma^{-1} \bar{K} \, || u - u' ||_0,$$

which proves the lemma. $\blacksquare$

APPENDIX C: THEOREM 2 PROOF

We prove this result by contradiction. Let $u \in C^{2,1}(\Omega \times [0, T])$ be a solution to (I) and assume that $u \not\subseteq < \tilde{u}, \hat{u} >$. There are six different cases:

(*i*) u equals the lower bound of $< \tilde{u}, \hat{u} >$ in at least one point (x_0, t_0), that is to say, $u(x_0, t_0) = \tilde{u}(x_0, t_0)$ and $\tilde{u}(x, t) < u(x, t)$ for all other $(x, t) \in \Omega \times [0, T] \setminus (x_0, t_0)$.

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(*ii*) u equals the upper bound of $< \tilde{u}, \hat{u} >$ in at least one point (x_0, t_0), that is to say $u(x_0, t_0) = \hat{u}(x_0, t_0)$ and $u(x, t) < \hat{u}(x, t)$ for all other $(x, t) \in \Omega \times [0, T] \setminus (x_0, t_0)$.
(*iii*) u is smaller than $(\tilde{u}_1, \tilde{u}_2)$ in at least one point.
(*iv*) u is larger than $(\hat{u}_1, \hat{u}_2)$ in at least one point.
(*v*) u is smaller than $(\tilde{u}_1, \tilde{u}_2)$ in all points.
(*vi*) u is larger than $(\hat{u}_1, \hat{u}_2)$ in all points.

We shall focus on (*i*), (*iii*), and (*v*). Similar reasoning can be applied to study the remaining cases.

(*i*) There exists a point (x_0, t_0) at which $\tilde{u}(x_0, t_0) = u(x_0, t_0)$ and $\tilde{u}(x, t) < u(x, t)$ for all other $(x, t) \in \Omega \times [0, T] \setminus (x_0, t_0)$. We define $v(x, t) \equiv u(x, t) - \tilde{u}(x, t)$. (x_0, t_0) is then a minimum for v. Therefore,

$$v_t(x_0, t_0) = 0,$$
$$v_{xx}(x_0, t_0) > 0.$$

Consequently, we have for $v = (v_1, v_2)$ that $\mathbb{L}_1 v_1 \mid_{(x_0,t_0)} < 0$ and $\mathbb{L}_2 v_2 \mid_{(x_0,t_0)} > 0$. However,

$$\mathbb{L}_1 v_1 = \mathbb{L}_1(u_1 - \tilde{u}_1) = \mathbb{L}_1 u_1 - \mathbb{L}_1 \tilde{u}_1 = f_1(u_1, u_2) - f_1(\tilde{u}_1, \tilde{u}_2) + \underline{a}_1(u_1 - \tilde{u}_1) \geq 0.$$

The expression above is positive under the assumptions because, as we have seen in the proof of Proposition B.1, either f_1 is increasing in both arguments, with $\underline{a}_1 \geq 0$ (which yields the result directly), or because $\underline{a}_2 \leq 0$. Notice that in the later case

$$\mathbb{L}_1 v_1(x, t) \geq -\underline{a}_2(u_2 - \tilde{u}_2) \geq 0.$$

Hence, we achieve a contradiction.
Similarly, one can verify that

$$\mathbb{L}_2 v_2 = \mathbb{L}_2(u_2 - \tilde{u}_2) = \mathbb{L}_2 u_2 - \mathbb{L}_2 \tilde{u}_2 = f_2(u_1, u_2) - f_2(\tilde{u}_1, \tilde{u}_2) - \bar{c}_2(u_2 - \tilde{u}_2) \leq 0,$$

which is negative under the assumptions of the theorem. As before, this result is directly confirmed if f_2 is decreasing in both arguments, with $\bar{c}_2 \geq 0$. Otherwise, we can show for $\bar{c}_1 \leq 0$ that

$$\mathbb{L}_2 v_2 \leq \bar{c}_1(u_1 - \tilde{u}_1) \leq 0.$$

A contradiction arises again. Then, we finally conclude that u and $\tilde{u}$ cannot coincide in one point.

(*iii*) When u is smaller than the lower bound of $< \tilde{u}, \hat{u} >$ in at least one point, by continuity, there must be at least a point (x_0, t_0) at which $u(x_0, t_0) = \tilde{u}(x_0, t_0)$. But we just showed in *i*) that this is impossible.

(*v*) Notice that u and $\tilde{u}$ are continuous functions on a compact space. The function v, defined as $v \equiv \tilde{u} - u$, achieves then a minimum at a point denoted by (x_0, t_0). Therefore,

$$v_t(x_0, t_0) = 0,$$
$$v_{xx}(x_0, t_0) > 0.$$

Hence, $\mathbb{L}_1 v_1 \mid_{(x_0,t_0)} < 0$ and $\mathbb{L}_2 v_2 \mid_{(x_0,t_0)} > 0$. However, as in case *i*),

$$\mathbb{L}_1 v_1 = \mathbb{L}_1(\tilde{u}_1 - u_1) = \mathbb{L}_1 \tilde{u}_1 - \mathbb{L}_1 u_1 = f_1(\tilde{u}_1, \tilde{u}_2) - f_1(u_1, u_2) + \underline{a}_1(u_1 - \tilde{u}_1) \geq 0,$$

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which leads to a contradiction under the assumptions of the theorem. Similarly,

$$\mathbb{L}_2 v_2 = \mathbb{L}_2(\tilde{u}_2 - u_2) = \mathbb{L}_2\tilde{u}_2 - \mathbb{L}_2 u_2 = f_2(\tilde{u}_1, \tilde{u}_2) - f_2(u_1, u_2) - \bar{c}_2(u_2 - \tilde{u}_2) \leq 0.$$

The contradiction arises because $\mathbb{L}_2 v_2 \,|_{(x_0,t_0)} > 0$. This finishes the proof.

APPENDIX D: THE SPATIAL AK MODEL

In the spatial AK model, the policy maker aims at maximizing welfare in (11), constrained by:

$$\begin{cases} k_t(x,t) - k_{xx}(x,t) = Ak(x,t) - \delta k(x,t) - c(x,t), \forall (x,t) \in \Omega \times [0,T] \\ \lim_{x \to \delta\Omega} k_x(x,t) = 0 \ \ \forall t \in [0,T], \\ k(x,0) = k_0(x) > 0 \ \text{ given } \ \forall x \in \Omega. \end{cases} \tag{D1}$$

Applying the Ekeland variation principle, we obtain that the co-state variable λ associated with the state equation for k in the Hamiltonian of our problem follows the equation below:

$$\begin{cases} \lambda_t(x,t) + \lambda_{xx}(x,t) = (\delta + \rho - A)\lambda(x,t), \forall (x,t) \in \Omega \times [0,T] \\ \lim_{x \to \delta\Omega} \lambda_x(x,t) = 0 \ \ \forall t \in [0,T], \\ \lambda(x,T) = \Psi'(k(x,T)), \forall x \in \Omega. \end{cases} \tag{D2}$$

and $\lambda = U'(c)$. As in Section 4.2, let us assume a CRRA utility function with parameter σ, that is, $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Under this assumption on the utility function, $c = \lambda^{-1/\sigma}$. In order to apply Theorem 1 to ensure the existence of a unique solution, we need first to check that the set of assumptions is fulfilled. Note that we can choose initial conditions that satisfy Assumption 1. Assumption 2 is satisfied because our functions are differentiable and they have bounded derivatives in the space defined between the upper and the lower solution. Then, Assumption 3, note that f_1 is increasing in both k and λ, and f_2 is decreasing in λ.

Finally, we need to find an upper and a lower solution for the couple (k, λ). According to the definition of upper and lower solutions provided in (5), we need to find two couples of functions $(\tilde{k}, \tilde{\lambda})$ and $(\hat{k}, \hat{\lambda})$ such that

$$\begin{aligned} &\tilde{k}_t(x,t) - \tilde{k}_{xx}(x,t) - f_1(\tilde{k}, \tilde{\lambda}) \leq 0 \leq \hat{k}_t(x,t) - \hat{k}_{xx}(x,t) - f_1(\hat{k}, \hat{\lambda}), \\ &\hat{\lambda}_t(x,t) + \hat{\lambda}_{xx}(x,t) - f_2(\hat{k}, \hat{\lambda}) \leq 0 \leq \tilde{\lambda}_t(x,t) + \tilde{\lambda}_{xx}(x,t) - f_2(\tilde{k}, \tilde{\lambda}), \\ &\lim_{x \to \delta\Omega} \hat{k}_x(x,t) = 0, \ \ \lim_{x \to \delta\Omega} \tilde{k}_x(x,t) = 0, \\ &\lim_{x \to \delta\Omega} \hat{\lambda}_x(x,t) = 0, \ \ \lim_{x \to \delta\Omega} \tilde{\lambda}_x(x,t) = 0. \end{aligned} \tag{D3}$$

Here, $f_1(k,\lambda) = Ak(x,t) - \delta k(x,t) - \lambda^{-1/\sigma}$ and $f_2(k,\lambda) = (\delta + \rho - A)\lambda(x,t)$. Note first that for any $\epsilon > 0$, $\tilde{\lambda} = \epsilon$ is a lower solution for λ and $\hat{\lambda} = e^{-\phi t}$, with $\phi > A - \delta - \rho$ is an upper solution. With these functions in hand, we can then turn to compute the upper and lower solutions for k, since they do depend on $\hat{\lambda}$ and $\tilde{\lambda}$. Looking at the first line of (D3), notice that $\tilde{k} = \epsilon$ is a lower solution for k. It only remains to find the upper solution for k. Again, from the first line of (D3), we conclude that the only requirement we need is that $\hat{k}$ satisfies that $0 < \hat{k}_t(x,t) - \hat{k}_{xx}(x,t) - A\hat{k}(x,t)$, since $\hat{\lambda}^{-1/\sigma}$ is always positive. Then, taking $\hat{k} = e^{\xi t}$ with $\xi - (A - \delta) > 0$ we obtain an upper solution for k.

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We have verified that all assumptions in Theorem 1 are fulfilled, and that there exists an upper and a lower solution for our problem. Then, Theorem 1 holds. Finally, considering Theorem 2, we can guarantee that every solution lies between the upper and lower solutions and, therefore, not missing any additional solution. Hence, we can ensure the existence of a unique solution for the spatial AK model.

APPENDIX E: UPPER AND LOWER SOLUTIONS FOR THE SPATIAL RAMSEY MODEL

In the more general spatial Ramsey model, we also need to find an upper and a lower solutions using (D3), where $f_1(k,\lambda) = Af(k(x,t)) - \delta k(x,t) - \lambda(x,t)^{-1/\sigma}$ and $f_2(k,\lambda) = \lambda\,(\delta + \rho - Af'(k(x,t)))$.

We know that if our problem has a solution (as we show later), the lower solution is actually a lower bound for this solution. Then, a good strategy to find a couple of lower solutions is to search a couple of constant functions in time and homogeneous in space that satisfy the conditions above. Let us assume that $(\tilde{k},\tilde{\lambda})$ is a couple of constant functions both in time and space, that is $(\tilde{k}(x,t),\tilde{\lambda}(x,t)) = (\tilde{k},\tilde{\lambda})$ for all t and x, then the two first conditions in (D3) regarding the lower solution become

$$\begin{aligned} -A\tilde{k}^{\alpha} + \delta\tilde{k} + \tilde{\lambda}^{-1/\sigma} &\leq 0, \\ 0 &\leq -\tilde{\lambda}\left(\delta + \rho - A\alpha\tilde{k}^{\alpha-1}\right), \end{aligned} \tag{E1}$$

with the boundary conditions being naturally satisfied since $(\tilde{k},\tilde{\lambda})$ are constant. From the second equation, if $\tilde{\lambda} > 0$, we obtain the following condition for $\tilde{k}$:

$$\delta + \rho - A\alpha\tilde{k}^{\alpha-1} \geq 0,$$

hence, if $\tilde{k} = \left(\frac{\alpha A}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} - \epsilon$, with $\epsilon > 0$ and sufficiently small, then the inequality is respected. Focusing on the first equation in (E1), we need that

$$A\tilde{k}^{\alpha} - \delta\tilde{k} - \tilde{\lambda}^{-1/\sigma} \geq 0,$$

or

$$A\tilde{k}^{\alpha} - \delta\tilde{k} \geq \tilde{\lambda}^{-1/\sigma}.$$

Then, if $\tilde{\lambda} = \left(A\tilde{k}^{\alpha} - \delta\tilde{k}\right)^{-\sigma} + \epsilon > 0$, the inequality is satisfied and

$$(\tilde{k},\tilde{\lambda}) = \left(\left(\frac{\alpha A}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} - \epsilon, \left(A\tilde{k}^{\alpha} - \delta\tilde{k}\right)^{-\sigma} + \epsilon\right),$$

with $\epsilon > 0$ and sufficiently small is a lower solution. Note that since $A\tilde{k}^{\alpha} - \delta\tilde{k}$ is always positive, and $\tilde{\lambda} > 0$ for ϵ sufficiently small, the proposed upper solution is well defined.

Similarly, an upper solution is an upper bound for the solution of a problem. Here, it seems reasonable to find functions of time that bound the evolution of the solution. For this reason and also for simplicity, let us look for a couple of exponential functions in time, independent of space, namely a couple of functions $(\hat{k},\hat{\lambda}) = (\varphi e^{\phi t}, \psi e^{\zeta t})$. We check next

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under which conditions this couple is an upper solution. According to (D3), we need that

$$\begin{aligned}&\hat{k}_t - A\hat{k}^\alpha + \delta\hat{k} + \hat{\lambda}^{-1/\sigma} \geq 0,\\ &\hat{\lambda}_t - \hat{\lambda}\left(\delta + \rho - A\alpha\hat{k}^{\alpha-1}\right) \leq 0.\end{aligned} \tag{E2}$$

Substituting for $(\hat{k}, \hat{\lambda})$, we can rewrite the second inequality above as

$$\zeta \leq \delta + \rho - A\alpha\varphi^{(\alpha-1)\phi}e^{(\alpha-1)\phi t}.$$

Then, since

$$\zeta \leq \delta + \rho - A\alpha\varphi^{(\alpha-1)\phi} \leq \delta + \rho - A\alpha\varphi^{(\alpha-1)\phi}e^{(\alpha-1)\phi t},$$

if $\zeta \leq \delta + \rho - A\alpha\varphi^{(\alpha-1)\phi}$ holds, then the second inequality in (E2) holds. With this in hand, let us work on the first inequality in (E2). Substituting for $(\hat{k}, \hat{\lambda})$:

$$\varphi(\phi - \delta)e^{\phi t} + \psi^{-\frac{1}{\sigma}}e^{-\frac{\zeta}{\sigma}t} > A\varphi^\alpha e^{\alpha\phi t}.$$

Note that $e^{\alpha\phi t} < e^{\phi t}$, for all t. Then,

$$\varphi(\phi - \delta)e^{\phi t} + \psi^{-\frac{1}{\sigma}}e^{-\frac{\zeta}{\sigma}t} > A\varphi^\alpha e^{\phi t} > A\varphi^\alpha e^{\alpha\phi t}.$$

If $\varphi(\phi - \delta)A\varphi^\alpha > 0$, then the first inequality in (E2) is satisfied and $(\hat{k}, \hat{\lambda})$ is a lower solution.

APPENDIX F: UPPER AND LOWER SOLUTIONS FOR THE SPATIAL RAMSEY MODEL WHEN A IS SPATIALLY DISTRIBUTED

Theorem 1 also holds when the model parameters are spatially distributed. In this appendix, we shall provide an upper and a lower solution in the case when A is spatially distributed, although with less detail than in Appendix E. Suppose then that A is a positive bounded spatially distributed function. Note that we do not need any further assumption on A. Let $\bar{A} = \max_\Omega A$ and $\underline{A} = \min_\Omega A$. Then, we can prove that $(\tilde{k}, \tilde{\lambda}) = \left(\left(\frac{\underline{A}\alpha}{\alpha+\rho}\right)^{\frac{1}{1-\alpha}}, \left(\underline{A}\tilde{k}^\alpha - \delta\tilde{k}\right)^{-\sigma}\right)$ is a lower solution. Since in this case $\tilde{k}$ is constant in time and space, we need to prove that

$$\begin{aligned}&-A(x)f(\tilde{k}) + \delta\tilde{k} + \tilde{\lambda}^{-1/\sigma} \leq 0,\\ &0 \leq \tilde{\lambda}\left(-\delta - \rho + A(x)f'(\tilde{k})\right) = \tilde{\lambda}\left(-\delta - \rho + A(x)\alpha\tilde{k}^{\alpha-1}\right).\end{aligned}$$

If $\tilde{\lambda} \geq 0$, we need that

$$-\delta - \rho + A(x)\alpha\tilde{k}^{\alpha-1} \geq 0,$$

in $\Omega \times [0, T]$. Then

$$\tilde{k}^{\alpha-1} \geq \frac{\delta + \rho}{A(x)\alpha} \iff \tilde{k}^{1-\alpha} \leq \frac{A(x)\alpha}{\delta + \rho}.$$

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Hence, if $\tilde{k}^{1-\alpha} \leq \frac{\underline{A}\alpha}{\delta+\rho} \leq \frac{A(x)\alpha}{\delta+\rho}$, then the condition is satisfied. Let us then simply take $\tilde{k} = \left(\frac{\underline{A}\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$. With this $\tilde{k}$ in hand, we can search for $\tilde{\lambda}$. For the previous definition of $\tilde{k}$, $\tilde{\lambda}$ must satisfy that

$$-A(x)f(\tilde{k}) + \delta\tilde{k} + \tilde{\lambda}^{-1/\sigma} \leq 0.$$

In particular, we could search for $\tilde{\lambda}$ such that

$$-A(x)f(\tilde{k}) + \delta\tilde{k} + \tilde{\lambda}^{-1/\sigma} \leq -\underline{A}f(\tilde{k}) + \delta\tilde{k} + \tilde{\lambda}^{-1/\sigma} \leq 0,$$

or $\tilde{\lambda}^{-1/\sigma} \leq \underline{A}f(\tilde{k}) - \delta\tilde{k}$, that is $\tilde{\lambda} \geq \left(\underline{A}f(\tilde{k}) - \delta\tilde{k}\right)^{-1\sigma}$. Hence, $(\tilde{k}, \tilde{\lambda}) = \left(\left(\frac{\underline{A}\alpha}{\alpha+\rho}\right)^{\frac{1}{1-\alpha}}, \left(\underline{A}\tilde{k}^{\alpha} - \delta\tilde{k}\right)^{-\sigma}\right)$ is indeed a lower solution.

The upper solution $(\hat{k}, \hat{\lambda})$ needs to satisfy that

$$\hat{k}_t - A\hat{k}^{\alpha} + \delta\hat{k} + \hat{\lambda}^{-1/\sigma} \geq 0,$$

$$\hat{\lambda}_t - \hat{\lambda}\left(\delta + \rho - A\alpha\hat{k}^{\alpha-1}\right) \leq 0.$$

We shall prove that under certain conditions, $(\hat{k}, \hat{\lambda}) = (\varphi e^{\phi t}, \psi e^{\zeta t})$ is an upper solution. Let us start by the inequality for $\hat{\lambda}$. If $\hat{\lambda} = \psi e^{\zeta t}$ is an upper solution, then

$$\psi e^{\zeta t}\left(\zeta - \delta - \rho + A\alpha\hat{k}^{\alpha-1}\right) \leq \psi e^{\zeta t}\left(\zeta - \delta - \rho + \bar{A}\alpha\hat{k}^{\alpha-1}\right) \leq 0.$$

Then, if $\hat{k}^{1-\alpha} \geq \frac{\bar{A}\alpha}{\delta+\rho-\zeta}$. Hence, $\hat{k} = \varphi e^{\phi t}$ is an upper solution if in particular, $\varphi > 0$, $\phi \geq 0$, and $\varphi \geq \frac{\bar{A}\alpha}{\delta+\rho-\zeta}$.

Next, let us turn our attention to $\hat{\lambda}$. Substituting $(\hat{k}, \hat{\lambda}) = (\varphi e^{\phi t}, \psi e^{\zeta t})$ in the first inequality that an upper solution needs to satisfy, we obtain that

$$-A\varphi^{\alpha}e^{\alpha\phi t} + \delta\varphi e^{\phi t} + \psi^{-1/\sigma}e^{-\zeta/\sigma t} \geq 0.$$

Then, if we find ψ, φ and ϕ such that

$$-\bar{A}\varphi^{\alpha}e^{\alpha\phi t} + \delta\varphi e^{\phi t} + \psi^{-1/\sigma}e^{-\zeta/\sigma t} \geq 0,$$

we would have found the upper solution. In particular, if the right-hand side is positive at $t = 0$ and increasing, then it will always be positive. Regarding the parameters of the upper solution, this constraints impose that

$$-\bar{A}\varphi^{\alpha} + \delta\varphi + \psi^{-1/\sigma} \geq 0,$$

$$-\bar{A}\varphi^{\alpha}\alpha\phi + \delta\varphi\phi - \tfrac{\zeta}{\sigma}\psi^{-1/\sigma} \geq 0,$$

or

$$\psi \leq \left(\bar{A}\varphi^{\alpha} - \delta\varphi\right)^{-\sigma},$$

$$\phi \geq \tfrac{\zeta}{\sigma}\tfrac{\psi^{-1/\sigma}}{\delta\varphi - \bar{A}\varphi^{\alpha}\alpha},$$

Under the above conditions for ψ and ϕ, plus the previously found $\varphi > 0$, $\phi \geq 0$, and $\varphi \geq \frac{\bar{A}\alpha}{\delta+\rho-\zeta}$, $(\hat{k}, \hat{\lambda}) = (\varphi e^{\phi t}, \psi e^{\zeta t})$ is an upper solution.

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APPENDIX G: NUMERICAL ALGORITHM

The algorithm is based on the iterative method presented in Theorem 1. It will then require a lower or an upper solution to start iterating. For illustration purposes, let us build sequences $\{u_1^k, u_2^k\}_{k\in\mathbb{N}}$ with $(u_1^0, u_2^0) = (\tilde{u}_1, \tilde{u}_2)$. We summarize the algorithm in the following three steps:

Step 1: Set a stopping criterion Δ and build a grid to represent the space-time set $\Omega \times [0, T]$, $\Omega \simeq \{Dx, 2 * Dx, ..., N * Dx\}$ and $[0, T] \simeq \{Dt, 2 * Dt, ..., J * Dt\}$. We initialize the sequences $\{u_1^k u_2^k\}_{k\in\mathbb{N}}$ with

$$(\underline{u}_1^0, \underline{u}_2^0) = (\tilde{u}_1, \tilde{u}_2),$$

$$(\underline{w}_1^0, \underline{w}_2^0) = (\tilde{w}_1, \tilde{w}_2).$$

Step 2: For every k, given (u_1^{k-1}, u_2^{k-1}) and $(w_1^{k-1}, w_2^{k-1})(x, t) = (u_1^{k-1}, u_2^{k-1})(x, T - t)$, compute

$$\begin{cases} \mathbb{L}_1 u_1^k = F_1(u_1^{k-1}, u_2^{k-1}), \\ u_1^k(x, 0) = \frac{\zeta}{k}\tilde{u}_1(x, 0) + (1 - \frac{\zeta}{k})u_1(x, 0), \\ \lim_{x\to\delta\Omega} u_{1,x}^k = 0, \end{cases}$$

with $0 < \zeta < 1$, and

$$\begin{cases} \mathbb{L}_2' w_2^k = -F_2(w_1^{k-1}, w_2^{k-1}), \\ w_2^k(x, 0) = \frac{\phi}{k}\tilde{w}_2(x, 0) + (1 - \frac{\phi}{k})\Psi_1'(w_1^{k-1}(x, 0), x), \\ \lim_{x\to\delta\Omega} w_{2,x}^k = 0, \end{cases}$$

with $0 < \psi < 1$.

Step 3: After every k, we measure the distance between (u_1^{k-1}, u_2^{k-1}) and (u_1^k, u_2^k). If the distance is smaller than Δ, then STOP. Set $k = k + 1$ and go to step 2 otherwise.

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