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Local structure of self-affine sets

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Abstract. The structure of a self-similar set with the open set condition does not change under magnification. For self-affine sets, the situation is completely different. We consider self-affine Cantor sets $E \subset \mathbb{R}^2$ of the type studied by Bedford, McMullen, Gatzouras and Lalley, for which the projection onto the horizontal axis is an interval. We show that in small square ε-neighborhoods N of almost each point x in E, with respect to many Bernoulli measures on the address space, $E \cap N$ is well approximated by product sets $[0, 1] \times C$, where C is a Cantor set. Even though E is totally disconnected, all tangent sets have a product structure with interval fibers, reminiscent of the view of attractors of chaotic differentiable dynamical systems. We also prove that E has uniformly scaling scenery in the sense of Furstenberg, Gavish and Hochman: the family of tangent sets is the same at almost all points x.

1. *Introduction*

The local fine structure of a fractal is intricate and regular at the same time. In general, it is not hard to show that there are no tangent planes (see [**2**, Theorem 1]). Successive magnification of the set around a given point x, the so-called *scenery flow* [**4, 9, 12**], will never lead to a unique 'tangent set'. On the other hand, there is a well-defined *family of tangent sets,* termed the 'tangential measure distribution' [**10, 16**] or 'gallery of micro-sets' [**7**] of the fractal. This family exists globally: it is obtained from the magnification flow at almost every point x. This has been proved for self-similar [**1, 10**] and self-conformal [**4**] measures, including random constructions [**7, 18**]. Recent papers by Furstenberg [**7**], Gavish [**9**] and Hochman [**12**] show that such a 'uniformly scaling scenery' implies a number of nice geometric properties of the given fractal set or measure.

For self-similar sets with the open set condition, this is easy to understand since the structure does not change under magnification. Tangent sets are essentially the same as

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parts of the set. In Furstenberg's terminology, such fractals are homogeneous since all micro-sets are mini-sets [**7**]. For self-conformal sets, the situation is only slightly different: tangent sets relate to parts of the fractal in a similar way as a tangent to a corresponding curve.

Here we study certain self-affine sets, and show in Theorem 2 that they have a 'uniformly scaling scenery'. Our main result, however, is that they undergo a metamorphosis when they are magnified. Disconnected sets will turn into connected tangent sets. We show that for certain totally disconnected self-affine sets, the tangent sets have a product structure with connected fibers, like attractors of differentiable dynamical systems. The important assertion is not that such tangent sets exist—this is fairly easy to see. The surprising fact is that essentially *all tangent sets have this form.* We conjecture that this is the typical local structure for large classes of self-affine sets, with and without the open set condition.

Basic definitions. We consider a family of plane self-affine Cantor sets in the unit square $Q = [0, 1]^2$, which includes the types studied by Bedford [**5**], McMullen [**15**], and Gatzouras and Lalley [**8**]. For $j \in J = \{1, \ldots, m\}$, let $f_j : Q \to Q$ denote a contractive map of the form

$$f_j(x_1, x_2) = (r_j x_1, s_j x_2) + (a_j, b_j) \quad \text{with } 0 < s_j < r_j < 1,$$

such that the rectangles $R_j = f_j(Q)$ are disjoint subsets of Q. The self-affine set generated by $f_1, \ldots, f_m$ is the unique closed non-empty subset E of Q which satisfies

$$E = f_1(E) \cup \cdots \cup f_m(E);$$

see [**3, 6**]. An example is given in Figure 1(a). The assumption $s_j < r_j$ will be essential for our results; see [**2**, Example 10].

For words $u = j_1 \cdots j_n \in J^n$, we consider the mapping $f_u = f_{j_1} \circ \cdots \circ f_{j_n}$ and the rectangle $R_u = f_u(Q)$. In the figure, the rectangles R_u are shaded in gray tones, depending only on the length n of the word. Thus, each gray tone colors a set $E^{(n)} = \bigcup_{u \in J^n} R_u$, and a darker color is used for larger n. The $E^{(n)}$ form a decreasing sequence with $E = \bigcap_{n=1}^{\infty} E^{(n)}$, so black indicates the set E. Since the R_j are disjoint, E is a Cantor set, i.e., totally disconnected without isolated points. Actually, the address map $\pi : J^\infty \to E$ defined by $\pi(j_1, j_2, \ldots) = \bigcap_{n=1}^{\infty} R_{j_1 \cdots j_n}$ is a homeomorphism; see [**3**].

If we magnify E, the local structure appears to be a product set of the interval $[0, 1]$ and a Cantor set $C \subset \mathbb{R}$, as indicated in Figure 1(c). Similar computer experiments show that such a local fiber structure exists virtually everywhere. At first sight, it seems surprising that a Cantor set will develop connected components when it is magnified. Our proof will show that this is a result of the different contraction factors in the horizontal and vertical directions. We have chosen simple assumptions to produce a clear argument. We conjecture, however, that the local fiber structure is typical for general self-affine sets where the f_i can involve different rotations and the R_i can heavily overlap.

Tangent sets. To formulate our result in rigorous terms, we use tangent sets, which are defined like tangential measures [**14**, Ch. 14], but in a topological setting. A tangent set is a bit more special than a Furstenberg micro-set [**7**]. The difference is that we fix a basic

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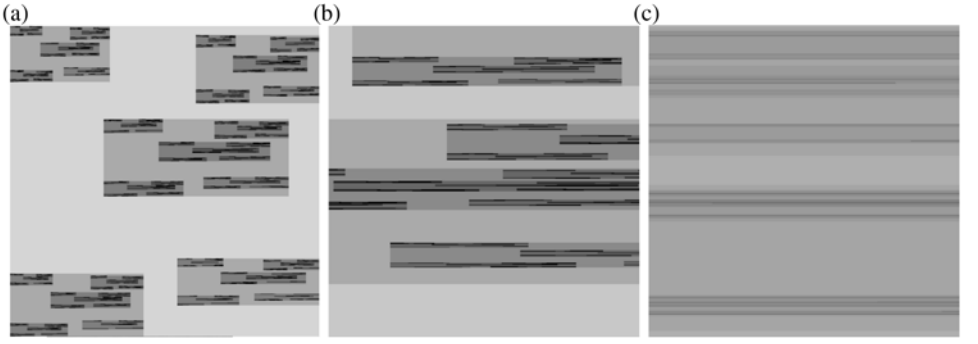


FIGURE 1. (a) $R_j = f_j(Q)$ in the unit square Q. (b) Magnification, $0.65 < x_1 < 0.77$, $0.54 < x_2 < 0.66$. (c) Magnification of a smaller window $[0.73687, 0.736873] \times [0.61145, 0.611453]$ shows the typical fiber structure, which remains preserved under further magnification.

point $x \in E$ at which magnification will proceed, as in [**4, 9, 12, 16**]. As a result of our rectangular setting, it will be convenient to define bounded tangent sets as subsets of the unit square Q, as in [**7**]. All tangent sets contain the center of Q, which corresponds to the basic point x.

For a point $x = (x_1, x_2)$ in E and a positive number $t < 1/2$, we consider the square neighborhood $Q_{x,t} = [x_1 - (t/2), x_1 + (t/2)] \times [x_2 - (t/2), x_2 + (t/2)]$ and the normalizing map

$$h_{x,t} = h : Q_{x,t} \to Q \quad \text{with } h(y) = \frac{y - x}{t} + \left(\frac{1}{2}, \frac{1}{2}\right).$$

The set

$$N_{x,t} = h(E \cap Q_{x,t}) \subset Q$$

is the *normalized view* of E in the neighborhood $Q_{x,t}$. A *tangent set* of E at x is a limit, with respect to the Hausdorff metric (see §2), of a sequence of normalized views N_{x,t_n} with $t_n \to 0$.

If E were a differentiable curve, there would be exactly one tangent set at x, namely the line segment through the center $(\frac{1}{2}, \frac{1}{2})$ of Q with the slope of the tangent line of E at x. As a result of compactness, tangent sets do exist at all points x of any set E. In the case of self-similar fractals, there is a large family of tangent sets at x, and these families will coincide for all 'typical' points x. All tangent sets of a self-similar fractal Cantor set are isometric to rather large subsets of E (see [**1**]).

Bernoulli measures. For truly self-affine Cantor sets E, we want to show that tangent sets contain connected fibers: all tangent sets at x have a product structure $[0, 1] \times C$, as in Figure 1(c). However, if x is on the left borderline of Q or of some R_u, then all tangent sets at x will contain no point (y_1, y_2) with $y_1 < \frac{1}{2}$. Thus, we have to confine ourselves to 'typical' points x.

To this end, we consider a Bernoulli measure ν_p on the address space J^∞ defined by some probability vector $p = (p_1, \ldots, p_m)$ with $p_j > 0$ and $\sum p_j = 1$, that is, $\nu_p\{(j_1, j_2 \cdots) \mid j_k = i_k \text{ for } k = 1, \ldots, n\} = p_{i_1} \cdot \cdots \cdot p_{i_n}$. Since the address map $\pi : J^\infty \to E$ is one-to-one, we can consider ν_p as a probability measure for the points x of E.

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Let M denote the maximum number of rectangles R_j, $j \in J$, which intersect a vertical segment $\{x_1\} \times [0, 1]$, taken over $0 \le x_1 \le 1$. Now we can formulate our main result.

THEOREM 1. *Let $E \subset Q$ be the self-affine Cantor set corresponding to the maps $f_j(x_1, x_2) = (r_j x_1, s_j x_2) + (a_j, b_j)$, $j = 1, \ldots, m$, where $0 < s_j < r_j < 1$, and the rectangles $R_j = f_j(Q)$ are disjoint subsets of Q. We assume that for some $\tilde{n} \ge 1$, each vertical segment $\{x_1\} \times [0, 1]$ intersects at least two rectangles R_u with $u \in J^{\tilde{n}}$. Moreover, let ν_p be the Bernoulli measure on $\{1, \ldots, m\}^\infty$ associated with an arbitrary probability vector $p = (p_1, \ldots, p_m)$, with $0 < p_j < 1/M$ for all j.*

Then, for ν_p almost all points x in E, all tangent sets of E at x have the form $[0, 1] \times C$, where $C \subset [0, 1]$ is a Cantor set.

In Figure 1 we have $M = 3$, and the condition on vertical segments is fulfilled for $\tilde{n} = 2$. This condition requires that two pairs of rectangles touch the right and left border of Q, respectively. This is used to guarantee that a line segment cannot be a tangent set (see Proposition 3). This condition as well as the requirement $p_j < 1/M$ can probably be relaxed by using measure-theoretic methods. We note that the theorem can be generalized to the d-dimensional setting, using maps $f_j(x_1, x_2, \ldots, x_d) = (r_j x_1, s_{2j} x_2, \ldots, s_{dj} x_d) + (b_{1j}, b_{2j}, \ldots, b_{dj})$ with $0 < s_{kj} < r_j < 1$. The proof of Theorem 1 is given in §2.

The next theorem says that our self-affine sets have 'uniformly scaling scenery' in the sense of Furstenberg, Gavish and Hochman.

THEOREM 2. *Under the assumptions of Theorem 1, there is a set $\tilde{E} \subset E$ with $\nu_p(\tilde{E}) = 1$ for every Bernoulli measure ν_p with $0 < p_j < 1/M$, such that for each $x \in \tilde{E}$, each tangent set T at x is a tangent set at each other point y of $\tilde{E}$. Thus, the family of all tangent sets depends on neither the point x nor on the particular measure ν_p.*

We prove that $E \setminus \tilde{E}$ is a zero set for many Bernoulli measures. One might ask if the Hausdorff dimension of this exceptional set is small. Unfortunately, our technique does not provide such estimates. It is also worth noting that there exist Bernoulli measures satisfying the assumptions of Theorem 2 and having an arbitrarily small Hausdorff dimension. Although zero sets of such measures can be large in other senses, we feel that the notion of smallness in Theorem 2 is natural since Bernoulli measures form a geometrically meaningful family of explicitly known measures for our setting of self-affine sets.

In §3, we prove Theorem 2 and discuss the structure of the Cantor sets C in the tangent set $T = [0, 1] \times C$. It will be seen that C is almost a subset of E. Under some additional conditions, we specify the distribution with respect to ν_p of the random Cantor set C.

2. *Structure of tangent sets*

We introduce some notation. Let

$$\delta = \min\{|x - y| \mid x \in R_i \text{ and } y \in R_j \text{ for some } i, j \in J \text{ with } i \ne j\},$$

$s^* = \max_{j=1}^m s_j$ and $s_* = \min_{j=1}^m s_j$. Similarly, r^*, r_* and p^*, p_* denote the maximum and minimum of r_j and p_j, respectively. We first prove that there is a uniform lower bound for the vertical extension of E in all normalized views of E.

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PROPOSITION 3. *Assume the conditions of Theorem 1.*

(i) *For $0 \le x_1 \le 1$, the set $(\{x_1\} \times [0, 1]) \cap E$ has a diameter of at least $\delta s_*^{\tilde{n}-1}$.*

(ii) *For a normalized view $N_{x,t}$, with $t < \delta s_*^{\tilde{n}-1}$, the set $(\{\frac{1}{2}\} \times [0, 1]) \cap N_{x,t}$ has a diameter of at least $\frac{1}{2}\delta s_*^{2\tilde{n}-1}$.*

Proof. (i) We use the condition on vertical segments. It implies that the sets $E^{(k\tilde{n})}$, $k = 1, 2, \ldots$, intersect each segment $\{x_1\} \times [0, 1]$. So this is also true for E, and by self-affinity for each set, $E \cap R_u$. Thus, if the segment intersects both R_u and R_v, with $u, v \in J^{\tilde{n}}$, it contains a point of E in both of the rectangles. The distance of R_u and R_v on the segment is at least δs_*^n, where n is the length of the common prefix of u and v. Since $n \le \tilde{n} - 1$, the estimate is true.

(ii) Let k be the largest integer for which $A := (\{x_1\} \times [0, 1]) \cap Q_{x,t}$ intersects only one rectangle R_u with $u \in J^{k\tilde{n}}$. The assumption $t < \delta s_*^{\tilde{n}-1}$, together with the proof of (i), implies $k \ge 1$. We apply the proof of (i) again to the two subrectangles of R_u of order $(k + 1)\tilde{n}$ which intersect A. We obtain

$$\operatorname{diam} A \cap E \ge s_u \cdot \delta s_*^{\tilde{n}-1}.$$

If (ii) did not hold for $N_{x,t}$, the above diameter would be smaller than $t \cdot (1/2)\delta s_*^{2\tilde{n}-1}$, which implies $t/2 > s_u/s_*^{\tilde{n}}$. Now, if u' denotes the prefix of u of length $(k - 1)\tilde{n}$, we have $t/2 > s_{u'}$. Since $x \in R_{u'}$, this means that $(\{x_1\} \times [0, 1]) \cap R_{u'}$ is contained in $Q_{x,t}$. On the other hand, the vertical segment condition says that as well as R_u, another subrectangle $R_v \subset R_{u'}$, with $v \in J^{k\tilde{n}}$, will intersect $\{x_1\} \times [0, 1]$. This contradicts the choice of k. □

Let us recall the definition of the Hausdorff distance between two sets A and B (see [**3**]). We have $d_H(A, B) \le \varepsilon$ if and only if for every point $x \in A$, there is $y \in B$ with $|x - y| \le \varepsilon$, and for each $y \in B$, there is $x \in A$ with $|x - y| \le \varepsilon$.

For a given $\varepsilon > 0$, a subset S of Q is called an *ε-pattern* if $S = [0, 1] \times \bigcup_{k=1}^{\ell} I_k$, where ℓ is a positive integer and I_k are disjoint closed intervals of positive length $\le \varepsilon$. Proposition 5 below will show that normalized views of E are near to such patterns. The following statement then implies that tangent sets have product structures.

PROPOSITION 4. *Let $\varepsilon_i > 0$, $i = 1, 2, \ldots$, be a sequence converging to 0, and let S_i be a sequence of ε_i-patterns, and $T \subset Q$ the limit set, with $d_H(S_i, T) < \varepsilon_i$ for all i. If the area of S_i converges to zero, T has the form $T = [0, 1] \times C$, where C is a nowhere dense subset of $[0, 1]$ with zero length.*

Proof. For each i, T is contained in the ε_i-neighborhood $S_i' = [0, 1] \times \bigcup_{k=1}^{\ell_i} I_k'$ of S_i. Here, $I_k' = [c_k - \varepsilon_i, d_k + \varepsilon_i]$, when $I_k = [c_k, d_k]$. Thus, $T \subset \bigcap_{i=1}^{\infty} S_i'$, which has the form $[0, 1] \times C$ with a closed set C. Since the area of S_i' is at most three times the area of S_i, the area of T is zero. So, C has length zero and is nowhere dense. □

Definition of approximate normalized views $P_{x,t}^K$. Let $N_{x,t} = h(E \cap Q_{x,t})$ with $x \in E$, and $t < \frac{1}{2}$ be a normalized view. Let $j_1 j_2 \cdots$ denote the address of x, and let $n = n(x, t)$ denote the largest integer for which $Q_{x,t}$ intersects only one rectangle R_u with $u = j_1 \cdots j_n \in J^n$. Then $Q_{x,t}$ intersects two different subrectangles $R_{uj_{n+1}}$ and R_{ui} with $i \ne j_{n+1}$. Since their distance is at least δs_u, we have

$$\delta s_*^n \le \delta s_u < t\sqrt{2}. \tag{1}$$

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The approximation of $N_{x,t}$ with approximation level K is now defined as

$$P_{x,t}^K = h(E^{(n+K)} \cap Q_{x,t}) = h\left(Q_{x,t} \cap \bigcup_{w \in J^{n+K}} R_w\right). \tag{2}$$

This is a union of rectangles in Q, as indicated in Figure 1 by one gray tone. The following technical statement says that for small t and almost all x, none of the rectangles will end within Q, and so $P_{x,t}^K$ will be an ε-pattern. This is the key to the proof of Theorem 1 at the end of this section.

PROPOSITION 5. *Let the conditions of Theorem 1 be fulfilled, and let* $\varepsilon_K = (s^*)^K \cdot \sqrt{2}/\delta$, *for some positive integer* K; $P_{x,t}^K$ *have the following properties.*

(i) *Each* $P_{x,t}^K$ *is a union of closed rectangles of height at most* ε_K.

(ii) $P_{x,t}^K$ *contains* $N_{x,t}$ *and* $d_H(N_{x,t}, P_{x,t}^K) < \varepsilon_K$.

(iii) *If* $P_{x,t}^K$ *is an* ε_K*-pattern, it has area at most* $(1-\delta)^{K-\tilde{k}}$, *where* $\tilde{k}$ *denotes the integer part of* $(\log \delta/2)/\log s^*$.

(iv) *The set* $B_K = \{x \in E \mid P_{x,t}^K$ *is not an* ε_K*-pattern for arbitrarily small* $t\}$ *has measure* $\nu_p(B_K) = 0$.

Proof. (i) $P_{x,t}^K$ is a union of closed rectangles in Q. Since each R_w has height s_w, and by (1),

$$s_w \le s_u \cdot (s^*)^K \le t \cdot \frac{\sqrt{2}}{\delta} \cdot (s^*)^K = t \cdot \varepsilon_K,$$

we conclude that the rectangles $h(Q_{x,t} \cap R_w)$ have height at most ε_K.

(ii) By construction, $P_{x,t}^K \supset N_{x,t}$, and for each point $y \in P_{x,t}^K$, there is a point $z \in N_{x,t}$ in the same $h(R_w)$ with $y_1 = z_1$, by the vertical segment condition. So the Hausdorff distance between $N_{x,t}$ and $P_{x,t}^K$ is bounded by ε_K.

(iii) We estimate the area of $P_{x,t}^K$. If this set is an ε_K-pattern, it consists of rectangles which have their endpoints at $x_1 = 0$ and $x_1 = 1$. The same holds for $P_{x,t}^{K'} = h(E^{(n+K')} \cap Q_{x,t})$ with $0 \le K' < K$. For each rectangle R_w of height s_w with $w \in J^{n+K'}$, there is at least one empty strip of height at least δs_u between the subrectangles R_{wi}, $i \in J$. Thus, when we go from $P_{x,t}^{K'}$ to $P_{x,t}^{K'+1}$, the area decreases by a factor smaller than or equal to $1-\delta$. When we start with an area of at most 1 for $P_{x,t}^0$, induction gives an of area at most $(1-\delta)^K$ for $P_{x,t}^K$.

However, there may be two rectangles at the upper and lower border of $Q_{x,t}$ for which this argument may not work since they are only partially contained in the view. If $s_u \le t$, we can do the induction with all R_u instead of the part in the square. If $s_u > t$, we start the induction with $\tilde{k}$, where $2(s^*)^{\tilde{k}} \le \delta$, and we estimate the area, including the two possible rectangles R_w, $w \in J^{n+\tilde{k}}$, on the boundary, and their subrectangles in the induction, which may be outside the square. The definition of $\tilde{k}$ guarantees that those two R_ws together have less area than the empty strip of height $\delta s_u > t\delta$. So we start with an area ≤ 1 at $K' = \tilde{k}$ and perform the induction as before. This completes the proof of (iii).

(iv) *Borel–Cantelli argument.* By (1), $n = n(x, t)$ tends to infinity for fixed x and $t \to 0$. Thus, we can write $P_{x,n}^K$ instead of $P_{x,t}^K$, and express B_K using n instead of t :

$$B_K = \{x \in E \mid P_{x,n}^K \text{ is not an } \varepsilon_K\text{-pattern for infinitely many } n\}.$$

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We shall determine an integer n_*, and for all $n \geq n_*$, an exceptional set $A_{n,K}$, such that

$$P_{x,n}^K \text{ is an } \varepsilon_K\text{-pattern for } x \notin A_{n,K}$$

and

$$\sum_{n=n_*}^{\infty} \nu_p(A_{n,K}) < \infty. \tag{3}$$

Then

$$B_K \subset \{x \in E \mid x \in A_{n,K} \text{ for infinitely many } n \geq n_*\}.$$

By the Borel–Cantelli lemma, the right-hand side is a ν_p null set. Thus, $\nu_p(B_K) = 0$, and (iv) is proved.

Construction of $A_{n,K}$. We investigate for which x and n the set $P_{x,n}^K$ is an ε_K-pattern. Since the height of the rectangles was at most ε_K, the requirement is that all rectangles reach from $x_1 = 0$ to $x_1 = 1$. This is not always true; see Figure 1(a) and (b). The idea is that for a larger n, the rectangle R_u becomes longer and narrower, whereas for fixed u and K, only the endpoints a_w, b_w of the $R_w = [a_w, b_w] \times [c_w, d_w]$ must be avoided by $Q_{x,t}$, where the number m^K of words $w \in J^{n+K}$ with prefix u is constant. Thus, for very large n, the set of exceptional points x should be small.

We choose a sufficiently large integer L, such that

$$(p^*M)^L \cdot 4m^K < 1. \tag{4}$$

This L exists since $p^*M < 1$ was assumed in Theorem 1. To define $A_{n,K}$, we fix $u \in J^n$ and consider the definition (2) of $P_{x,n}$ as a union of the R_ws, where w is a word of length $n + K$ with prefix u. We say that $v = ui_1 \cdots i_L$ is a forbidden word if R_v intersects a vertical segment $\{c\} \times [0, 1]$, where c is one of the $4m^K$ values $a_w - (t/2)$, $a_w + (t/2)$, $b_w - (t/2)$, $b_w + (t/2)$. Each vertical segment intersects at most M^L rectangles R_v, and $\nu_p(R_v) \leq \nu_p(R_u) \cdot p^{*L}$ for each v. (For a Bernoulli measure, $\nu_p(R_v) = \nu_p(R_v \cap E) = \prod_{k=1}^{n+L} p_{v_k}$.) By (4), the union of all R_v over all forbidden words v has ν_p-measure smaller than $\nu_p(R_u)$.

Now, let $A_{n,K}$ be the union of $R_v \cap E$ over all $u \in J^n$ and all forbidden words $v = ui_1 \cdots i_L$. Note that by self-affinity, the set of forbidden words for each $u \in J^n$ is defined by the same suffixes $i_1 \cdots i_L$. Then $\sum_{u \in J^n} \nu_p(R_u) = 1$ and (4) imply $\nu_p(A_{n,K}) < 1$. The set $A_{n,K}$ has the following property. When $x \in E \backslash A_{n,K}$ and t is chosen so that $Q_{x,t}$ intersects only one rectangle R_u with $u \in J^n$, but two rectangles of order $n + 1$, then $P_{x,t}^K$ is an ε_K-pattern.

Convergence of the sum. To complete the above argument, we show that all rectangles R_v of order $n + L$ have length greater than t. This will work only for sufficiently large n. Let

$$q = \min_{j=1}^{m} \frac{r_j}{s_j}.$$

By assumption, $q > 1$. For the rectangle R_u, the length is at least q^n larger than the height. By Proposition 3 (ii), the height s_u of R_u is at least $(t/2)\delta s_*^{2\tilde{n}-1}$. So, the length of R_u is

$$r_u \geq q^n s_u \geq q^n \frac{t}{2} \delta s_*^{2\tilde{n}-1}.$$

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The condition $r_v \geq r_u r_*^L > t$ is now implied by the following restriction on n:

$$q^n \cdot r_*^L > 2\delta^{-1} s_*^{1-2\tilde{n}}. \tag{5}$$

Let n_* denote the smallest integer n for which this inequality is fulfilled, with L defined by (4). Then the exceptional set $A_{n,K}$ could be properly defined for $n \geq n_*$. However, to obtain convergence in (3), we now modify the definition. We define $A_{n,K}$ with $\nu_p(A_{n,K}) < 1$ in the above way only for $n_* \leq n < n_* + \ell$, where ℓ is the smallest integer with $q^\ell r_* \geq 1$. For $n_* + \ell \leq n < n_* + 2\ell$, we define $A_{n,K}$ by forbidden suffixes of length $L+1$ instead of L. The condition (5) will hold and guarantee that R_v have length larger than t, while modification by (4) shows that $\nu_p(A_{n,K}) < p^* M$.

In general, we define $A_{n,K}$, for $n_* + k\ell \leq n < n_* + (k+1)\ell$, by forbidden suffixes of length $L + k$, and obtain $\nu_p(A_{n,K}) < (p^* M)^k$ from (5). This implies

$$\sum_{n=n_*}^{\infty} \nu_p(A_{n,K}) < \frac{\ell}{1 - p^* M} < \infty,$$

so the above Borel–Cantelli argument works. □

Proof of Theorem 1. The set $B = \bigcup_{K=1}^{\infty} B_K$ fulfills $\nu_p(B) = 0$. Take x outside this set, and take a tangent set $T = \lim_{t_i \to 0} N_{x,t_i}$ of E at x. Since x is not in B_K, we find for every K a number t_i, such that P^K_{x,t_i} is an ε_K-pattern and $d_H(T, N_{x,t_i}) \leq \varepsilon_K$, hence $d_H(T, P^K_{x,t_i}) \leq 2\varepsilon_K$. The areas of these patterns converge to zero. Proposition 4, with $2\varepsilon_K$ instead of ε_i, now shows that $T = [0, 1] \times C$, where C is nowhere dense and has length zero.

It remains to show that C cannot contain isolated points and thus is a Cantor set. Let us assume that C contains an isolated point c, with distance $4\alpha > 0$ from the rest of C. The following calculation verifies that there are normalized views of E contained in arbitrary thin horizontal strips, which contradicts Proposition 5.

For each given $\beta \in (0, \alpha]$, there is a number t such that the normalized view $N_{x,t}$ satisfies $d_H(T, N_{x,t}) < \beta$. Thus,

$$\{y \in N_{x,t} \mid |y_2 - c| < 3\alpha\} = \{y \in N_{x,t} \mid |y_2 - c| < \beta\} \neq \emptyset.$$

We take a point y from this set and let $z = h_{x,t}^{-1}(y)$ and $s = t(\alpha + \beta)$. Then $N_{z,s} \subset [0, 1] \times [1/2 - (2\beta/(\alpha+\beta)), 1/2 + (2\beta/(\alpha+\beta))]$. For small β, this contradicts Proposition 3(ii). □

3. *Uniformly scaling scenery*

After proving Theorem 1, we can focus on finding the structure of the Cantor set $C \subset [0, 1]$. We restrict ourselves to points x outside the exceptional set $B = \bigcup_{K=1}^{\infty} B_K$ in the proof of Theorem 1.

PROPOSITION 6. *The set C in each tangent set $T = [0, 1] \times C$ at a point $x \in E \backslash B$ is a limit of sets C_k which are normalized intersections of vertical segments with E.*

Proof. T is a limit of normalized views N_{x,t_k}. We show that C is the limit of a subsequence of the vertical sections $C_k = N_{x,t_k} \cap (\{\frac{1}{2}\} \times [0, 1])$. Using Proposition 5(iv), we choose

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the subsequence (C_K), such that $d_H(T, N_{x,t_K}) < \varepsilon_K$ and P^K_{x,t_K} is an ε_K-pattern. Then $P^K_{x,t_K} = [0, 1] \times C_K$. Since $d_H(N_{x,t_K}, P^K_{x,t_K}) < \varepsilon_K$ by Proposition 5(ii), P^K_{x,t_K} converge to T. Hence C_K converges to C. □

Below we shall consider examples where all sets C are themselves normalized vertical sections of E. However, it is not hard to find examples where this is not always the case. The following argument indicates, however, that these can be considered as exceptions. Given C_K, let V_K denote a vertical segment in the unit square, such that C_K is a normalized image of $V_K \cap E$. Then the midpoint of V_K must belong to E since $(\frac{1}{2}, \frac{1}{2})$ belongs to each tangent set. If $V_K \subset R_j$ for some rectangle, then we can take the larger homothetic image $f_j^{-1}(V_K)$ instead of V_K. In other words, we can assume that V_K intersects at least two of the rectangles $R_1, \ldots, R_m$. So V_K and C_K do not differ much in size.

Theorem 1 and the above argument can be roughly summarized as follows: *essentially all tangent sets are products of the horizontal component* $[0, 1]$ *and a vertical component* C*, which appears in* E *on a rather large scale.* For the vertical component, the case is almost the same as for self-similar Cantor sets [**1**]. Now we prove Theorem 2, which says that each of these tangent sets can be found at essentially all points of E.

Proof of Theorem 2. Definition of $\tilde{E}$. The set $B = \bigcup_{K=1}^{\infty} B_K$ in the proof of Theorem 1 has measure zero for all Bernoulli measures ν_p with $0 < p_j < 1/M$. We define $\tilde{E}$ as the set of all $x \in E \setminus B$ for which the address contains all words w from the alphabet J. Of course, if all words of arbitrary length are contained in $j_1 j_2 j_3 \cdots$, then each word w is contained infinitely often. The ergodic theorem for Bernoulli shifts implies that this property holds for ν_p almost all addresses $j_1 j_2 j_3 \cdots$. Thus, $\nu_p(\tilde{E}) = 1$ for all the measures ν_p.

Main idea. Now let T be a tangent set of E at some point $x \in \tilde{E}$, and y another point in $\tilde{E}$. We want to show that T is a tangent set at y. For an arbitrary K, we choose, with Proposition 5, a normalized view $N_{x,t}$, such that $d_H(T, N_{x,t}) < \varepsilon_K$, and $P = P^K_{x,t}$ is an ε_K-pattern. We shall find arbitrarily small $s > 0$, such that $N_{y,s}$ satisfies $d_H(P, P^K_{y,s}) < \varepsilon_K$. For small s, Proposition 5 implies

$$d_H(T, N_{y,s}) \le d_H(T, N_{x,t}) + d_H(N_{x,t}, P) + d_H(P, P^K_{y,s}) + d_H(P^K_{y,s}, N_{y,s}) < 4\varepsilon_K.$$

Performing this construction for every ε_K, we see that T is a tangent set at y, and Theorem 2 is true.

Finding $N_{y,s}$. We make an assumption on boundary effects which will be verified below. We assume that the rectangles R_w in the definition (2) of the ε_K-pattern $P = P^K_{x,t}$ do not end exactly at the left, right, upper or lower border of the square $Q_{x,t}$. When this condition holds, let $\delta < \varepsilon_K$ be smaller than the distance of any side of a rectangle R_w to a parallel border of the square $Q_{x,t}$. By assumption $\delta > 0$, and for $N_{z,t}$ with $|z - x| < \delta$, the approximation $P^K_{z,t}$ satisfies $d_H(P_{z,t}, P) < \delta$. Moreover, we find a prefix w of the address of x such that for all $z \in R_w$, we have $|z - x| < \delta$.

Now we use self-affinity: for all y with an address of the form uwv, where u is a word and v a sequence, and for $s = r_u t$, the approximation $P^K_{y,s}$ must coincide with some $P_{z,t}$. Since $f_u(R_w) = R_{uw}$, the vertical structure of the strips is the same, and the horizontal structure consists of lines anyway. By the definition of $\tilde{E}$, the address of the given point y contains the word w infinitely many times, so we find $N_{y,s}$ with arbitrarily small s.

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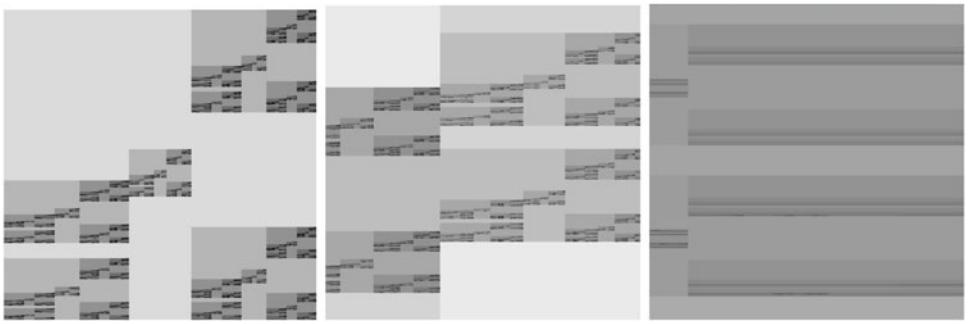


FIGURE 2. Example fulfilling the conditions of Proposition 7. Magnifications of squares with side length 0.12 and 2×10^{-8}, respectively. All vertical sections of E are self-similar random Cantor sets.

Removing boundary effects. If the assumption on $N_{x,t}$ is not satisfied, we use the fact that the 'magnification flow acts on the tangent sets' [**12**]. We take a tangent set T' which contains T in the following sense: $T = h(T' \cap [\alpha/2, 1 - (\alpha/2)]^2)$, where $h(z) = (z - m)/(1 - \alpha) + m$, with $m = (1/2, 1/2)$ and some $\alpha > 0$. Since the space of compact subsets of Q with d_H is compact, T' can be taken as the limit of a subsequence of N_{x,t'_k}, where $t'_k = t_k/(1 - \alpha)$ and N_{x,t_k} converges to T. When we require that the level-K approximation of N_{x,t'_k} is an ε_K-pattern, then the rectangles of P^K will extend beyond the left and right borders for every $N_{x,\overline{t_k}}$ with $t_k < \overline{t_k} < t'_k$. Moreover, for each k, there is at most a finite number of exceptional values $\overline{t_k}$ between t_k and t'_k for which the upper or lower border of the square $Q_{x,\overline{t_k}}$ coincides with the upper or lower border of a rectangle of approximation level K. Thus, we find a number $\beta \in (0, \alpha)$ such that $\overline{t_k} = t'_k(1 - \beta)$ is not an exception for any k. The above proof now works for $\overline{T} = h(T' \cap [\beta/2, 1 - (\beta/2)]^2)$, with $h(z) = (z - m)/(1 - \beta) + m$ instead of T. This shows that $N_{y,\overline{s_k}}$ converges to $\overline{T}$ for some sequence $(\overline{s_k})$. Then N_{y,s_k} with $s_k = (1 - \alpha)/(1 - \beta) \cdot \overline{s_k}$ converges to T. □

Attractors of hyperbolic dynamical systems have the local structure $[0, 1] \times C$, where the Cantor set C changes continuously when we run with x along a trajectory. For our self-affine sets, there are no trajectories. When we change x_1, the set C will change its topological structure whenever x passes the boundary of a rectangle. We shall briefly explain why in our case C should be interpreted as a random Cantor set. Instead of the family of all tangent sets, we could also have studied the distribution of the sets T or C with respect to ν_p, see [**1, 10, 16**]; but we restrict ourselves to a simple class of examples.

We assume the following condition due to Gatzouras and Lalley [**8**]. Let $\pi(x_1, x_2) = x_1$ denote the projection on the horizontal axis. We assume that the intervals $\pi(R_j)$, $j \in J$, either coincide or have no interior points in common. We also require that the union of the projections is $[0, 1]$, which is weaker than the vertical segment condition of Theorem 1. Figure 2 shows an example.

Thus, we have numbers $0 = a_1 < a_2 < \cdots < a_k < a_{k+1} = 1$, such that for each $j \in J$, there is an i_j with $\pi(R_j) = [a_{i_j}, a_{i_j+1}]$. Let $J_i = \{j \in J \mid i_j = i\}$ for $i = 1, \ldots, k$. Then each f_j with $j \in J_i$ has the form

$$f_j(x_1, x_2) = (r_i x_1 + a_i, s_j x_2 + b_j) = (h_i(x_1), g_j(x_2)). \tag{6}$$

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Here, $r_i = a_{i+1} - a_i$, so the one-dimensional self-similar set generated by $h_1, \ldots, h_k$ is the unit interval. The intervals $h_i([0, 1])$ are disjoint, so every point $x_1 \in [0, 1]$ has an address $i_1 i_2 \cdots \in \{1, \ldots, k\}^\infty$, and only the endpoints of subintervals have two addresses. Now we can accurately describe the Cantor sets C.

PROPOSITION 7. (Cf. [**5**]) *Let the self-affine set E be generated by the mappings f_j defined in (6). Then for each $x_1 \in [0, 1]$ with address $i_1 i_2 \cdots \in \{1, \ldots, k\}^\infty$, the set $C = E \cap (\{x_1\} \times [0, 1])$ has the representation*

$$C = \{x_2 \mid \text{ there are } j_k \in J_{i_k},\ k = 1, 2, \ldots, \text{ with } \lim_{n\to\infty} g_{j_1} \circ \cdots \circ g_{j_n}(0) = x_2\}.$$

In other words, the sets J_i, $i = 1, \ldots, k$, define k families of one-dimensional functions $g_j(t) = s_j t + b_j$, $j \in J_i$. The set C is the random self-similar subset of $[0, 1]$ obtained by applying a function in J_{i_1} to a function in J_{i_2} to a function in J_{i_3}, etc. When x_1 is one of the countably many points with two addresses, we obtain the union of the two corresponding Cantor sets. The proof is a simple calculation.

To speak of a random construction, we still need a probability measure. For a Bernoulli measure ν_p on J^∞ with probability vector $(p_1, \ldots, p_m)$, we define $P_i = \sum_{j\in J_i} p_j$ for $i = 1, \ldots, k$. Then at each level, we choose an $i \in \{1, \ldots, k\}$ with probability P_i, and apply all the mappings f_j with $j \in J_i$. This is equivalent to saying that the probability space is $\Omega = [0, 1]$, and the probability measure is the self-similar measure generated on $[0, 1]$ by the similarity maps h_i and weights P_i, with $i = 1, \ldots, k$. A good choice of p_j depends on the size of R_j, but we do not go into details here.

Fine dimension estimates for random fractal constructions are known to involve logarithmic corrections [**11**]. In fact, the detailed Hausdorff dimension calculations of self-affine Gatzouras–Lalley carpets by Peres [**17**] involve logarithmic corrections, different from those in [**11**], as a result of the different probabilistic construction.

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