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NOTES ON GENERALISED INTEGRAL POLYNOMIAL PELL EQUATIONS

NGUYEN XUAN THO

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Abstract

Given a nonzero integer n, Gupta and Saha ['Integer solutions of the generalised polynomial Pell equations and their finiteness: the quadratic case', *Canad. Math. Bull.*, to appear] classified all polynomials $x^2 + ax + b \in \mathbb{Z}[x]$ for which the polynomial Pell equation $P^2 - (x^2 + ax + b)Q^2 = n$ has solutions $P, Q \in \mathbb{Z}[x]$ with $Q \neq 0$. We generalise their work to the equation $P^2 - (f^2 + af + b)Q^2 = nR$, where f is a fixed polynomial in $\mathbb{Z}[x]$. As an application of our results, we study the equation $P^2 - D(f)Q^2 = n$, where D is a monic, quartic and non square-free polynomial in $\mathbb{Z}[x]$. This extends Theorem 1.4 of Scherr and Thompson ['Quartic integral polynomial Pell equations', *J. Number Theory* **259** (2024), 38–56].

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1. Introduction

Let R be a ring or field. A polynomial Pell equation over $R[x]$ is an equation of the form

$$P^2 - DQ^2 = 1, \tag{1.1}$$

where D is a fixed polynomial in $R[x]$. A nontrivial solution (P, Q) of (1.1) is a solution with $P, Q \in R[x]$ and $Q \neq 0$. The study of (1.1) over $\mathbb{C}[x]$ dates back to Abel [1] in 1826. In contrast, the study of (1.1) over $\mathbb{Z}[x]$ began much later, in 1976, with the seminal paper of Nathanson [11], which answered a question of Chowla and proved that the equation $P^2 - (x^2 + d)Q^2 = 1, d \in \mathbb{Z}$, has a nontrivial solution in $\mathbb{Z}[x]$ if and only if $d \in \{\pm 1, \pm 2\}$. Classifying those $D \in \mathbb{Z}[x]$ for which (1.1) admits a nontrivial solution in $\mathbb{Z}[x]$ remains an open problem. Since Nathanson's work, several authors have studied (1.1) for other classes of polynomials D.

- Gaunet [4] extended Nathanson's result to the equation

$$P^2 - (x^2 + ax + b)Q^2 = \pm 1, \quad a, b \in \mathbb{Z}.$$

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- Dubickas and Steuding [3] extended Nathanson's result to the equation

$$P^2 - (x^{2k} + d)Q^2 = 1, \quad k \in \mathbb{Z}_{>0},\ d \in \mathbb{Z}.$$

- Webb and Yokota [15–17] studied the case where D is a monic quartic polynomial in $\mathbb{Z}[x]$.
- Jedrzejak [7] investigated the equation $P^2 - (x^{2m} + ax + b)Q^2 = 1$, $m \in \mathbb{Z}_{>0}, a, b \in \mathbb{Z}$.
- Murru [10] generalised Nathanson's work to the equation

$$P^2 - (f^2 + d)Q^2 = 1, \quad d \in \mathbb{Z},\ f \in \mathbb{Z}[x]. \tag{1.2}$$

Murru [10, Theorem 2] proved that (1.2) has a nontrivial solution in $\mathbb{Z}[x]$ for any nonconstant polynomial f if and only if $d = \pm 1, \pm 2$. Unfortunately, this is false: when $d = 4 \notin \{\pm 1, \pm 2\}$, (1.2) admits the nontrivial solution $(P, Q) = (2f^2 + 1, f)$. A corrected version of Murru's theorem is given in Remark 1.4. See also the works of Hazama [6], McLaughlin [8], Mollin [9] and Schinzel [13, 14]. More recently, Gupta and Saha [5] generalised Nathanson's result to the equation

$$P^2 - (x^2 + ax + b)Q^2 = n, \tag{1.3}$$

where n is a nonzero integer and $a, b \in \mathbb{Z}$.

For the rest of this paper, Δ denotes $a^2 - 4b$ and $v_p(x)$ for a prime number p and nonzero integer x denotes the highest integer k such that $p^k \mid x$ and $p^{k+1} \nmid x$.

THEOREM 1.1 [5, Theorems 1, 2]. *Let n be a nonzero integer. Equation (1.3) has a nontrivial solution in $\mathbb{Z}[x]$ if and only if one of the following holds:*

(i) *n, Δ are both nonzero squares in $\mathbb{Z}$ and $\Delta \mid 4n$;*
(ii) *n is a nonzero square in $\mathbb{Z}$, Δ is not a square in $\mathbb{Z}$ and $\Delta \mid 8\sqrt{n}$;*
(iii) *n is not a square in $\mathbb{Z}$ and $4n/\Delta$ is a square in $\mathbb{Z}$.*

In this paper, we shall further generalise Gupta and Saha's work to the equation

$$P^2 - (f^2 + af + b)Q^2 = n,$$

where f is a fixed polynomial in $\mathbb{Z}[x]$ and n is a nonzero integer. Our main results are the following theorems.

THEOREM 1.2. *Let n be a positive integer and $f \in \mathbb{Z}[x]$ be a nonconstant polynomial with leading coefficient α, where α is a square-free positive integer coprime to $2n$. The equation*

$$P^2 - (f^2 + af + b)Q^2 = n^2, \quad a, b \in \mathbb{Z}, \tag{1.4}$$

has a nontrivial solution in $\mathbb{Z}[x]$ if and only if one of the following holds:

(i) *Δ is a nonzero square in $\mathbb{Z}$ and $\Delta \mid 4n^2$;*
(ii) *Δ is not a square in $\mathbb{Z}$, $\Delta \mid 8n\alpha$ and $f + a/2 = 0$ in $\mathbb{F}_p[x]$ for any prime p dividing both Δ and α.*

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THEOREM 1.3. *Let n be a nonsquare integer and $f \in \mathbb{Z}[x]$ be a nonconstant polynomial in $\mathbb{Z}[x]$ with leading coefficient α, where α is a square-free positive integer coprime to $2n$. The equation*

$$P^2 - (f^2 + af + b)Q^2 = n, \quad a, b \in \mathbb{Z}, \tag{1.5}$$

has a nontrivial solution in $\mathbb{Z}[x]$ if and only if $4n/\Delta$ is a nonzero square in $\mathbb{Z}$.

REMARK 1.4. In Theorems 1.2 and 1.3, if $f(x) = x$, we recover Theorem 1.1; if $f(x) = x^k$ and $n = 1$, we recover the result of Dubickas and Steuding [3]; and if $f(x) \in \mathbb{Z}[x]$ is monic and $n = 1$, we obtain a corrected version of Murru [10, Theorem 2].

As an application, we will prove two theorems which generalise a recent result of Scherr and Thompson [12, Theorem 1.4] on quartic polynomial Pell equations.

THEOREM 1.5. *Let n be a positive integer and $f \in \mathbb{Z}[x]$ be a nonconstant polynomial with leading coefficient α, where α is a square-free positive integer coprime to $2n$. Let $D \in \mathbb{Z}[x]$ be monic, quartic and non square-free. The equation*

$$P^2 - D(f)Q^2 = n^2$$

has a nontrivial solution in $\mathbb{Z}[x]$ if and only if, up to a $\mathbb{Z}$-linear change of variables $x \mapsto x + d$, D is one of:

(i) $x^2(x^2 - c^2)$, *where $c \in \mathbb{Z}$ and $c \mid n$;*
(ii) $x^2(x^2 + 2cx - 3c^2)$, *where $c \in \mathbb{Z}$ and $2c^3 \mid n$;*
(iii) $x^2(x^2 + b)$, *where $b \in \mathbb{Z}$, $b \mid 2n\alpha$ and $f = 0$ in $\mathbb{F}_p[x]$ for any prime number p dividing both b and α;*
(iv) $x^2(x^2 - 6cx - 3c^2)$, *where $c \in \mathbb{Z}$ and $54c^6 \mid n$;*
(v) $x^2(x^2 + 2cx - c^2)$, *where $c \in \mathbb{Z}$ and $c^4 \mid 2n$.*

THEOREM 1.6. *Let n be a nonsquare integer and $f \in \mathbb{Z}[x]$ be a nonconstant polynomial with leading coefficient α, where α is a square-free positive integer coprime to $2n$. Let $D \in \mathbb{Z}[x]$ be monic, quartic and non square-free. Then, the equation*

$$P^2 - D(f)Q^2 = n$$

has no solutions in $\mathbb{Z}[x]$.

REMARK 1.7. In Theorem 1.5, if $f(x) = x$ and $n = 1$, we recover [12, Theorem 1.4]. In Theorem 1.6, if $f(x) = x$ and n is a nonsquare integer, we obtain the following result.

COROLLARY 1.8. *Let n be a nonsquare integer. Then, for any monic, quartic and non square-free polynomial $D \in \mathbb{Z}[x]$, the equation*

$$P^2 - DQ^2 = n$$

has no solutions in $\mathbb{Z}[x]$.

The next theorem is the fundamental tool in our paper.

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THEOREM 1.9 (Gupta and Saha [5, Propositions 1, 2]). *Let n be a nonzero integer, R a subring of $\mathbb{C}$ containing $\sqrt{n}$ and its inverse, and D a nonconstant polynomial in $R[x]$. If the equation*

$$P^2 - DQ^2 = n \tag{1.6}$$

has a nontrivial solution in $R[x]$, then it has precisely four minimal nontrivial solutions. If (p, q) is one of these minimal solutions, then all nontrivial solutions of (1.6) in $R[x]$ are obtained as $(\pm P_k, \pm Q_k)$, where

$$P_k + Q_k\sqrt{D} = \sqrt{n}\left(\frac{p}{\sqrt{n}} + \frac{q}{\sqrt{n}}\sqrt{D}\right)^k, \quad k \geq 1.$$

Suppose that the leading coefficients of p and q are a and b, respectively. Then, the leading coefficients of P_k and Q_k are respectively

$$\sqrt{n}2^{k-1}\left(\frac{a}{\sqrt{n}}\right)^k \quad \text{and} \quad \sqrt{n}2^{k-1}\left(\frac{a}{\sqrt{n}}\right)^{k-1}\left(\frac{b}{\sqrt{n}}\right).$$

2. Proofs of the main theorems

PROOF OF THEOREM 1.2. A minimal solution of (1.4) in $\mathbb{C}[x]$ is

$$(P, Q) = \left(\frac{n(2f + a)}{\sqrt{\Delta}}, \frac{2n}{\sqrt{\Delta}}\right).$$

By Theorem 1.9, all nontrivial solutions of (1.4) in $\mathbb{C}[x]$ are $(\pm P_k, \pm Q_k)$, where

$$P_k + Q_k\sqrt{f^2 + af + b} = n\left(\frac{2f + a + 2\sqrt{f^2 + af + b}}{\sqrt{\Delta}}\right)^k, \quad k \geq 1. \tag{2.1}$$

Since α is the leading coefficient of f, by Theorem 1.9, the leading coefficients of P_k and Q_k are $n2^{k-1}(2\alpha)^k/\sqrt{\Delta}^k$ and $n2^{k-1}(2\alpha)^{k-1}/\sqrt{\Delta}^k$, respectively. Suppose (1.4) has a nontrivial solution in $\mathbb{Z}[x]$. Then, there exists a positive integer k such that P_k, $Q_k \in \mathbb{Z}[x]$. Therefore,

$$n2^{k-1}\frac{(2\alpha)^k}{\sqrt{\Delta}^k} \in \mathbb{Z} \quad \text{and} \quad n2^{k-1}\frac{(2\alpha)^{k-1}}{\sqrt{\Delta}^k} \in \mathbb{Z}. \tag{2.2}$$

Case 1: $\sqrt{\Delta} \in \mathbb{Z}$. Then, $x^2 + ax + b$ can be factorised as $x^2 + ax + b = (x - u)(x - v)$, with $u, v \in \mathbb{Z}$ and $\sqrt{\Delta} = u - v > 0$. By (2.2),

$$\frac{n2^{2k-2}\alpha^{k-1}}{(u - v)^k} = n2^{k-1}\frac{(2\alpha)^{k-1}}{\sqrt{\Delta}^k} \in \mathbb{Z}. \tag{2.3}$$

We claim that $(u - v) \mid 2n$. The case $u - v = 1$ is immediate. Consider the case $u - v \geq 2$. Let p be a prime factor of $u - v$. We consider three subcases.

Subcase (i): $p = 2$. Because $2 \nmid \alpha$, it follows from (2.3) that

$$kv_2(u - v) \leq 2k - 2 + v_2(n) < 2k + v_2(n).$$

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Hence,

$$v_2(u-v) \le 1 + v_2(n). \tag{2.4}$$

Subcase (ii): $p > 2$ and $p \mid n$. Then, $p \nmid \alpha$ and it follows from (2.3) that

$$v_p(u-v) \le k v_p(u-v) \le v_p(n). \tag{2.5}$$

Subcase (iii): $p > 2$ and $p \mid \alpha$. Since α is square-free and $\gcd(n, \alpha) = 1$, we have $v_p(\alpha) = 1$ and $v_p(n) = 0$. It follows from (2.3) that

$$k \le k v_p(u-v) \le v_p(\alpha^{k-1}) = k - 1,$$

which is impossible.

Inequalities (2.4) and (2.5) show that $(u-v) \mid 2n$. Conversely, assume $\sqrt{\Delta} = (u-v) \mid 2n$. Then, (1.4) has a nontrivial solution in $\mathbb{Q}[x]$ with $Q \in \mathbb{Z}[x]$:

$$(P, Q) = \left(\frac{2nf + na}{\sqrt{\Delta}}, \frac{2n}{\sqrt{\Delta}} \right).$$

If $2 \mid a$, then $P \in \mathbb{Z}[x]$. If $2 \nmid a$, then since $a^2 - 4b = \Delta = (u-v)^2 \mid 4n^2$, we have $u - v \mid n$. Thus, $P \in \mathbb{Z}[x]$.

Case 2: $\sqrt{\Delta} \notin \mathbb{Z}$. It follows from (2.2) that k is even. Let $k = 2m$, with $m \in \mathbb{Z}_{>0}$. Then,

$$\frac{n 2^{4m-1} \alpha^{2m-1}}{\Delta^m} = n 2^k \frac{(2\alpha)^{k-1}}{\sqrt{\Delta}^k} \in \mathbb{Z}. \tag{2.6}$$

Let p be a prime divisor of Δ. We claim that

$$v_p(\Delta) \le v_p(8n\alpha). \tag{2.7}$$

Subcase (i): $p = 2$. Then, $p \nmid \alpha$. By (2.6), we have $m v_2(\Delta) \le v_2(n) + (4m - 1)$. Hence, $v_2(\Delta) < 4 + v_2(n)$ and it follows that

$$v_2(\Delta) \le 3 + v_2(n). \tag{2.8}$$

Subcase (ii): $p > 2$ and $p \nmid \alpha$. Then, $p \mid n$. By (2.6),

$$v_p(\Delta) \le m v_p(\Delta) \le v_p(n). \tag{2.9}$$

Subcase (iii): $p > 2$ and $p \mid \alpha$. Then, $p \nmid 2n$. Since α is square-free, $v_p(\alpha) = 1$. By (2.6), $m v_p(\Delta) \le (2m - 1) v_p(\alpha) = 2m - 1$. Hence,

$$v_p(\Delta) = 1 = v_p(\alpha). \tag{2.10}$$

Inequalities (2.8), (2.9) and (2.10) show that (2.7) holds. Hence,

$$\Delta \mid 8n\alpha. \tag{2.11}$$

In $\mathbb{F}_p[x]$, we have $f^2 + af + b = (f + a/2)^2$. Note that $P_{2m} \in \mathbb{Z}[x]$ and $p \nmid 2n$. We use (2.1) and work in $\mathbb{F}_p[x]$:

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$$0 = \frac{2\Delta^m P_{2m}}{n} = \left(2f + a + 2\sqrt{f^2 + af + b}\right)^{2m} + \left(2f + a - 2\sqrt{f^2 + af + b}\right)^{2m}$$
$$= (2f + a + 2f + a)^{2m} + (2f + a - 2f - a)^{2m}$$
$$= 4^k (2f + a)^{2m}.$$

Hence,

$$f + a/2 = 0 \text{ in } \mathbb{F}_p[x] \quad \text{for all primes } p \mid \Delta. \tag{2.12}$$

Conversely, assume that both (2.11) and (2.12) hold. In (2.1), let $k = 2$. Then, (1.4) has a nontrivial solution in $\mathbb{Q}[x]$:

$$(P_2, Q_2) = \left(\frac{n(8f^2 + 8af + a^2 + 4b)}{\Delta}, \frac{n(8f + 4a)}{\Delta}\right).$$

To show $P_2, Q_2 \in \mathbb{Z}[x]$, it suffices to show that

$$P_2, Q_2 \in \mathbb{Z}_p[x] \tag{2.13}$$

for all prime divisors p of Δ, where $\mathbb{Z}_p$ is the ring of p-adic integers.

Case (i): $p = 2$. Then, $p \nmid \alpha$. By (2.8), $v_2(\Delta) \le 3 + v_2(n)$. If a is even, then

$$P_2 = \frac{n(8f^2 + 8af + \Delta + 8b)}{\Delta} \in \mathbb{Z}_2[x] \quad \text{and} \quad Q_2 = \frac{8n(f + a/2)}{\Delta} \in \mathbb{Z}_2[x].$$

Case (ii): $p > 2$ and $p \nmid \alpha$. By (2.9), $v_p(\Delta) \le v_p(n)$. Thus, $P_2, Q_2 \in \mathbb{Z}_p[x]$.

Case (iii): $p > 2$ *and* $p \mid \alpha$. We use (2.12) and work in $\mathbb{F}_p[x]$:

$$8f^2 + 8af + a^2 + 4b = 2(2f + a)^2 + 4b - a^2 = 0.$$

By (2.10), $v_p(\Delta) = 1$. It follows that $P_2, Q_2 \in \mathbb{Z}_p[x]$. Thus, (2.13) holds.

Theorem 1.2 is proved. □

PROOF OF THEOREM 1.3. Equation (1.5) has a minimal solution in $\mathbb{C}[x]$:

$$(P, Q) = \left(\frac{2\sqrt{n}f + \sqrt{n}a}{\sqrt{\Delta}}, \frac{2\sqrt{n}}{\sqrt{\Delta}}\right).$$

Hence, all nontrivial solutions of (1.5) in $\mathbb{C}[x]$ are $(\pm P_k, \pm Q_k)$, where

$$P_k + Q_k\sqrt{f^2 + af + b} = \sqrt{n}\left(\frac{2f + a + 2\sqrt{f^2 + af + b}}{\sqrt{\Delta}}\right)^k, \quad k \in \mathbb{Z}_{>0}. \tag{2.14}$$

Since α is the leading coefficient of f, by Theorem 1.9, the leading coefficients of P_k and Q_k in (2.14) are $\sqrt{n}2^{k-1}(2\alpha)^k/\sqrt{\Delta}^k$ and $\sqrt{n}2^k(2\alpha)^{k-1}/\sqrt{\Delta}^k$, respectively. Assume that (1.5) has a nontrivial solution in $\mathbb{Z}[x]$. Then, there exists a positive integer k such that $P_k, Q_k \in \mathbb{Z}[x]$. Therefore,

$$\sqrt{n}\frac{2^{2k-1}\alpha^{k-1}}{\sqrt{\Delta}^k} \in \mathbb{Z}.$$

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Since $\sqrt{n} \notin \mathbb{Z}$, we must have k odd. Let $k = 2m + 1$, with $m \in \mathbb{Z}_{\geq 0}$. Then,

$$\frac{\sqrt{n}}{\sqrt{\Delta}} \cdot \frac{2^{4m+1}\alpha^{2m-1}}{\Delta^m} \in \mathbb{Z}. \tag{2.15}$$

Let $\sqrt{n}/\sqrt{\Delta} = r/s$, where $r, s \in \mathbb{Z}_{>0}$ and $\gcd(r, s) = 1$. By (2.15),

$$\frac{2r}{s} \cdot \frac{16^m\alpha^{2m-1}}{\Delta^m} \in \mathbb{Z}. \tag{2.16}$$

We claim that

$$\frac{2r}{s} \in \mathbb{Z}. \tag{2.17}$$

If $s = 1$, then (2.17) is obvious. Assume that $s > 1$. Let p be a prime factor of s.

Case (i): $p = 2$. Let $v_2(s) = t \in \mathbb{Z}_{>0}$. If $t \geq 2$, then $v_2(\Delta) \geq 2t \geq 4$. Thus,

$$v_2(s\Delta^m) \geq 2 + 4m > v_2(2r \cdot 16^m\alpha^{2m-1}).$$

Hence, (2.16) is impossible. Therefore,

$$v_2(s) = 1 \leq v_2(2r). \tag{2.18}$$

Case (ii): $p > 2$. Since $p \nmid r$, by (2.16), we must have $p \mid \alpha$. Since α is square-free, we have $v_p(\alpha) = 1$. Since $p \mid s$, $p^2 \mid \Delta$. Thus,

$$v_p(\Delta^m) \geq 2m > v_p(\alpha^{2m-1}) = v_p(2r \cdot 16^m\alpha^{2m-1}).$$

Hence, (2.16) is impossible. Therefore,

$$v_p(s) = 0 \quad \text{for all prime } p > 2. \tag{2.19}$$

Now (2.17) follows from (2.18) and (2.19), which shows that $4n/\Delta$ is a nonzero integer square. Conversely, assume that $4n/\Delta$ is a nonzero integer square. Then, (1.5) has a nontrivial solution in $\mathbb{Q}[x]$ with $Q \in \mathbb{Z}[x]$:

$$(P, Q) = \left(\frac{2\sqrt{n}f + \sqrt{n}a}{\sqrt{\Delta}}, \frac{2\sqrt{n}}{\sqrt{\Delta}}\right).$$

If a is even, then $P \in \mathbb{Z}[x]$. If a is odd, since $4n/\Delta$ is an integer square, n/Δ is also an integer square. Hence, $P \in \mathbb{Z}[x]$.

Theorem 1.3 is proved. □

3. Proofs of Theorems 1.5 and 1.6

LEMMA 3.1. *Consider the Pell equation*

$$P^2 - (f^2 + af + b)Q^2 = n,$$

where n is a nonzero integer and a, b are integers with $a^2 - 4b \neq 0$. Let P_k, Q_k be polynomials in $\mathbb{C}[x]$ defined by

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$$P_k + Q_k\sqrt{f^2+af+b} = \sqrt{n}\left(\frac{2f+a}{\sqrt{\Delta}} + \frac{2}{\sqrt{\Delta}}\sqrt{f^2+af+b}\right)^k, \quad k \geq 1. \tag{3.1}$$

Assume that there exists a positive integer k such that $f \mid Q_k$. Then, (a, b) is one of:

- $(0, b)$, *with* $b \in \mathbb{Z} - \{0\}$ *and* $2 \mid k$;
- $(2c, -c^2)$, *with* $c \in \mathbb{Z} - \{0\}$ *and* $4 \mid k$;
- $(6c, -3c^2)$, *with* $c \in \mathbb{Z} - \{0\}$ *and* $6 \mid k$;
- $(2c, -3c^2)$, *with* $c \in \mathbb{Z} - \{0\}$ *and* $3 \mid k$.

PROOF. By (3.1),

$$\begin{aligned}\frac{Q_k}{\sqrt{n}} &= \frac{1}{2}\left(\left(\frac{2f+a}{\sqrt{\Delta}} + \frac{2}{\sqrt{\Delta}}\sqrt{f^2+af+b}\right)^k - \left(\frac{2f+a}{\sqrt{\Delta}} - \frac{2}{\sqrt{\Delta}}\sqrt{f^2+af+b}\right)^k\right) \\ &= \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1}\left(\frac{2f+a}{\sqrt{\Delta}}\right)^{k-2j-1}\left(\frac{4(f^2+af+b)}{\Delta}\right)^j. \end{aligned} \tag{3.2}$$

If $b = 0$, then $a \neq 0$ because $a^2 - 4b \neq 0$. Reducing (3.2) mod f gives

$$\frac{Q_k}{\sqrt{n}} \equiv k\left(\frac{a}{\sqrt{\Delta}}\right)^{k-1} \not\equiv 0 \pmod{f}, \quad k \geq 1.$$

If $b \neq 0$, reducing (3.2) mod f gives

$$\begin{aligned}\frac{Q_k}{\sqrt{n}} &\equiv \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1}\left(\frac{a}{\sqrt{\Delta}}\right)^{k-2j-1}\left(\frac{4b}{\Delta}\right)^j \pmod{f} \\ &\equiv \frac{1}{2}\left(\left(\frac{a+2\sqrt{b}}{\sqrt{\Delta}}\right)^k - \left(\frac{a-2\sqrt{b}}{\sqrt{\Delta}}\right)^k\right) \pmod{f}. \end{aligned} \tag{3.3}$$

Suppose there exists a positive integer k such that $f \mid Q_k$. Then, (3.3) yields

$$\left(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\right)^k = 1. \tag{3.4}$$

If $a = 0$, then (3.4) forces $2 \mid k$. Consider now $a \neq 0$.

Case 1: $b > 0$. In this case, both $a + 2\sqrt{b}$ and $a - 2\sqrt{b}$ are real numbers. Since $ab \neq 0$, $|a + 2\sqrt{b}| \neq |a - 2\sqrt{b}|$. Hence, (3.4) is impossible.

Case 2: $b < 0$. Let $\omega = (a + 2\sqrt{b})/(a - 2\sqrt{b})$. Then,

$$\omega^k = 1 \quad \text{and} \quad \omega = \frac{a^2+4b}{a^2-4b} + \frac{4a\sqrt{b}}{a^2-4b^2}. \tag{3.5}$$

Hence, ω is a root of unity of degree 2 over $\mathbb{Q}$. Therefore, $\omega \in \{\pm i, \frac{1}{2}(\pm 1 \pm \sqrt{-3})\}$.

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Subcase (i): $\omega = \pm i$. Then, (3.5) forces $4 \mid k$ and $a^2 + 4b = 0$. Thus, $a = 2c$ and $b = -c^2$, where $c \in \mathbb{Z} - \{0\}$.

Subcase (ii): $\omega = \frac{1}{2}(1 \pm \sqrt{-3})$. Then, (3.5) forces $6 \mid k$ and

$$\frac{a^2 + 4b}{a^2 - 4b} = \frac{1}{2}.$$

Hence, $4b = -a^2/3$. Thus, $a = 6c$ and $b = -3c^2$, where $c \in \mathbb{Z} - \{0\}$.

Subcase (iii): $\omega = \frac{1}{2}(-1 \pm \sqrt{-3})$. Then, (3.5) forces $3 \mid k$ and

$$\frac{a^2 + 4b}{a^2 - 4b} = -\frac{1}{2}.$$

Hence, $4b = -3a^2$. Thus, $a = 2c$ and $b = -3c^2$, where $c \in \mathbb{Z} - \{0\}$. □

The next lemma is the same as [12, Lemma 3.1].

LEMMA 3.2. *Let n be a nonzero integer and $f \in \mathbb{Z}[x]$ be a nonconstant polynomial. Let $D(x) = (x - c)^2(x^2 + ax + b) \in \mathbb{Z}[x]$. Then, $P^2 - D(f)Q^2 = n$ has a nontrivial solution in $\mathbb{Z}[x]$ if and only if $P^2 - (f^2 + af + b)Q^2 = n$ has a nontrivial solution in $\mathbb{Z}[x]$ with $(f - c) \mid Q$.*

PROOF OF THEOREM 1.5. By Lemma 3.2 and a $\mathbb{Z}$-linear change of variable, it suffices to consider nontrivial solutions in $\mathbb{Z}[x]$ of

$$P^2 - (f^2 + af + b)Q^2 = n^2 \tag{3.6}$$

with $f \mid Q$. Equation (3.6) has a minimal solution in $\mathbb{C}[x]$:

$$(P, Q) = \left(\frac{n(2f + a)}{\sqrt{\Delta}}, \frac{2n}{\sqrt{\Delta}}\right).$$

Hence, all nontrivial solutions in $\mathbb{C}[x]$ of (3.6) are $(\pm P_k, \pm Q_k)$, where

$$P_k + \sqrt{f^2 + af + b}\,Q_k = n\left(\frac{2f + a}{\sqrt{\Delta}} + \frac{2}{\sqrt{\Delta}}\sqrt{f^2 + af + b}\right)^k, \quad k \geq 1.$$

Assume that there exists a positive integer k such that $P_k, Q_k \in \mathbb{Z}[x]$ and $f \mid Q_k$.

Case 1: $\sqrt{\Delta} \in \mathbb{Z}$. By Theorem 1.2, $\sqrt{\Delta} \mid 2n$.

Case 1.1: $a = 0$. Then, $b = -c^2$, where c is an integer divisor of n. Lemma 3.1 shows that $2 \mid k$. Let $k = 2m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{2m} is $n2^{2m-1}(\alpha/c)^{2m}$. Since $c \mid n$ and $\gcd(n, \alpha) = 1$, it follows that

$$\frac{n2^{2m-1}}{c^{2m}} \in \mathbb{Z}. \tag{3.7}$$

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We claim $c^2 \mid 2n$. This is obvious if $m = 1$. Assume $m \geq 2$. Let p be a prime divisor of c.

Subcase (i): $p = 2$. From (3.7),

$$\begin{aligned} v_2(n) &\geq 2m(v_2(c) - 1) + 1 \geq 4(v_2(c) - 1) + 1 \\ &\geq 2v_2(c) - 1 + 2v_2(c) - 2 \geq 2v_2(c) - 1. \end{aligned} \tag{3.8}$$

Subcase (ii): $p > 2$. From (3.7),

$$v_p(n) \geq v_p(c^2). \tag{3.9}$$

Inequalities (3.8) and (3.9) show that $c^2 \mid 2n$. Conversely, if $(a, b) = (0, -c^2)$, where $c \in \mathbb{Z} - \{0\}$ and $c^2 \mid 2n$, then (3.6) has a nontrivial solution in $\mathbb{Z}[x]$:

$$(P_2, Q_2) = \left(\frac{n(2f^2 - c^2)}{c^2}, \frac{2nf}{c^2}\right).$$

Case 1.2: $a \neq 0$. Since $\sqrt{\Delta} \in \mathbb{Z}$, by Lemma 3.1, $(a, b) = (2c, -3c^2)$ and $3 \mid k$, where $c \in \mathbb{Z} - \{0\}$ and $2c \mid n$. Let $k = 3m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{3m} is $n2^{3m-1}(\alpha/2c)^{3m}$. Since $2c \mid n$ and $(n, \alpha) = 1$, it follows that $2c^{3m} \mid n$. Hence, $2c^3 \mid n$. Conversely, if $(a, b) = (2c, -3c^2)$, where $c \in \mathbb{Z} - \{0\}$ and $2c^3 \mid n$, we verify in MAGMA [2] that (3.6) has a nontrivial solution in $\mathbb{Z}[x]$:

$$(P_3, Q_3) = \left(\frac{n(f + c)(f^2 + 2cf - 2c^2)}{2c^3}, \frac{nf(f + 2c)}{2c^3}\right).$$

Case 2: $\sqrt{\Delta} \notin \mathbb{Z}$. By Theorem 1.2, $\Delta \mid 8n\alpha$ and for any prime divisor p dividing both Δ and α, $f + a/2 = 0$ in $\mathbb{F}_p[x]$.

Case 2.1: $a = 0$. Then, $b \mid 2n\alpha$ and $-b$ is not a square in $\mathbb{Z}$, and $f = 0$ in $\mathbb{F}_p[x]$ for any prime p dividing both b and α. Conversely, in this case, (3.6) has a nontrivial solution in $\mathbb{Z}[x]$:

$$(P_2, Q_2) = \left(\frac{n(2f^2 + b)}{b}, \frac{2nf}{b}\right).$$

Case 2.2: $a \neq 0$. By Lemma 3.1, (a, b) is $(6c, -3c^2)$ and $6 \mid k$ or $(2c, -c^2)$ and $4 \mid k$, where c is a nonzero integer.

Subcase (i): $(a, b) = (6c, -3c^2)$ *and* $6 \mid k$. Since $48c^2 = \Delta \mid 8n\alpha$ and α is square-free, we have $6c^2 \mid n$. Let $k = 6m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{6m} is

$$2^{6m-1} n \left(\frac{\alpha}{2\sqrt{3}c}\right)^{6m} = \frac{n\alpha^{6m}}{2 \cdot 3^{3m} \cdot c^{6m}}.$$

Since $6c^2 \mid n$ and $\gcd(n, \alpha) = 1$, we have $\gcd(2 \cdot 3^{3m} \cdot c^{6m}, \alpha) = 1$. It then follows that $2 \cdot 3^{3m} \cdot c^{6m} \mid n$. Hence, $54c^6 \mid n$. Conversely, if $(a, b) = (6c, -3c^2)$, where $c \in \mathbb{Z} - \{0\}$ and $54c^6 \mid n$, we verify in MAGMA [2] that (3.6) has a nontrivial solution in $\mathbb{Z}[x]$:

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$$(P_6, Q_6) = \Big(\frac{n(-54c^6 + 324c^4f^2 + 324c^3f^3 + 117c^2f^4 + 18cf^5 + f^6)}{54c^6}, \frac{nf(3c + f)(6c + f)(6c^2 + 6cf + f^2)}{54c^6}\Big).$$

Subcase (ii): $(a, b) = (2c, -c^2)$ *and* $3 \mid k$. Since $8c^2 = \Delta \mid 8n\alpha$ and α is square-free, we have $c^2 \mid n$. Let $k = 4m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{4m} is

$$n2^{4m-1}\Big(\frac{\alpha}{\sqrt{2}c}\Big)^{4m} = \frac{n2^{2m-1}\alpha^{4m}}{c^{4m}}.$$

Since $c^2 \mid n$ and $\gcd(n, \alpha) = 1$, we have $\gcd(c^{4m}, \alpha) = 1$. It follows that

$$\frac{n2^{2m-1}}{c^{4m}} \in \mathbb{Z}. \tag{3.10}$$

We claim $c^4 \mid 2n$. This is obvious if $m = 1$, so we assume $m \geq 2$. Let p be a prime divisor of c.

Subcase (ii)(a): $p = 2$. From (3.10),

$$v_2(n) \geq 4mv_2(c) - (2m - 1) \geq 8v_2(c) - (2m - 1) > 4v_2(c). \tag{3.11}$$

Subcase (ii)(b): $p > 2$. From (3.10),

$$v_p(n) \geq 4mv_p(c) > 4v_p(c). \tag{3.12}$$

Inequalities (3.11) and (3.12) show that $c^4 \mid 2n$. Conversely, if $(a, b) = (2c, -c^2)$, where $c \in \mathbb{Z} - \{0\}$ and $c^4 \mid 2n$, we verify in MAGMA [2] that (3.6) has a nontrivial solution in $\mathbb{Z}[x]$:

$$(P_4, Q_4) = \Big(\frac{n(2f^4 + 8cf^3 + 8c^2f^2 - c^4)}{c^4}, \frac{2nf(f + c)(f + 2c)}{c^4}\Big). \qquad \square$$

Proof of Theorem 1.6. By Lemma 3.2 and a $\mathbb{Z}$-linear change of variable, it suffices to consider nontrivial solutions in $\mathbb{Z}[x]$ of

$$P^2 - (f^2 + af + b)Q^2 = n \tag{3.13}$$

with $f \mid Q$. Equation (3.13) has a minimal solution in $\mathbb{C}[x]$:

$$(P, Q) = \sqrt{n}\Big(\frac{2f + a}{\sqrt{\Delta}}, \frac{2}{\sqrt{\Delta}}\Big).$$

Hence, all nontrivial solutions in $\mathbb{C}[x]$ of (3.13) are $(\pm P_k, \pm Q_k)$, where

$$P_k + \sqrt{f^2 + af + b}Q_k = \sqrt{n}\Big(\frac{2f + a}{\sqrt{\Delta}} + \frac{2}{\sqrt{\Delta}}\sqrt{f^2 + af + b}\Big)^k, \quad k \geq 1.$$

Assume that there exists a positive integer k such that $P_k, Q_k \in \mathbb{Z}[x]$ and $f \mid Q_k$.

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Case 1: $a = 0$. Lemma 3.1 shows that $2 \mid k$. Let $k = 2m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{2m} is $\sqrt{n}2^{2m-1}(\alpha/\sqrt{-b})^{2m}$, which is an irrational number. Hence, $P_k \notin \mathbb{Z}[x]$.

Case 2: $a \neq 0$. By Lemma 3.1, (a, b) is one of:

- $(2c, -c^2)$, where $c \in \mathbb{Z}$, $c \neq 0$ and $4 \mid k$;
- $(6c, -3c^2)$, where $c \in \mathbb{Z}$, $c \neq 0$ and $6 \mid k$;
- $(2c, -3c^2)$, where $c \in \mathbb{Z}$, $c \neq 0$ and $3 \mid k$.

Case 2.1: $(2c, -c^2)$ and $4 \mid k$. Let $k = 4m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{4m} is $\sqrt{n}2^{4m-1}(\alpha/\sqrt{2}c)^{4m}$, an irrational number. Hence, $P_k \notin \mathbb{Z}[x]$.

Case 2.2: $(6c, -3c^2)$ *and* $6 \mid k$. Let $k = 6m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{6m} is $\sqrt{n}2^{6m-1}(\alpha/2\sqrt{3}c)^{6m}$, an irrational number. Hence, $P_k \notin \mathbb{Z}[x]$.

Case 2.3: $(2c, -3c^2)$ *and* $3 \mid k$. Let $k = 3m$, where $m \in \mathbb{Z}_{>0}$. The leading coefficient of P_{3m} is $\sqrt{n}2^{3m-1}(\alpha/2c)^{3m}$, an irrational number. Hence, $P_k \notin \mathbb{Z}[x]$.

In all cases, $P_k \notin \mathbb{Z}[x]$. Hence, (3.13) has no nontrivial solutions in $\mathbb{Z}[x]$. Theorem 1.9 is proved. □

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NGUYEN XUAN THO, Faculty of Mathematics and Informatics,
Hanoi University of Science and Technology,
1 Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam
e-mail: tho.nguyenxuan1@hust.edu.vn

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