Teaching Notes

Projectiles from circular motion

A mathematical model

A uniform rod AB of length $2L$ metres and mass M kg can rotate freely in a vertical plane about a horizontal axis through a point O of the rod. A heavy particle of mass M' kg is attached to the rod at A. One end of a light inextensible string of length ℓ metres is attached to the rod at B. The other end of the string is attached to a particle P of mass m kg.

Initially, there is no motion and the system is held so that the string is taut and lying in the same vertical plane below the rod. The system is released at time $t = 0$ and moves under gravity. The particle P moves under the action of two forces: its weight and the tension in the string, T Newtons.

In the ensuing motion, particle P and the string sweep around O until they separate from the rod. The particle then continues its flight through the air passing above O and lands on horizontal ground h metres lower than O with the string attached. Air resistance is ignored.

The diagrams

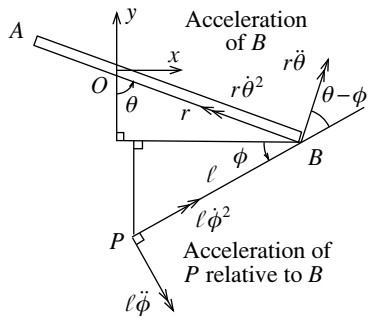


FIGURE 1

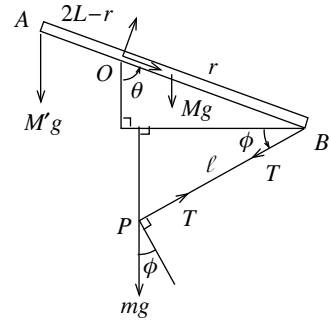


FIGURE 2

Figure 1 shows the circular motion components of the accelerations of B and of P relative to B where $OB = r$ metres, θ is the angle in radians which OB makes with the downward vertical and ϕ is the angle in radians which the string BP makes with the negative x-direction.

Figure 2 shows the external forces acting on the system together with the tension in the string. The weight of the uniform rod is shown as a single force acting through its midpoint.

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The equations of motion

For P, using Newton's Second Law $F = ma$ perpendicular to the string gives

$$mg\cos\phi = m\left(\ell\ddot{\phi} - r\ddot{\theta}\sin(\theta-\phi) - r\dot{\theta}^2\cos(\theta-\phi)\right)$$

which is equivalent to

$$\ell\ddot{\phi} = r\ddot{\theta}\sin(\theta-\phi) + r\dot{\theta}^2\cos(\theta-\phi) + g\cos\phi \tag{1}$$

and using $F = ma$ along PB gives

$$T - mg\sin\phi = m\left(\ell\dot{\phi}^2 + r\ddot{\theta}\cos(\theta-\phi) - r\dot{\theta}^2\sin(\theta-\phi)\right). \tag{2}$$

For the rod and mass M' using $C = I\ddot{\theta}$ about the horizontal axis through O gives

$$M'g(2L-r)\sin\theta - Mg(r-L)\sin\theta - Tr\cos(\theta-\phi) = I\ddot{\theta}, \tag{3}$$

where $I = \frac{1}{3}ML^2 + M(r-L)^2 + M'(2L-r)^2$.

Eliminating T from (2) and (3) leads to

$$\begin{aligned}\left(I + mr^2\cos^2(\theta-\phi)\right)\ddot{\theta} = {} & mr\cos(\theta-\phi)\left(r\dot{\theta}^2\sin(\theta-\phi) - \ell\dot{\phi}^2 - g\sin\phi\right)\\ & + g\sin\theta\left(M'(2L-r) - M(r-L)\right).\end{aligned} \tag{4}$$

A numerical solution method

The initial angular accelerations $\ddot{\theta}_0$ and $\ddot{\phi}_0$ when $t = 0$ can be found from (4) and (1) using the initial values $\theta = \theta_0$, $\dot{\theta}_0 = 0$, $\phi = \phi_0$, $\dot{\phi}_0 = 0$. Substituting these values into the equations for constant angular acceleration

$$\delta\theta = \dot{\theta}_0(\delta t) + \tfrac{1}{2}\ddot{\theta}_0(\delta t)^2 \qquad \text{and} \qquad \dot{\theta} = \dot{\theta}_0 + \ddot{\theta}_0(\delta t) \tag{5}$$

give approximations for $\theta = \theta_0 + \delta\theta$, $\dot{\theta}$ and similarly for ϕ, $\dot{\phi}$ at time δt seconds.

The values of θ, $\dot{\theta}$, ϕ and $\dot{\phi}$ at time $t = \delta t$ are then used with (4), (1) and (5) to obtain approximations for θ, $\dot{\theta}$, ϕ and $\dot{\phi}$ at time $t = 2(\delta t)$. This iterative process is repeated many times until the instant is reached when the string is released and particle P becomes a projectile. The approximations obtained in this way can be made very accurate by choosing a very small value for δt and working with 15 significant figures.

The position and velocity of the particle

Figure 1 shows horizontal and vertical x- and y-axes passing through O. The coordinates of $P(x, y)$ are given in terms of θ and ϕ by

$$x = r\sin\theta - \ell\cos\phi; \qquad y = -r\cos\theta - \ell\sin\phi \tag{6}$$

and, by differentiating with respect to t, the components of the velocity of P are given in terms of θ, $\dot{\theta}$, ϕ and $\dot{\phi}$ by

$$\dot{x} = r\cos\theta\,\dot{\theta} + \ell\sin\phi\,\dot{\phi}; \qquad \dot{y} = r\sin\theta\,\dot{\theta} - \ell\cos\phi\,\dot{\phi}. \tag{7}$$

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One further differentiation enables the circular motion accelerations used in the first two equations of motion to be checked.

Finding the range

Consider a particle projected from the origin at time $t = 0$ with components of velocity $\dot{x}$ and $\dot{y}$. If the origin is h metres above the ground, the particle lands at time t where

$$-h = \dot{y}t - \tfrac{1}{2}gt^2 \qquad \left[s = ut + \tfrac{1}{2}at^2\right]$$

$$\Rightarrow t = \frac{\dot{y} + \sqrt{\dot{y}^2 + 2gh}}{g} \qquad (t > 0).$$

In this time, the particle travels $\dot{x}\left(\dfrac{\dot{y} + \sqrt{\dot{y}^2 + 2gh}}{g}\right)$ horizontally.

However, as particle P starts from (x, y) instead of the origin, it lands at a horizontal distance of

$$-\left(x + \dot{x}\left(\frac{\dot{y} + \sqrt{\dot{y}^2 + 2g(y + h)}}{g}\right)\right) \tag{8}$$

in the negative x-direction from the point on the ground directly below O.

The θ, $\dot{\theta}$, ϕ and $\dot{\phi}$ values can be used at all times while the string remains taut to find the position, velocity and corresponding range of P using (6), (7) and (8). From these ranges, the maximum range can be found with associated θ and ϕ values.

Improving the maximum range

To find the range of the particle, numerical values are needed for the masses M', M, m, the lengths L, ℓ, r, h and the angles θ_0, ϕ_0.

$$\text{Let } M' = 0.15, M = 0.01, m = 0.005, L = 0.15, \ell = 0.2, h = 1, \tag{9}$$

leaving θ_0, ϕ_0 and r to be chosen. The value of g is taken to be 9.8. Considering motion while $\theta_0 \leqslant \theta \leqslant \dfrac{5\pi}{3}$, $\phi_0 \leqslant \phi \leqslant 2\pi$ and $T \geqslant 0$, let $\theta_0 = \dfrac{\pi}{18} = 10°$, $\phi_0 = \dfrac{\pi}{4} = 45°$ and $r = 0.15$.

Using a programmable calculator and taking $\delta t = 0.001$, the maximum range is found to be approximately 3.0 metres. Figure 3 shows a sequence of graphs leading to a maximum range of 6.3 metres for

$$\theta_0 = \frac{53\pi}{180} = 53°, \phi_0 = 0, r = 0.2. \tag{10}$$

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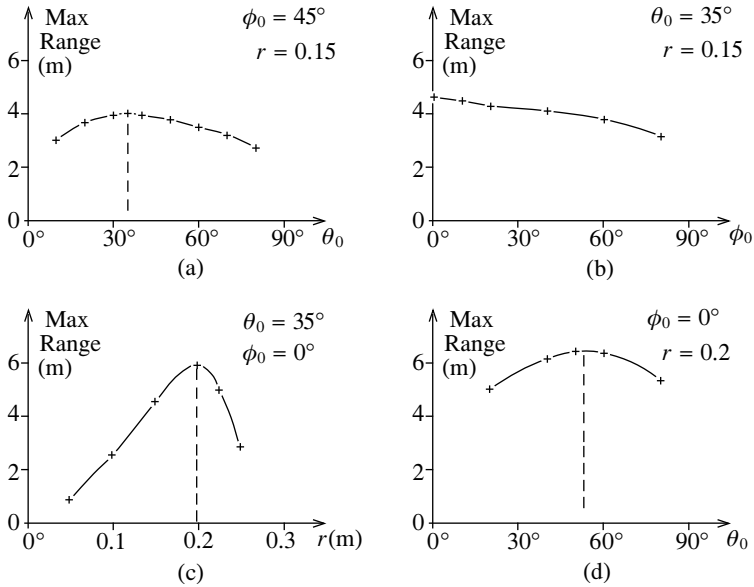


FIGURE 3

The following sequence of statements shows the framework for the construction of the program that obtained the coordinates of the points plotted in graph (a) of Figure 3.

Assign values to M', M, m, L, ℓ and h using (9).
Let $\delta t = 0.001$, $\phi_0 = \frac{1}{4}\pi$ and $r = 0.15$.
Prompt 'θ_0 in degrees = ?' and convert to radians to obtain θ_0.
Reset values $\theta = \theta_0$, $\dot{\theta} = 0$, $\phi = \phi_0$, $\dot{\phi} = 0$, MaxRange = 0.
Loop for values of t from δt to $3 + \frac{1}{2}\delta t$ in steps of δt.
    $\ddot{\theta} = \ldots$ using equation (4)
    $\ddot{\phi} = \ldots$ using equation (1)
    $T = \ldots$ using equation (2)
    $\theta = \theta + \dot{\theta}(\delta t) + \frac{1}{2}\ddot{\theta}(\delta t)^2$, $\dot{\theta} = \dot{\theta} + \ddot{\theta}(\delta t)$
    Similarly $\phi = \ldots$, $\dot{\phi} = \ldots$.
    If $\theta > \frac{5}{3}\pi$, $\theta > 2\pi$ or $T < 0$ then leave the loop.
    Range = … using (6), (7) and (8).
    If Range > MaxRange then change MaxRange.
Take the next t value through the loop.
Display the MaxRange value.

With $\delta t = 0.0001$ and the values given in (9) and (10), the maximum range is 6.28 metres to 3 significant figures and occurs when $\theta \approx 168.3°$ and $\phi \approx 227.8°$. Figure 4 provides more details of this trajectory.

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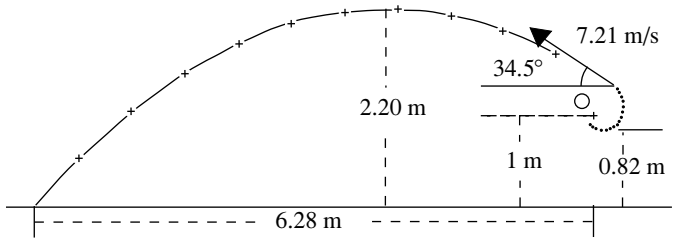


FIGURE 4

Releasing the string

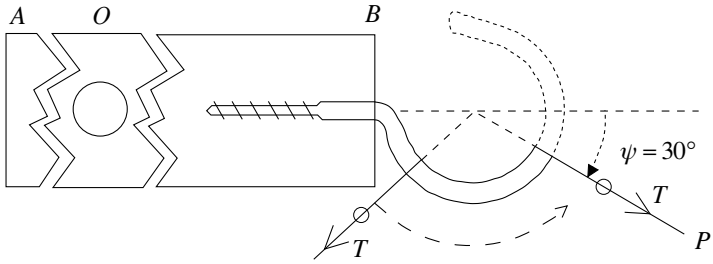


FIGURE 5

Let the string have a loop at its end which passes over a smooth circular arc at the end B of the rod (Figure 5). Let ψ be the angle between the directions of the rod AB and the string BP. Then

$$\psi = 90° + \theta - \phi.$$

Figure 6 shows ψ decreasing with time. For the maximum range,

$$\psi = 90° + 168.3° - 227.8° = 30.5° \approx 30°.$$

As ψ decreases, the tension in the string (Figure 7) pulls the loop along the smooth arc. It will slip off when $\psi = 30°$ if the arc ends at this point.

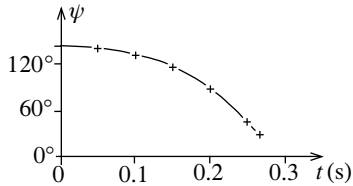


FIGURE 6

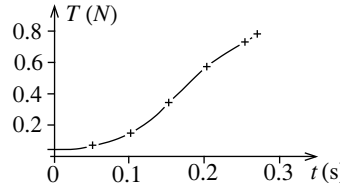


FIGURE 7

Other ranges

The range depends on the final value of ψ and the values of the variables given in (9) and (10). When the final value of ψ is 30°, Figure 8 shows the effect of changing the length of the string while keeping the other variables constant and Figure 9 shows the effect of changing only the mass M'.

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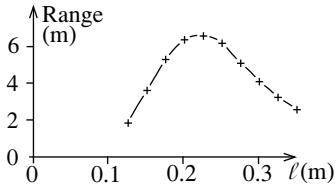


FIGURE 8

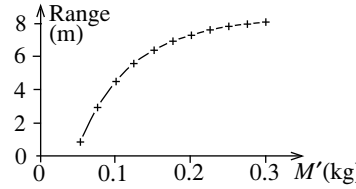


FIGURE 9

An indicator of accuracy

A maximum range obtained using an extra term of the Taylor series in the iterative formulae

$$\theta_{n+1} = \theta_n + \dot{\theta}_n \delta t + \frac{1}{2}\ddot{\theta}_n (\delta t)^2 + \frac{1}{3}\dddot{\theta}_n (\delta t)^3$$

$$\dot{\theta}_{n+1} = \dot{\theta}_n + \ddot{\theta}_n \delta t + \frac{1}{2}\dddot{\theta}_n (\delta t)^2 \qquad (11)$$

and similarly for ϕ_{n+1}, $\dot{\phi}_{n+1}$ for $n = 0, 1, 2, \ldots$ could be compared to the maximum range found using (5) to give an indicator of accuracy. The values of $\ddot{\theta}_n$ and $\ddot{\phi}_n$ can be found by differentiating (1) and (4) with respect to t and substituting the values of the constants and θ_n, $\dot{\theta}_n$, $\ddot{\theta}_n$, ϕ_n, $\dot{\phi}_n$, $\ddot{\phi}_n$ into the two derived equations.

Using $\ddot{\theta} = 3\dot{\theta} - 2\theta + 4$ given that $\theta = 1$, $\dot{\theta} = 0$ when $t = 0$ as an example, the programmed calculator gave the 6 approximations obtained for θ when $t = 3$ shown in the table below.

δt	0.01	0.001	0.0001
θ using method (5)	337.14	362.29	364.96
θ using method (11)	365.05	365.2556	365.25770

The exact solution $\theta = e^{2t} - 2e^t + 2$ provides the true value of $e^6 - 2e^3 + 2 \approx 365.25772$.

Practical construction

Wooden dowelling, strong cotton thread and a shortened smooth hook (Figure 5) can be used with light particles such as those referred to in (9). The most suitable values for ℓ, r, θ_0 and the final value of ψ can be determined experimentally. The values will very much depend on the design, efficiency and timing of the machine.

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