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A note on magnetohydrodynamic duct flow†

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Abstract. This paper is concerned with the problem of the flow of an incompressible electrically conducting fluid along a rectangular duct under a transverse magnetic field. The case in which the walls perpendicular to the field are perfectly conducting and those parallel to the field are non-conducting has been considered by Hunt (1), who derives the solution in two ways; as the limiting cases of the flows with (*a*) non-conducting walls parallel and thin walls of arbitrary conductivity perpendicular to the field, and (*b*) thin walls of arbitrary conductivity parallel and perfectly conducting walls perpendicular to the field. We show that these two limiting solutions derived by Hunt are in fact equivalent. In addition, we extend the solution of case (*b*) above by removing the thin wall restriction.

1. *Introduction.* In this paper we are concerned with the flow of an incompressible electrically conducting fluid flowing under the action of a pressure gradient along a rectangular duct with an imposed transverse magnetic field. Hunt (1) examines the following two cases, defined with reference to Fig. 1.

(*a*) Non-conducting walls, AA, parallel to the field and walls, BB, of arbitrary conductivity perpendicular to the field.

(*b*) Perfectly conducting walls, BB, perpendicular to the field and walls, AA, of arbitrary conductivity parallel to the field.

For both configurations Hunt obtains an exact solution for the flow velocity and induced field, but with the restriction that the arbitrarily conducting walls must be thin. Sloan and Smith (2) remove the thin-wall restriction in their treatment of case (*a*) and using similar ideas the restriction may be lifted for case (*b*) also. This more general solution for (*b*) is presented here. Both the solution of Sloan and Smith for (*a*) and the one given here for (*b*) reduce to those of Hunt as the thickness of the appropriate walls tends to zero.

The limiting cases of (*a*), when the conductivity of BB tends to infinity and (*b*), when the conductivity of AA tends to zero, represent the same configuration, namely: BB perfectly conducting, AA non-conducting. It would be desirable, as Hunt (1) points out, to check that the two solutions so derived are the same. Since the forms of the expressions differ considerably, however, this proves complicated and Hunt does not, in fact, carry out the necessary analysis. In the final section of this paper the identity

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of these two solutions is established. (For a qualitative discussion of the limiting configuration, see Broadbent (3).)

2. *Formulation of the problem.* Let the (x, y)-plane be normal to the axis of the duct with origin at its centre (see Fig. 1), and the z-axis be along the duct to complete the usual right-handed system. In section, the fluid occupies $|x| < b$, $|y| < a$ and the walls occupy $|x| < bp$, $a < |y| < aq$ and $b < |x| < bp$, $|y| < a$, where $p, q > 1$. Under

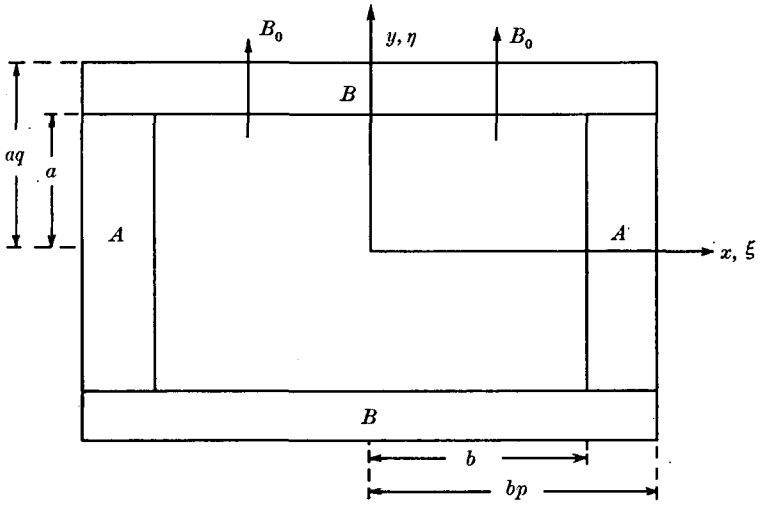


Fig. 1. Duct cross-section.

the assumption that there is no secondary flow and no variation in duct cross-section or applied magnetic field, it can be shown (4) that the velocity and induced field have components only in the z-direction. The equations describing the system are

$$\overline{\eta}\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}\right)+B_0\frac{\partial H_z}{\partial y}=-P, \tag{1}$$

$$\frac{1}{\sigma_1}\left(\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}\right)+B_0\frac{\partial V_z}{\partial x}=0, \tag{2}$$

in the fluid, and

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}=0, \tag{3}$$

in the walls, where B_0 is the flux density of the imposed field, V_z is the fluid velocity, H_z is the induced field and σ_1, $\overline{\eta}$, $-P$ are the fluid conductivity, viscosity and pressure gradient respectively.

It is convenient at this stage to introduce the following dimensionless variables,

$$V=\frac{V_z\overline{\eta}}{Pa^2}, \quad H=\frac{H_z\overline{\eta}^{\frac{1}{2}}}{Pa^2\sigma_1^{\frac{1}{2}}}, \quad \xi=\frac{x}{a}, \quad \eta=\frac{y}{a}.$$

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In terms of these variables (1), (2) and (3) become

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+M\frac{\partial H_1}{\partial \eta}=-1, \tag{4}$$

$$\frac{\partial^2 H_1}{\partial \xi^2}+\frac{\partial^2 H_1}{\partial \eta^2}+M\frac{\partial V}{\partial \eta}=0, \tag{5}$$

$$\frac{\partial^2 H_2}{\partial \xi^2}+\frac{\partial^2 H_2}{\partial \eta^2}=0, \tag{6}$$

where subscripts 1 and 2 distinguish the magnetic field in the fluid and the walls respectively. The Hartmann number, $M = B_0 a(\sigma_1/\overline{\eta})^{\frac{1}{2}}$, is the ratio of the Lorentz to the viscous force.

We now examine the solution for the two cases mentioned in the introduction.

3. *Case (a). Non-conducting walls parallel to the field.* In case (*a*), with the outer boundary of the system insulated, (4), (5) and (6) are subject to the following boundary conditions

(i) $V = 0$ on $|\xi| = l\,(|\eta| < 1)$ and on $|\eta| = 1$,

(ii) $H_1 = 0$ on $|\xi| = l\,(|\eta| < 1)$,

(iii) $\sigma_2 \dfrac{\partial H_1}{\partial \eta} = \sigma_1 \dfrac{\partial H_2}{\partial \eta}$, $\quad H_1 = H_2$ on $|\eta| = 1\ (|\xi| < l)$,

(iv) $H_2 = 0$ on $|\xi| = lp\,(1 < |\eta| < q)$, on $|\eta| = q\,(|\xi| < lp)$ and on $|\eta| = 1\,(l < |\xi| < lp)$,

where $l = b/a$, and the suffix 2 refers to walls BB. The induced field throughout the non-conducting walls is, of course, zero.

The standard method of solving (4), (5) and (6) involves the expansion of the dependent variables as Fourier series in either ξ or η, the boundary conditions governing the coordinate dependence. In this case to satisfy the boundary conditions on $|\xi| = l$, the appropriate expansions are Fourier cosine series in ξ. Sloan and Smith (2) give the solution for equations (4), (5) and (6) subject to boundary conditions (i), (ii) and (iii), but with (iv) replaced by

(v) $H_2 = 0$ on $|\xi| = l\,(1 < |\eta| < q)$ and on $|\eta| = q\,(|\xi| < l)$.

Their solution for the velocity and induced field distribution is

$$V = \sum_{k=0}^{\infty} \frac{2(-1)^k}{l\alpha_k^3} \cos \alpha_k \xi \left\{1 - \frac{A_k \cosh \gamma_k \eta - B_k \cosh \beta_k \eta}{A_k \cosh \gamma_k - B_k \cosh \beta_k}\right\}, \tag{7}$$

$$H_1 = \sum_{k=0}^{\infty} \frac{2(-1)^k}{l\alpha_k^3} \cos \alpha_k \xi \left\{\frac{B_k \sinh \beta_k \eta - A_k \sinh \gamma_k \eta}{A_k \cosh \gamma_k - B_k \cosh \beta_k}\right\}, \tag{8}$$

$$H_2 = \sum_{k=0}^{\infty} \frac{2(-1)^k}{l\alpha_k^3} \cos \alpha_k \xi \left\{\frac{B_k \sinh \beta_k - A_k \sinh \gamma_k}{A_k \cosh \gamma_k - B_k \cosh \beta_k}\right\} \operatorname{cosech} \{a_k(1-q)\} \sinh \{\eta \mp q\} \tag{9}$$

for $\qquad 1 < \eta < q, \quad -q < \eta < -1\,(|\xi| < l)$,

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where
$$\alpha_k = (k+\tfrac{1}{2})\pi/l, \quad \beta_k, \gamma_k = \tfrac{1}{2}\{-M \mp (M^2+4\alpha_k^2)^{\frac{1}{2}}\},$$
$$A_k = \sigma_1\alpha_k \sinh\beta_k - \sigma_2\beta_k \tanh\{\alpha_k(1-q)\}\cosh\beta_k,$$
$$B_k = \sigma_1\alpha_k \sinh\gamma_k - \sigma_2\gamma_k \tanh\{\alpha_k(1-q)\}\cosh\gamma_k.$$

A brief examination of (iv), (v) and (9) shows that the solution satisfying (i)–(iv) is the same as that given above with, in addition, $H_2 = 0$ in the corner regions $1 < |\eta| < q$, $l < |\xi| < lp$. The velocity and induced field distribution in the fluid are therefore still given by (7) and (8). By letting $q \to 1$, in which case $\tanh\{\alpha_k(1-q)\} \sim \alpha_k(1-q)$, (7) and (8) become identical with the results of Hunt (1) for thin walls, BB.

4. *Case* (*b*). *Perfectly conducting walls perpendicular to the field.* The solution to case (*b*) is derived by Hunt (1) with the restriction that the walls of arbitrary conductivity, AA, are thin. Since, to a good approximation for a thin wall, the harmonic function, H_2, can be shown to vary linearly across the wall (5), this allows him to consider the fluid equations (4) and (5) only. Here, this restriction is dispensed with and the field is matched exactly between the side walls and the fluid.

We consider (4), (5) and (6) subject to the following boundary conditions, (the outer boundary of the system is again assumed to be insulated):

(i) $V = 0$ on $|\xi| = l$ ($|\eta| < 1$) and on $|\eta| = 1$ ($|\xi| < l$);

(ii) $\dfrac{\partial H_1}{\partial \eta} = 0$ on $|\eta| = 1$, $|\xi| < l$;

(iii) $\sigma_2 \dfrac{\partial H_1}{\partial \xi} = \sigma_1 \dfrac{\partial H_2}{\partial \xi}$, $\quad H_1 = H_2$ on $|\xi| = l$, $\quad |\eta| < 1$;

(iv) $\dfrac{\partial H_2}{\partial \eta} = 0$ on $|\eta| = 1$, $\quad l < |\xi| < lp$;

(v) $H_2 = 0$ on $|\xi| = lp$, $\quad |\eta| < 1$;

where suffix 2 refers to the walls AA.

Once again the standard Fourier series method of solving (4), (5) and (6) can be used. In order to satisfy the boundary conditions on $|\eta| = 1$, the dependent variables are expanded for $|\eta| < 1$ as

$$V = \sum_{j=0}^{\infty} v_j \cos\alpha_j\eta, \qquad H_1 = \sum_{j=0}^{\infty} h_j \sin\alpha_j\eta,$$
$$H_2 = \sum_{j=0}^{\infty} s_j \sin\alpha_j\eta \quad (l < \xi < lp), \qquad H_2 = \sum_{j=0}^{\infty} t_j \sin\alpha_j\eta \quad (-lp < \xi < -l),$$

where $\alpha_j = (j+\tfrac{1}{2})\pi$.

From the boundary condition on $\xi = lp$ and (6), we must have, for all j

$$s_j = G_j \sinh\{\alpha_j(\xi - lp)\} \quad (l < \xi < lp),$$

where the G_j are constants. Similarly, by symmetry

$$t_j = -G_j \sinh\{\alpha_j(\xi + lp)\} \quad (-lp < \xi < -l).$$

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Then boundary condition (iii) gives

$$\frac{\partial h_j}{\partial \xi} = \pm \frac{\sigma_1}{\sigma_2} G_j \alpha_j \cosh\{\alpha_j l(1-p)\}$$

and $$h_j = G_j \sinh\{\alpha_j l(1-p)\} \quad \text{at} \quad \xi = \pm l,$$

whence $$\frac{\partial h_j}{\partial \xi} = \pm \frac{\sigma_1}{\sigma_2} \alpha_j h_j \coth\{\alpha_j l(1-p)\} \quad \text{at} \quad \xi = \pm l.$$

The corresponding condition in the work of Hunt (1) is

$$\frac{\partial h_j}{\partial \xi} = \mp \frac{h_j}{d_A}$$

where $d_A = (\sigma_2/\sigma_1) l(p-1)$ is a constant for all j. Hence by replacing d_A, whenever it occurs in Hunt's solution, by

$$d_j = \frac{\sigma_2}{\sigma_1 \alpha_j} \tanh\{\alpha_j l(p-1)\},$$

we obtain the solution to the more general case as

$$V = \sum_{j=0}^{\infty} \frac{2(-1)^j \cos \alpha_j \eta}{\alpha_j(\alpha_j^2 + M^2)} \left\{1 - \frac{C_j(\xi) - (M/\alpha_j) D_j(\xi)}{K_j} - \frac{d_j[\beta_j E_j(\xi) - \gamma_j F_j(\xi)]}{K_j}\right\}, \tag{10}$$

$$H_1 = \sum_{j=0}^{\infty} \frac{2(-1)^j \sin \alpha_j \eta}{\alpha_j(\alpha_j^2 + M^2)} \left\{-\left(\frac{M}{\alpha_j}\right) + \frac{(M/\alpha_j) C_j(\xi) + D_j(\xi)}{K_j} + \frac{d_j\{\gamma_j E_j(\xi) + \beta_j F_j(\xi)\}}{K_j}\right\}, \tag{11}$$

where $$\beta_j, \gamma_j = (\tfrac{1}{2}\alpha_j)^{\frac{1}{2}} [\pm \alpha_j + (\alpha_j^2 + M^2)^{\frac{1}{2}}]^{\frac{1}{2}},$$

$$d_j = \frac{\sigma_2}{\sigma_1 \alpha_j} \tanh\{\alpha_j l(p-1)\},$$

$$K_j = \cosh 2\beta_j l + \cos 2\gamma_j l + d_j(\beta_j \sinh 2\beta_j l - \gamma_j \sin 2\gamma_j l),$$

$$C_j(\xi) = \cos \gamma_j(l-\xi) \cosh \beta_j(l+\xi) + \cos \gamma_j(l+\xi) \cosh \beta_j(l-\xi),$$

$$D_j(\xi) = \sin \gamma_j(l-\xi) \sinh \beta_j(l+\xi) + \sin \gamma_j(l+\xi) \sinh \beta_j(l-\xi),$$

$$E_j(\xi) = \cos \gamma_j(l-\xi) \sinh \beta_j(l+\xi) + \cos \gamma_j(l+\xi) \sinh \beta_j(l-\xi),$$

$$F_j(\xi) = \sin \gamma_j(l-\xi) \cosh \beta_j(l+\xi) + \sin \gamma_j(l+\xi) \cosh \beta_j(l-\xi).$$

As in section 4, when $p \to 1$, $\tanh\{\alpha_j l(p-1)\} \sim \alpha_j l(p-1)$ and (10) and (11) become identical with the results of Hunt. To complete the solution for the general case, G_j may be determined from (iii) and (11) to give

$$H_2 = \sum_{j=0}^{\infty} \frac{2(-1)^j d_j}{K_j \alpha_j(\alpha_j^2 + M^2)} \sin \alpha_j \eta \left\{\left(\frac{M\beta_j}{\alpha_j} - \gamma_j\right) \sinh 2\beta_j l - \left(\frac{M\gamma_j}{\alpha_j} + \beta_j\right) \sin 2\gamma_j l \right\} \sinh\{\alpha_j(lp \mp \xi)\},$$

for $l < \xi < lp$, $-lp < \xi < -l$ ($|\eta| < 1$).

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5. *Limiting cases.* In the limit as $\sigma_2 \to \infty$, the solution of (a) given by (7) and (8) reduces to

$$V = \sum_{k=0}^{\infty} \frac{2(-1)^k \cos \alpha_k \xi}{l\alpha_k^3} \left\{ 1 - \frac{\gamma_k \cosh \beta_k \eta}{(\gamma_k - \beta_k) \cosh \beta_k} + \frac{\beta_k \cosh \gamma_k \eta}{(\gamma_k - \beta_k) \cosh \gamma_k} \right\}, \tag{12}$$

$$H_1 = \sum_{k=0}^{\infty} \frac{2(-1)^k \cos \alpha_k \xi}{l\alpha_k^3} \left\{ \frac{\beta_k \sinh \gamma_k \eta}{(\gamma_k - \beta_k) \cosh \gamma_k} - \frac{\gamma_k \sinh \beta_k \eta}{(\gamma_k - \beta_k) \cosh \beta_k} \right\}, \tag{13}$$

where α_k, β_k, γ_k are as defined previously.

As $\sigma_2 \to 0$, the solution of (b) given by (10) and (11) becomes

$$V = \sum_{j=0}^{\infty} \frac{2(-1)^j \cos \alpha_j \eta}{\alpha_j(\alpha_j^2 + M^2)} \left\{ 1 - \frac{C_j(\xi) - (M/\alpha_j) D_j(\xi)}{\cosh 2\beta_j l + \cos 2\gamma_j l} \right\}, \tag{14}$$

$$H_1 = \sum_{j=0}^{\infty} \frac{2(-1)^j \sin \alpha_j \eta}{\alpha_j(\alpha_j^2 + M^2)} \left\{ -\left(\frac{M}{\alpha_j}\right) + \frac{(M/\alpha_j) C_j(\xi) + D_j(\xi)}{\cosh 2\beta_j l + \cos 2\gamma_j l} \right\}, \tag{15}$$

with α_j, β_j, γ_j, $C_j(\xi)$, $D_j(\xi)$ as defined previously. We wish to show that (12) and (13) are equivalent to (14) and (15) respectively, since they represent solutions for the same physical situation.

Consider first equations (14) and (15). By expanding the expressions for $C_j(\xi)$, $D_j(\xi)$ and using the Fourier series expansions for $\cos \gamma_j \xi \cosh \beta_j \xi$, $\sin \gamma_j \xi \sinh \beta_j \xi$ derived in the Appendix, we obtain

$$C_j(\xi) = \sum_{k=0}^{\infty} \frac{2(-1)^k}{l} \cos \alpha_k \xi \left\{ (\cos^2 \gamma_j l \cosh^2 \beta_j l + \sin^2 \gamma_j l \sinh^2 \beta_j l) \left(\frac{\Phi}{\beta_j^2 + \Phi^2} - \frac{\Psi}{\beta_j^2 + \Psi^2} \right) \right\},$$

$$D_j(\xi) = \sum_{k=0}^{\infty} \frac{2(-1)^k}{l} \cos \alpha_k \xi \left\{ (\cos^2 \gamma_j l \cosh^2 \beta_j l + \sin^2 \gamma_j l \sinh^2 \beta_j l) \left(\frac{\beta_j}{\beta_j^2 + \Psi^2} - \frac{\beta_j}{\beta_j^2 + \Phi^2} \right) \right\},$$

where $$\Phi = \gamma_j + \alpha_k, \quad \Psi = \gamma_j - \alpha_k.$$

Noting the identities:

$$1 = \sum_{k=0}^{\infty} \frac{2(-1)^k}{l\alpha_k} \cos \alpha_k \xi \quad \text{for} \quad |\xi| < l,$$

$$2(\cos^2 \gamma_j l \cosh^2 \beta_j l + \sin^2 \gamma_j l \sinh^2 \beta_j l) = \cosh 2\beta_j l + \cos 2\gamma_j l,$$

we have

$$\left\{ 1 - \frac{C_j(\xi) - (M/\alpha_j) D_j(\xi)}{\cosh 2\beta_j l + \cos 2\gamma_j l} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{l} \cos \alpha_k \xi \left\{ \frac{2}{\alpha_k} - \frac{\Phi}{\beta_j^2 + \Phi^2} + \frac{\Psi}{\beta_j^2 + \Psi^2} - \frac{M}{\alpha_j} \left(\frac{\beta_j}{\beta_j^2 + \Phi^2} - \frac{\beta_j}{\beta_j^2 + \Psi^2} \right) \right\}$$

$$= \sum_{k=0}^{\infty} \frac{2(-1)^k}{l} \cos \alpha_k \xi \left\{ \frac{(\alpha_j^2 + \alpha_k^2)(\alpha_j^2 + M^2)}{\alpha_k [(\alpha_j^2 + \alpha_k^2)^2 + M^2 \alpha_j^2]} \right\},$$

whence $$V = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{4(-1)^{j+k}}{l} \left\{ \frac{\alpha_j^2 + \alpha_k^2}{\alpha_j \alpha_k [(\alpha_j^2 + \alpha_k^2)^2 + M^2 \alpha_j^2]} \right\} \cos \alpha_j \eta \cos \alpha_k \xi. \tag{16}$$

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Similarly

$$\left\{-\frac{M}{\alpha_j}+\frac{(M/\alpha_j)C_j(\xi)+D_j(\xi)}{\cosh 2\beta_j l+\cos 2\gamma_j l}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{l}\cos\alpha_k\xi\left\{-\frac{2M}{\alpha_j\alpha_k}+\frac{M}{\alpha_j}\left(\frac{\Phi}{\beta_j^2+\Phi^2}-\frac{\Psi}{\beta_j^2+\Psi^2}\right)+\frac{\beta_j}{\beta_j^2+\Psi^2}-\frac{\beta_j}{\beta_j^2+\Phi^2}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{2(-1)^k}{l}\cos\alpha_k\xi\left\{\frac{-M\alpha_j(\alpha_j^2+M^2)}{\alpha_k[(\alpha_j^2+\alpha_k^2)^2+M^2\alpha_j^2]}\right\},$$

and hence
$$H_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{4(-1)^{j+k}}{l}\left\{\frac{-M}{\alpha_k[(\alpha_j^2+\alpha_k^2)^2+M^2\alpha_j^2]}\right\}\sin\alpha_j\eta\cos\alpha_k\xi. \tag{17}$$

The expressions (16) and (17) may also be obtained by taking the limit $t\to\infty$ in the time dependent solution for the development of the flow from rest given by Lundgren, Atabek and Chang (6).

We now examine the solution given by (12) and (13). Noting the relationship

$$1=\sum_{j=0}^{\infty}\frac{2(-1)^j}{\alpha_j}\cos\alpha_j\eta\quad\text{for}\quad|\eta|<1,$$

and using the appropriate Fourier expansion for $\cosh\beta_k\eta$, $\cosh\gamma_k\eta$ given in the Appendix, we have

$$\left\{1-\frac{\gamma_k\cosh\beta_k\eta}{(\gamma_k-\beta_k)\cosh\beta_k}+\frac{\beta_k\cosh\gamma_k\eta}{(\gamma_k-\beta_k)\cosh\gamma_k}\right\}$$

$$=\sum_{j=0}^{\infty}2(-1)^j\cos\alpha_j\eta\left\{\frac{1}{\alpha_j}-\frac{\alpha_j\beta_k}{(M^2+4\alpha_k^2)^{\frac{1}{2}}(\beta_k^2+\alpha_j^2)}+\frac{\alpha_j\beta_k}{(M^2+4\alpha_k^2)^{\frac{1}{2}}(\gamma_k^2+\alpha_j^2)}\right\}$$

$$=\sum_{j=0}^{\infty}\frac{2(-1)^j\cos\alpha_j\eta}{\alpha_j}\left\{\frac{\alpha_k^2(\alpha_k^2+\alpha_j^2)}{(\alpha_k^2+\alpha_j^2)^2+\alpha_j^2M^2}\right\}.$$

Therefore, from (12)

$$V=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{4(-1)^{j+k}}{l}\left\{\frac{\alpha_j^2+\alpha_k^2}{\alpha_j\alpha_k[(\alpha_j^2+\alpha_k^2)^2+M^2\alpha_j^2]}\right\}\cos\alpha_j\eta\cos\alpha_k\xi. \tag{18}$$

Similarly,

$$\left\{\frac{\beta_k\sinh\gamma_k\eta}{(\gamma_k-\beta_k)\cosh\gamma_k}-\frac{\gamma_k\sinh\beta_k\eta}{(\gamma_k-\beta_k)\cosh\beta_k}\right\}$$

$$=\sum_{j=0}^{\infty}2(-1)^j\sin\alpha_j\eta\left\{\frac{\beta_k\gamma_k}{(M^2+4\alpha_k^2)^{\frac{1}{2}}(\gamma_k^2+\alpha_j^2)}-\frac{\beta_k\gamma_k}{(M^2+4\alpha_k^2)^{\frac{1}{2}}(\beta_k^2+\alpha_j^2)}\right\}$$

$$=\sum_{j=0}^{\infty}2(-1)^j\sin\alpha_j\eta\left\{\frac{-\alpha_k^2M}{(\alpha_j^2+\alpha_k^2)^2+M^2\alpha_j^2}\right\}$$

whence,
$$H_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{4(-1)^{j+k}}{l}\left\{\frac{-M}{\alpha_k[(\alpha_j^2+\alpha_k^2)^2+M^2\alpha_j^2]}\right\}\sin\alpha_j\eta\cos\alpha_k\xi. \tag{19}$$

Equations (18) and (19) are identical with equations (16) and (17) thus establishing the desired result.

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APPENDIX. EVALUATION OF FOURIER COEFFICIENTS

(*a*) $\cos\gamma_j\xi\cosh\beta_j\xi$. We may express this in the form,

$$\cos\gamma_j\xi\cosh\beta_j\xi = \sum_{k=0}^{\infty} b_k\cos\alpha_k\xi \quad \text{for} \quad |\xi| < l,$$

where $\alpha_k = (k+\frac{1}{2})(\pi/l)$, and the Fourier coefficients, b_k, are given by

$$b_k = \frac{2}{l}\int_0^l \cos\gamma_j\xi\cosh\beta_j\xi\cos\alpha_k\xi\,d\xi.$$

Setting $\Phi = \gamma_j+\alpha_k$, $\Psi = \gamma_j-\alpha_k$, and integrating by parts we have,

$$b_k = \frac{(-1)^k}{l}\left\{\cosh\beta_j l\cos\gamma_j l\left(\frac{\Phi}{\beta_j^2+\Phi^2}-\frac{\Psi}{\beta_j^2+\Psi^2}\right) - \sinh\beta_j l\sin\gamma_j l\left(\frac{\beta_j}{\beta_j^2+\Phi^2}-\frac{\beta_j}{\beta_j^2+\Psi^2}\right)\right\}.$$

(*b*) $\sin\gamma_j\xi\sinh\beta_j\xi$. Similarly,

$$\sin\gamma_j\xi\sinh\beta_j\xi = \sum_{k=0}^{\infty} b_k\cos\alpha_k\xi \quad \text{for} \quad |\xi| < l,$$

where

$$b_k = \frac{(-1)^k}{l}\left\{\cosh\beta_j l\cos\gamma_j l\left(\frac{\beta_j}{\beta_j^2+\Phi^2}-\frac{\beta_j}{\beta_j^2+\Psi^2}\right)+\sinh\beta_j l\sin\gamma_j l\left(\frac{\Phi}{\beta_j^2+\Phi^2}-\frac{\Psi}{\beta_j^2+\Psi^2}\right)\right\}.$$

(*c*) $\cosh\beta_k\eta$.

$$\cosh\beta_k\eta = \sum_{j=0}^{\infty}\frac{2(-1)^j\alpha_j\cosh\beta_k}{\beta_k^2+\alpha_j^2}\cos\alpha_j\eta \quad \text{for} \quad |\eta| < 1.$$

(*d*) $\sinh\beta_k\eta$.

$$\sinh\beta_k\eta = \sum_{j=0}^{\infty}\frac{2(-1)^j\beta_k\cosh\beta_k}{\beta_k^2+\alpha_j^2}\sin\alpha_j\eta \quad \text{for} \quad |\eta| < 1.$$

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