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On Heiberg's Euclid

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Argument

In two articles published in 1881 and 1884, the young academics, Martin Klamroth and Johan L. Heiberg, engaged in a brief debate on the textual choices that should govern the publication of a new critical edition of Euclid's *Elements*. This short debate seemed to settle the problem in Heiberg's favor as to what should be taken as the definitive text of Euclid's *Elements*. But the issue ought to be considered once again for there are good reasons for the claim that Klamroth was right, and that Heiberg was wrong. If so, we have been consulting, and continue to consult, the wrong text in our efforts to interpret the Euclidean tradition. In order to substantiate this claim, the textual issue debated by Klamroth and Heiberg is rehearsed again, and the principal reasons brought forth by Heiberg against Klamroth's position are reconstructed. Specimens from three broad areas of evidence – structural, linguistic, and technical – will be considered. They reveal how the medieval tradition of the text advocated by Klamroth displays textual superiority to the Greek tradition promoted by Heiberg. Such a reorientation of the texts has the potential to change significantly our understanding of ancient mathematics.

More than a century ago two young academics engaged in a brief debate. One was a *Gymnasiallehrer* at Altona, Martin Klamroth, a specialist in Arabic studies. The other, Johan Ludwig Heiberg, only five years beyond his doctoral dissertation at Copenhagen, had already completed a new critical edition of Archimedes and was in the process of issuing a new critical edition of Euclid. The debate was short – only one paper by each. But the outcome was significant. For it settled, as it seems, once and for all, the issue of what one should take as the definitive text of Euclid's *Elements*. Since Heiberg determined the matter in 1884, no one has seriously scrutinized the arguments he deployed against Klamroth's alternative view in 1881.

My reason for reviving the issue is that I am convinced Klamroth was right and Heiberg was wrong. If that is the case, we have been consulting, and continue to consult, the wrong text in our efforts to interpret the Euclidean tradition.

[1] The late Wilbur R. Knorr could not be asked for the text of an "Argument." Material from this paper and from Knorr 1996 were used in phrasing this "Argument" as well as the captions to the figures (the editors).

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What I propose to do, then, is

(1) outline the textual issue debated by Klamroth and Heiberg, then
(2) describe some of the documents that, although known by them both, were not assessed properly.
(3) I will then consider a few items used by Heiberg to support his position, but show that, to the contrary, they actually lead to the opposite conclusion. The present account is only a summary of the main points; my complete discussion has appeared in a paper published in *Centaurus*.[2]

Early in the nineteenth century, the French philologist François Peyrard observed that one of the manuscripts of the *Elements* which, unlike the others, lacked the title describing it as in the "edition of Theon" (that is, of Theon of Alexandria, *ca.* 360 AD) also happened to lack a certain corollary at the end of Book VI that Theon himself explicitly states elsewhere to be his own addition. Concerning this manuscript, Vatican graec. 190, Peyrard said, "I would not absurdly conjecture it to contain an emended text of Euclid," in contrast with the other manuscripts that would be "nothing other than the vulgate recension of Theon."

Peyrard's view gained ready acceptance among the specialists. Editions of the *Elements* in the nineteenth century made progressively more substantial reference to the Vatican manuscript, culminating in the new critical edition (1883–88) by Heiberg, who chose it as his primary witness. Indeed, Heiberg dubbed the Vatican manuscript P in acknowledgment of Peyrard. All subsequent editions, translations, commentaries, and discussions of the *Elements* have followed Heiberg's example.

But even before Heiberg's edition appeared, a very different proposal was put forward. In 1881 Klamroth, in a major review of the Arabic versions of the *Elements*, noticed that they too lacked Theon's corollary. Upon further review he concluded that the medieval translation must faithfully represent the older form of the text, antedating not only Theon's version, but also that held in Peyrard's manuscript.

In 1884, however, Heiberg responded against Klamroth. In brief, he argued that the Arabic text was missing too much; that, far from being an authentic, older form of the text, it must have been based on an abridged version of the Greek text, made by a Byzantine adaptor, no later than the eighth century. In view of these changes – both the abridgments by the Greek editor and the paraphrasing tendencies of the Arabic translators and editors – Heiberg maintained, it would thus become impossible to use this evidence for meaningful text criticism.

Heiberg's case against Klamroth's proposal is perfunctory – indeed, not good at all. But Klamroth never attempted to respond to it. As background for my rebuttal here, I will first summarize the data on the ancient and medieval manuscripts; my analysis of Heiberg's argument will then be set out under four headings:

[2] Knorr, Wilbur R. 1996. "The Wrong Text of Euclid: On Heiberg's Text and its Alternatives." *Centaurus* 38:208–276. The references mentioned in the present paper can be found in the bibliography of Knorr 1996 (the editors).

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1. structural parallels confirming the medieval text,
2. aspects of linguistic usage,
3. examples of mathematical differences,
4. ancient testimonia.

P denotes the Vatican manuscript (gr. 190), identified by Peyrard and preferred by Heiberg as preserving the authentic Euclidean text.

T denotes the family of manuscripts holding Theon's version (six additional manuscripts collated by Heiberg in Oxford, Vienna, Florence, Bologna, and Paris).

B denotes a Bologna manuscript (A 18–19), dated from the eleventh century which holds a variant text for the last 4 propositions of Book XI and the whole of Book XII. Otherwise it is included in the T group.

There are two major Arabic versions:

al-Ḥajjāj: two editions were produced by al-Ḥajjāj (late eighth – early ninth century); the first a paraphrase version (late eighth century), the second a scholarly edition (early ninth century). Only Version II seems to have interest for text critical purposes.

Isḥāq: a translation was produced by Isḥāq ibn Ḥunayn around the mid-ninth century, and was revised by Thābit ibn Qurra a few decades later.

There are two major Latin versions:

A denotes the translation made by Adelard of Bath early in the twelfth century, the so-called "Version I" (to distinguish it from adaptations of it, Versions II and III, made by other scholars later). In the judgment of Hubertus L. L. Busard, it appears to rest on al-Ḥajjāj II.

G denotes the translation made by Gerard of Cremona at Toledo, after the mid-twelfth century. In the judgment of Busard, it is based on Isḥāq-Thābit. Gerard's version includes added paragraphs he assigns to "Thābit"; these items agree with the Theonine text of the Greek.

In my analysis I derive my text of the "medieval text" mainly from Busard's editions of Adelard and Gerard (A and G). Of course, it would be better to consult the Arabic directly. But this would be difficult, given that critical editions of the key Arabic versions are not yet available, and specialists do not even agree on which manuscripts transmit which versions (indeed, De Young and others assume that most of the two dozen or so Arabic manuscripts hold composite texts).

Collations of the Latin texts with selected Greek and Arabic texts provide justification for this choice, as I will next show.

Figure 1 displays the structural comparisons of PT with the Bologna variant and the Latin AG. For the last 4 propositions of Book XI and the first 11 propositions of Book XII, the Bologna text agrees with AG (the differences in numbering of XI 38–41 is immaterial). Note their shared omissions of XII 6, and of the lemmas after XII 2 and 4, and of the porisms after XII 7 and 8; as well as the changes of order of XII 7 and 10. It is also the case that AG agree practically *verbatim* with the Bologna

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	P and T	B	A and G	Comment
XI	36	36	38	B = AG
	37	37	39	B = AG
	38	38	40	B = AG
	39	39	41	B = AG
XII	1	1	1	B = AG
	2	2	2	B = AG
	Lemma	Omit	omit	
	3	3	3	B = AG
	4	4	4	B = AG
	Lemma	Omit	omit	
	5	5	5	B = AG
	6	Omit	omit	
	7	6	6	B = AG
	Porism	Omit	omit	
	8	8	8	B = AG
	Porism	Omit	omit	
	9	7	7	B = AG
	10	9	9	B = AG
	11	11	11	B = AG
	12	10	10	B = AG

Fig. 1. Structural comparison (Greek and Latin versions of Books XI, XII).

text, except for obvious scribal discrepancies. By contrast, PT present a systematically different wording.

This *verbatim* agreement of AG with Bologna permits two major conclusions at once:

1. Since the Latin agrees with the Greek in Bologna, it must be the case that the Arabic underlying the Latin agrees at least as closely with the same Greek version. This gives us our warrant to use the Latin as our evidence of the medieval version.

2. The *verbatim* agreement means that all the translators – al-Ḥajjāj, Isḥāq, Adelard and Gerard – subscribed to a literal, *de verbo ad verbum* style of translating, and executed it with reasonable success. Heiberg denied this, insisting that the medieval translations were so loose and inaccurate that they could not be used at all for text criticism. But this judgment was based on collations of selected Arabic items against *Theonine* texts.

Ironically, Heiberg was the one who identified the Bologna variant text as being in agreement with two parts of the Arabic of Book XII he knew from Klamroth's paper (for Klamroth did not know of the Bologna text). But Heiberg dismissed its significance, claiming that the Bologna text was merely a Byzantine adaptation.

For the rest of Book XII the correspondence is more complex.

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	P and T	Bologna	A and G	Comment
XII	13	12	Omit	B = PT
	14	13	Omit	B = PT
	15	14	12	B ~ PT
	16	15	13	B ~ PT
	17	16	14	B ~ PT
	18	17	15	B = PT

Fig. 2. Structural comparison (Prop. XII.13–18).

Here AG omit propositions 13–14, where Bologna agrees with PT. For the next three propositions Bologna seems to hold a text intermediate between AG and PT. The three versions are all in close agreement on the last proposition of Book XII (see fig. 2).

We will consider an example from the penultimate proposition later.

By collating T with P, Heiberg compiled a detailed account of what he took to be the editorial changes made by Theon in his source. The differences, however, are remarkably insubstantial. Heiberg admitted this and Heath, echoing Heiberg, judged that the changes were "unworthy of Theon." Astonishing – that Theon's editorial style would not well be represented by his edition of the *Elements*!

As an example, consider this page (fig. 3) from XII 2 (taken from Heath's translation). Here, according to Heiberg's apparatus, P and T do not differ at all.

The same page, if amended in accordance with the Bologna text, looks like this (fig. 4):

The medieval versions (the Latin AG and also the Arabic, as I confirmed from the Copenhagen manuscript) agree with Bologna. The most striking difference here is Bologna's omission of two longish passages, both beginning with the word "inasmuch as" (Greek: *epeidêper*). The first explains how it is that the square inscribed in a given circle is more than half the circle; the second explains, analogously, that the isosceles triangle inscribed in any segment of the circle is greater than half the segment.

In Book XII of PT comparable passages with *epeidêper* occur 15 times; these are all absent from Bologna and absent from AG. Furthermore, there are several cross references in PT to these passages; all the cross references are likewise absent from Bologna and AG.

Throughout the rest of the *Elements*, *epeidêper* passages occur only 23 times more (namely, 8 times in Book XIII, 15 times in all the remaining 11 books). All these passages are in both P and T, although Heiberg brackets a few of them as interpolations – it is, after all, a phrase that annotators might use, and does sometimes occur in scholia. But as before, none of these 23 passages occurs in the medieval version (AG).

This pattern makes no sense under Heiberg's hypothesis of the poriority of P. For it would mean that the Bologna editor has systematically removed passages that had

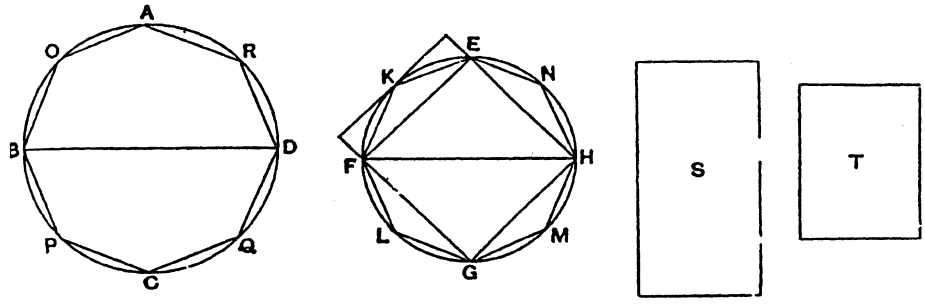
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PROPOSITION 2.

Circles are to one another as the squares on the diameters.

Let *ABCD*, *EFGH* be circles, and *BD*, *FH* their diameters;

I say that, as the circle *ABCD* is to the circle *EFGH*, so is the square on *BD* to the square on *FH*.



For, if the square on *BD* is not to the square on *FH* as the circle *ABCD* is to the circle *EFGH*,

then, as the square on *BD* is to the square on *FH*, so will the circle *ABCD* be either to some less area than the circle *EFGH*, or to a greater.

First, let it be in that ratio to a less area *S*.

Let the square *EFGH* be inscribed in the circle *EFGH*; then the inscribed square is greater than the half of the circle *EFGH*, inasmuch as, if through the points *E*, *F*, *G*, *H* we draw tangents to the circle, the square *EFGH* is half the square circumscribed about the circle, and the circle is less than the circumscribed square;

hence the inscribed square *EFGH* is greater than the half of the circle *EFGH*.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *K*, *L*, *M*, *N*,

and let *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be joined;

therefore each of the triangles *EKF*, *FLG*, *GMH*, *HNE* is also greater than the half of the segment of the circle about it, inasmuch as, if through the points *K*, *L*, *M*, *N* we draw tangents to the circle and complete the parallelograms on the straight lines *EF*, *FG*, *GH*, *HE*, each of the triangles *EKF*,

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Fig. 3. Prop. XII.2 in Heath translation.

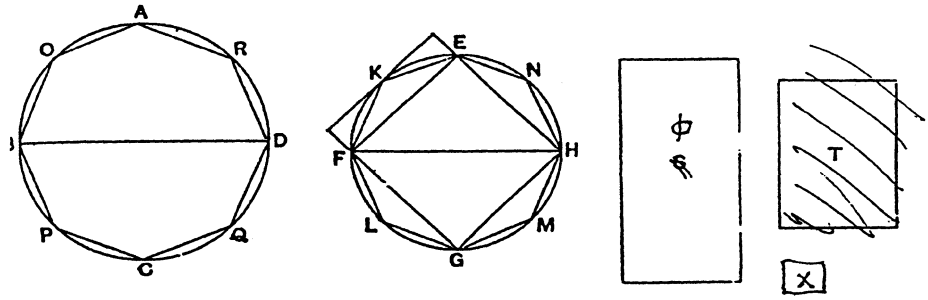
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For, if the square on *BD* is not to the square on *FH* as the circle *ABCD* is to the circle *EFGH*,

then, as the square on *BD* is to the square on *FH*, so will the circle *ABCD* be either to some less area than the circle *EFGH*, or to a greater.

First, let it be in that ratio to a less area *S*.

Let the square *EFGH* be inscribed in the circle *EFGH*; then the inscribed square is greater than the half of the circle *EFGH*, inasmuch as, if through the points *E*, *F*, *G*, *H* we draw tangents to the circle, the square *EFGH* is half the square circumscribed about the circle, and the circle is less than the circumscribed square;

hence the inscribed square *EFGH* is greater than the half of the circle *EFGH*.

Let the circumferences *EF*, *FG*, *GH*, *HE* be bisected at the points *K*, *L*, *M*, *N*,

and let *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be joined;

therefore each of the triangles *EKF*, *FLG*, *GMH*, *HNE* is also greater than the half of the segment of the circle about it, inasmuch as, if through the points *K*, *L*, *M*, *N* we draw tangents to the circle and complete the parallelograms on the straight lines *EF*, *FG*, *GH*, *HE*, each of the triangles *EKF*,

Fig. 4. Prop. XII.2 amended in accordance with B,

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the *appearance* of being interpolations (that is, because of the presence of the introductory word *epeidêper*); and that this editor carefully eliminated all the later passages that referred to them. But such behavior would be practically perverse for a Byzantine editor: to systematically *remove* such items in this way, when they were just the sort of explanation that the editors usually *added*. To the contrary, given the Bologna text, an editor might well feel the need for such explanations; the high occurrence of *epeidêper* in Book XII could reflect one particular editor's verbal preference, and we could expect him to insert cross references to his own prior additions, as appropriate.

There are about a dozen or so other *types* of interpolative idioms like this, commonly used by the ancient annotators, and one can count more than 2000 different passages in PT where they appear. They are all absent from the medieval text.

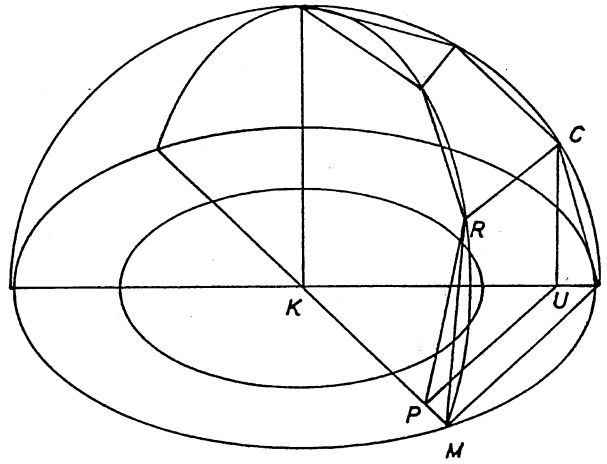
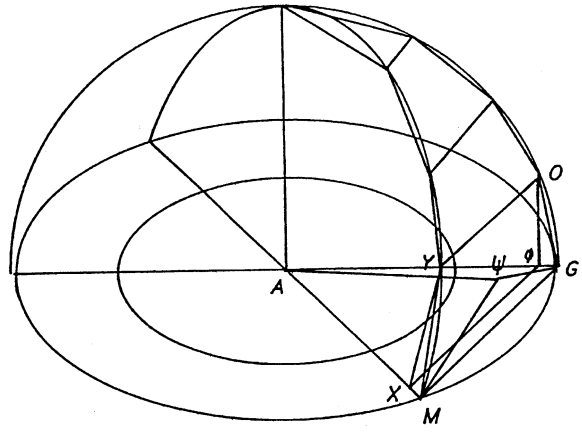


Fig. 5. Prop. XII.14 in G.

I would like to illustrate how the differences of PT, B, and AG are not always merely linguistic differences, but sometimes are of real mathematical interest. My example is taken from the penultimate proposition of Book XII. The project here is to construct within a given sphere a certain polyhedron that lies entirely outside a second smaller sphere, concentric with the first. Consider first the diagram in AG (specifically, Gerard). A key step of the construction relates to the trapezium CLMR. Since by the conditions of the construction the sides CL, LM and MR all lie outside the inner sphere, and since CR does the same; the text asserts that *therefore* the whole trapezium likewise does not touch the inner sphere.

This inference, however, is a mistake. For one can readily produce cases where the sides of the trapezium lie outside the inner sphere, but that points of the interior of the trapezium might intersect that sphere.

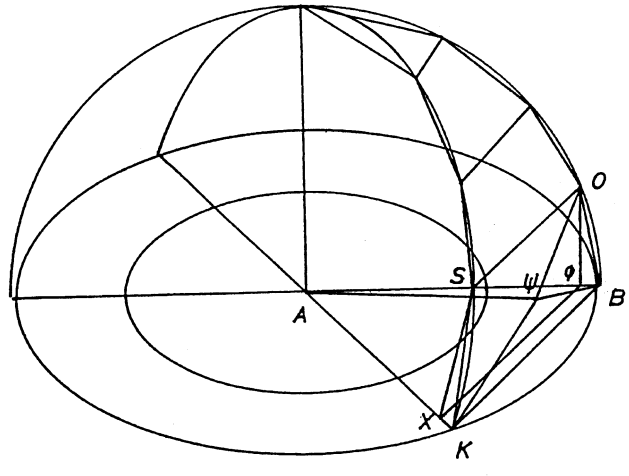
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XII 16 (BOLOGNA)

Fig. 6. Prop. XII.16 in B.

In the Bologna text, a correct argument appears in place of this error. It is asserted (without proof) that if Aψ is drawn perpendicular to the plane of the trapezium MGOY, then ψ is the center of the circle circumscribing the trapezium. From this it is shown that Aψ > Aϕ, which in turn is greater than the radius of the inner sphere; hence, the trapezium lies entirely outside the inner sphere.



XII 17 (P & T)

Fig. 7. Prop. XII.17 in PT.

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The argument in PT partially inverts this order: it does not assume the inscription of the trapezium (here OBKS) in a circle, but instead proves it. For if, as before, Aψ is perpendicular to the plane of OBKS, then it is argued from the "Pythagorean theorem" that the distance from ψ to each of the vertices O, B, K, S is the same. The argument then continues essentially as in Bologna, to conclude that Aψ > Aϕ, whence the trapezium lies entirely outside the inner sphere.

In place of the incorrect argument in AG, the Bologna presents a correct proof. But it makes two assumptions: that a circle can be circumscribed about an equilateral trapezium, and that the line Aψ is perpendicular to the plane of the trapezium. These are not proved in the *Elements*, but they are familiar in textbooks on spherics, as in the *Spherics* of Theodosius, and so might be assumed by the Bologna editor, correcting AG. On the other hand, PT revise the proof so that only prior results from the *Elements* are assumed. But with these changes, the circle becomes superfluous. For all that the proof requires is that Aψ is perpendicular to the trapezium, and that the four lines ψO, ψB, ψK ψS are equal. Apparently, PT presuppose a form of the proof like that in Bologna.

Corroboration is provided by a certain scholium, labelled "lêmma," that appears in P. This proves an inequality that is assumed in the main proof of P, namely, that $KB^2 > 2\ \psi B^2$. But in the lêmma the inequality is framed as $MG^2 > 2\ \psi G^2$, in agreement with the diagram lettering in Bologna. This is incompatible with Heiberg's supposition of the priority of P, for there would be no reason for an annotator of P to base his "lêmma" on a text like the Bologna variant.

Heiberg surveys testimonia from ancient authorities that confirm PT against the medieval text, so that, in his view, its prototype, such as the Bologna text, must be a later derivative version. But the passages he cites from Proclus (fifth century), Simplicius and Eutocius (sixth century) need only reflect their use of Theon's edition from the sixth century.

Heiberg also cites passages from the *Collection* of Pappus (early fourth century), where propositions from Book XIII are cited in agreement with the number scheme in PT, but different from that in the medieval text. These passages, however, are all phrases of the form "as in the N^{th} proposition of Book XIII of the *Elements*." The manuscripts of ancient geometric texts abound with annotations of this form; indeed, it is the most common form encountered in the scholia to the *Elements*. The extant manuscript tradition of Pappus' *Collection* stems from a single Byzantine copy (Vatican, graec. 218, eleventh-twelfth century). All modern accounts of Pappus (including the editions of Hultsch and Jones) agree that the extant text includes interpolations (although there is some disagreement, of course, about the extent). Thus, the passages cited by Heiberg stand out as likely interpolations.

Heiberg omits to mention that the end of the same part (Book V) of Pappus' *Collection* presents a complete lemma (proving that only five regular solids are possible) which duplicates entirely the argument at the end of Book XIII in PT. Presumably, the argument was not in the version of the *Elements* consulted by Pappus

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himself; for he typically presumes his reader's access to the *Elements*, and so would have no reason to repeat what the reader would find there.

The medieval version (as in Adelard) lacks this lemma to Book XIII, and so seems better to represent the form of the text implied by Pappus.

Conclusions

Heiberg's position is a conjecture. As such it requires scrutiny. As I have set out, his argument against Klamroth is essentially circular.

(1) Heiberg dismisses the medieval translations as too loose, *because they differ from the text in P.*

(2) Heiberg insists that Euclid would never have produced a text like the Bologna or medieval versions, for they omit certain results (like XII.6 or various lemmas and corollaries) that are indispensable, since they are assumed in later propositions. But *it is only through the example of P* that Heiberg can assume what sort of text Euclid would have produced – such as, that Euclid would have required explicit statements and proofs of all such results, no matter how obvious they might be.

To the contrary, we can establish, without fear of circular reasoning, what sort of text the ancient editors and commentators wished to produce: a text in which explanations of minor steps are regularly interpolated for the benefit of students – in short, a text like PT. Lacking such material, the Bologna and medieval versions seem more like the form of the text we would expect, before it had undergone this glossing. It is a text that assumes a better trained and more committed student than P and T assume.

In the light of the medieval evidence, P and T are more like each other than different. It thus becomes difficult to view them as separate recensions. I think that P was just a preliminary version by Theon, who later produced T as a slightly revised version.

As for the medieval versions (embracing those in Arabic, Hebrew, and Latin): I hope that specialists will now realize the considerable value these have for establishing the ancient text. Perhaps too, they may accept that the effort to disentangle the Arabic evidence can be greatly assisted by referring to the Latin evidence, in conjunction with Greek evidence, like the Bologna manuscript.

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