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SOME LIMITING BEHAVIOR FOR ASYMPTOTICALLY NEGATIVE ASSOCIATED RANDOM VARIABLES

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In this paper, we study the almost sure convergence for sequences of asymptotically negative associated (ANA) random variables. As a result, we extend the classical Khintchine–Kolmogorov convergence theorem, Marcinkiewicz strong law of large numbers, and the three series theorem for sequences of independent random variables to sequences of ANA random variables without necessarily adding any extra conditions.

Keywords: applied probability, operations research, simulation, stochastic dynamic programming

1. INTRODUCTION AND LEMMAS

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The random variables we deal with are all defined on $(\Omega, \mathcal{F}, P)$. Let $\{X_n; n \geq 1\}$ be a sequence of random variables.

Random variables $X_1, X_2, \ldots, X_n, n \geq 2$, are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)) \leq 0,$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable) functions such that this covariance exists. A sequence of random variables $\{X_i; i \geq 1\}$ is said to be NA if its every finite subfamily is NA.

The definition of NA was introduced by [1]. A statistical test depends greatly on sampling. The random sampling without replacement from a finite population is NA, but is not independent. As pointed out and proved by [1], a number of well-known multivariate distributions possess the NA property, such as (a) multinomial, (b) convolution of unlike multinomials, (c) multivariate hypergeometric, (d) Dirichlet, (e) Dirichlet compound multinomial, (f) negatively correlated normal distribution, (g) permutation distribution,

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(h) random sampling without replacement, and (i) joint distribution of ranks. NA sampling has wide applications such as in multivariate statistical analysis and reliability theory. Because of the wide applications of NA sampling, the limit behaviors of NA random variables have received more and more attention recently. One can refer to [1] for fundamental properties, [4] for the three series theorem, and [7,8], for the Berry–Esseen-type bound and the strong convergence, etc.

A new kind of dependence structures called asymptotically negative association was proposed by [10,11], which is a useful weakening of the definition of NA.

Let N be a set of natural numbers. For two non-empty disjoint sets $S, T \subset N$, we define $\mathrm{dist}(S,T)$ to be $\inf\{|s-t|; s \in S, t \in T\}$.

DEFINITION 1.1: *A sequence $\{X_n; n \geq 1\}$ of random variables is said to be asymptotically negative associated (ANA) if*

$$\rho^-(n) = \sup\{\rho^-(S,T); S,T \subset N,\ dist(S,T) \geq n\} \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$\rho^-(S,T) \hat{=} 0 \vee \left\{ \frac{\mathrm{Cov}(f(X_i, i \in S), g(X_j, j \in T))}{\sqrt{\mathrm{Var}(f(X_i, i \in S))\mathrm{Var}(g(X_j, j \in T))}}; f, g \in \mathcal{C} \right\},$$

and $\mathcal{C}$ is the set of increasing for every variable functions.

It is obvious that a sequence of ANA random variables is NA if and only if $\rho^-(1) = 0$. Compared to NA, ANA defines a strictly larger class of random variables (for detailed examples; see [10]. Consequently, the study of the limit theorems for ANA random variables is of much interest. A number of statisticians have studied ANA random variables and a series of useful results have been established. We refer to [10,11], for the central limit theorem, [2,3,5,9], for some inequalities of the maximum of partial sums and some limiting behavior, [6] for the law of the iterated logarithm.

The main purpose of this paper is to study the strong limit theorems of partial sums of ANA random variables and try to obtain some new results. We establish Khiatchine–Kolmogorov convergence theorem, the three series theorem and Marcinkiewicz strong law of large numbers for ANA random variables. Our results in this paper extend the classical Khintchine–Kolmogorov convergence theorem, Marcinkiewicz strong law of large numbers, and the three series theorem for independent sequences to ANA sequences without necessarily adding any extra conditions.

From the definition of an ANA sequence, we have

LEMMA 1.2: *Decreasing or increasing functions defined on disjoint subsets of an ANA sequence $\{X_n; n \geq 1\}$ with mixing coefficients $\rho^-(s)$ is also ANA with mixing coefficients not greater than $\rho^-(s)$.*

LEMMA 1.3 [5]: *Let $0 \leq r < 1/12$ and K be a positive integer. Let $\{X_n; n \geq 1\}$ be a sequence of ANA random variables with $\rho^-(K) \leq r$, $\mathbb{E}X_n = 0$, and $\mathbb{E}X_n^2 < \infty$. Then for all $n \geq 1$ there exists a positive constant $D = D(2,K,r)$ such that*

$$\mathbb{E}\left(\max_{1 \leq i \leq n} |S_i|^2\right) \leq D \sum_{i=1}^{n} \mathbb{E}X_i^2,$$

where $S_i = \sum_{j=1}^{i} X_j$.

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2. MAIN RESULTS AND THE PROOFS

THEOREM 2.1 (Khintchine–Kolmogorov convergence theorem): *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of random variables which satisfies*

$$\sum_{n=1}^{\infty} \mathrm{Var} X_n < \infty, \tag{2.1}$$

then $\sum_{n=1}^{\infty}(X_n - \mathbb{E}X_n)$ *converges almost surely and in quadratic mean.*

In the following, for some $c > 0$, let $X_n^{(c)} = -cI_{(X_n<-c)} + X_n I_{(|X_n|\leq c)} + cI_{(X_n>c)}$.

THEOREM 2.2 (The three series theorem): *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of random variables, for some* $c > 0$*, if the following three series converge, that is,*

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \mathbb{E}X_n^{(c)} \quad \textit{converges}, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \mathrm{Var} X_n^{(c)} < \infty, \tag{2.4}$$

then $\sum_{n=1}^{\infty} X_n$ *converges almost surely.*

COROLLARY 2.3: *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of random variables satisfying* $\mathbb{E}X_n = 0, n \geq 1$*, and for some* $c > 0$

$$\sum_{n=1}^{\infty} \mathbb{E}\left[X_n^2 I_{(|X_n|\leq c)} + |X_n| I_{(|X_n|>c)}\right] < \infty, \tag{2.5}$$

then $\sum_{n=1}^{\infty} X_n$ *converges a.s.*

COROLLARY 2.4: *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of random variables, for some constants* $0 < \alpha_n \leq 2, \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{\alpha_n} < \infty$*, where* $\mathbb{E}X_n = 0$ *when* $1 < \alpha_n \leq 2$*, then* $\sum_{n=1}^{\infty} X_n$ *converges a.s..*

Taking $\alpha_n = p, 0 < p \leq 2$ in Corollary 2.4 we can obtain the following corollary.

COROLLARY 2.5: *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of random variables, for* $0 < p \leq 2, \sum_{n=1}^{\infty} \mathbb{E}|X_n|^p < \infty$*, and* $\mathbb{E}X_n = 0$ *for* $1 < p \leq 2$*, then* $\sum_{n=1}^{\infty} X_n$ *converges a.s..*

THEOREM 2.6: *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of random variables, and let* $\{g_n(x); n \geq 1\}$ *be a sequence of even functions, positive and increasing in the half-line* $x > 0$*. If one or the other of the following conditions is satisfied for every* $n \geq 1$:

(i) $x/g_n(x)$ *is a increasing function of* x *on the half-line* $x > 0$ *for all* $n \geq 1$,

(ii) $x/g_n(x)$ *and* $g_n(x)/x^r, 1 < r \leq 2$ *are decreasing functions of* x *on half-line* $x > 0$ *for all* $n \geq 1$*, and also* $\mathbb{E}X_n = 0$.

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Then for any positive number sequence $\{a_n; n \geq 1\}$ *with* $a_n \uparrow \infty$ *satisfying*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}g_n(X_n)}{g_n(a_n)} < \infty, \tag{2.6}$$

the series $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ *converges almost surely and therefore* $a_n^{-1} \sum_{i=1}^{n} X_i \to 0$ *a.s.*

Taking $g_n(x) = |x|^p, 0 < p \leq 2$ and $r = 2$ in Theorem 2.6 we can immediately obtain the following corollary.

COROLLARY 2.7: *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of random variables, and let* $\{a_n; n \geq 1\}$ *be a sequence of positive numbers with* $a_n \uparrow \infty$, *if* $\sum_{n=1}^{\infty} \frac{\mathbb{E}|X_n|^p}{a_n^p} < \infty$, *and for* $1 < p \leq 2, \mathbb{E}X_n = 0$, *then*

$$a_n^{-1} \sum_{k=1}^{n} X_k \to 0 \text{ a.s., } n \to \infty.$$

THEOREM 2.8 (Marcinkiewicz strong law of large numbers): *Let* $\{X_n; n \geq 1\}$ *be an ANA sequence of identically distributed random variables, for* $0 < p < 2$,

$$\mathbb{E}|X_1|^p < \infty, \tag{2.7}$$

and for $1 \leq p < 2, \mathbb{E}X_1 = 0$, *then*

$$n^{-1/p} \sum_{k=1}^{n} X_k \to 0 \quad \text{a.s.} \quad n \to \infty. \tag{2.8}$$

Remark 2.9: Theorems 2.1, 2.2, 2.6, and 2.8 extend corresponding results for independent sequences to ANA sequences of random variables without necessarily adding any extra conditions.

Remark 2.10: Theorems 2.1 and 2.2 are the important tools in establishing various almost sure results for ANA sequences of random variables.

In the following, the symbol c stands for a generic positive constant which may differ from one place to another. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n, $S_n = \sum_{j=1}^{n} X_j$, and I denote an indicator function.

PROOF OF THEOREM 2.1: Without loss of generality, assume that $\mathbb{E}X_n = 0$. From the definition of ANA, $\rho^-(n) \to 0$ as $n \to \infty$, thus, for $0 < r < 1/12$, there is a positive integer K such that $\rho^-(K) \leq r$. Therefore, for $m \geq n \to \infty$, by Lemma 1.2 and (2.1) we have

$$\mathbb{E}(S_m - S_n)^2 \ll \sum_{k=n+1}^{m} \mathbb{E}X_k^2 \to 0.$$

Whence $\{S_n; n \geq 1\}$ is a Cauchy sequence in L_2, according to the Cauchy Convergence Criterion, there exists a random variable S such that $\mathbb{E}(S_n - S)^2 \to 0$, that is, $S_n \xrightarrow{L_2} S$.

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A fortiori, $S_n \xrightarrow{P} S$ and so there exist positive integers $n_k \to \infty$ such that

$$S_{n_k} \to S \quad \text{a.s.} \quad k \to \infty. \tag{2.9}$$

On the other hand, again employing Lemma 1.2 and (2.1), for any given $\varepsilon > 0$ (letting $n_0 = 0$)

$$\begin{aligned}\sum_{k=1}^{\infty} P\left(\max_{n_{k-1}<j\le n_k} |S_j - S_{n_{k-1}}| > \varepsilon\right) &\ll \sum_{k=1}^{\infty} \mathbb{E}\left(\max_{n_{k-1}<j\le n_k} |S_j - S_{n_{k-1}}|\right)^2 \\ &\ll \sum_{k=1}^{\infty} \sum_{j=n_{k-1}+1}^{n_k} \mathbb{E}X_j^2 = \sum_{j=1}^{\infty} \mathbb{E}X_j^2 \\ &< \infty.\end{aligned}$$

By the Borel–Cantilli lemma,

$$\max_{n_{k-1}<j\le n_k} |S_j - S_{n_{k-1}}| \to 0 \quad \text{a.s.} \quad k \to \infty,$$

which together with (2.9), according to the method of subsequence gives

$$S_n \to S \quad \text{a.s.}, \quad n \to \infty. \qquad \blacksquare$$

Proof of Theorem 2.2.: Obviously, $X_n^{(c)}$ is increasing on X_n, thus by Lemma 1.2, $\{X_n^{(c)}; n \ge 1\}$ is also an ANA sequence of random variables. Hence, if (2.4) holds, then $\sum_{n=1}^{\infty}(X_n^{(c)} - \mathbb{E}X_n^{(c)})$ converges a.s. by Theorem 2.1, whence (2.3) implies that

$$\sum_{n=1}^{\infty} X_n^{(c)} \quad \text{converges a.s.} \tag{2.10}$$

According to (2.2),

$$\sum_{n=1}^{\infty} P(X_n \ne X_n^{(c)}) = \sum_{n=1}^{\infty} P(|X_n| > c) < \infty.$$

By the Borel–Cantilli lemma,

$$P(X_n \ne X_n^{(c)};\ \text{i.o.}) = 0,$$

which together with (2.10), gives that $\sum_{n=1}^{\infty} X_n$ converges a.s. $\blacksquare$

Proof of Corollary 2.3: By the Markov inequality and (2.5),

$$\sum_{n=1}^{\infty} P(|X_n| > c) = \sum_{n=1}^{\infty} P(|X_n| I_{(|X_n|>c)} > c) \le \sum_{n=1}^{\infty} c^{-1}\mathbb{E}|X_n| I_{(|X_n|>c)} < \infty.$$

Thus, by $\mathbb{E}X_n = 0$, (2.5) and the c_r inequality,

$$\sum_{n=1}^{\infty} |\mathbb{E}X_n^{(c)}| \le \sum_{n=1}^{\infty} \left(cP(|X_n| > c) + |\mathbb{E}X_n I_{(|X_n|\le c)}|\right) \ll \sum_{n=1}^{\infty} \mathbb{E}|X_n| I_{(|X_n|>c)} < \infty,$$

and

$$\sum_{n=1}^{\infty} \mathrm{Var}X_n^{(c)} \ll \sum_{n=1}^{\infty} \left(c^2 P(|X_n| > c) + \mathbb{E}X_n^2 I_{(|X_n|\le c)}|\right) < \infty,$$

whence Corollary 2.3 follows from Theorem 2.2. $\blacksquare$

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PROOF OF COROLLARY 2.4: It suffices to consider separately three cases: (i) $1 < \alpha_n \le 2$, $n \ge 1$, (ii) $0 < \alpha_n \le 1, n \ge 1$, and (iii) $1 < \alpha_n \le 2$ for some n and $0 < \alpha_n \le 1$ for other n.

(i) If $1 < \alpha_n \le 2$ for all $n \ge 1$, then

$$X_n^2 I_{(|X_n| \le 1)} + |X_n| I_{(|X_n| > 1)} \le |X_n|^{\alpha_n} I_{(|X_n| \le 1)} + |X_n|^{\alpha_n} I_{(|X_n| > 1)} = |X_n|^{\alpha_n},$$

whence (2.5) is obtained from $\sum_{n=1}^{\infty} \mathbb{E}|X_n|^{\alpha_n} < \infty$. Therefore, $\sum_{n=1}^{\infty} X_n$ converges a.s. by Corollary 2.3.

(ii) If $0 < \alpha_n \le 1$ for all $n \ge 1$, then

$$\sum_{n=1}^{\infty} \mathbb{E}((X_n)^2 + |X_n|) I_{(|X_n| \le 1)} \le 2 \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{\alpha_n} < \infty,$$

and

$$\sum_{n=1}^{\infty} P(|X_n| > 1) \le \sum_{n=1}^{\infty} \mathbb{E}|X_n|^{\alpha_n} < \infty,$$

so that the three series of Theorem 2.2 converge for $c = 1$. Hence $\sum_{n=1}^{\infty} X_n$ converges a.s. by Theorem 2.2.

(iii) If $1 < \alpha_n \le 2$ for some n and $0 < \alpha_n \le 1$ for other n, let

$$N_1 = \{n; 1 < \alpha_n \le 2\}, \quad N_2 = \{n; 0 < \alpha_n \le 1\}.$$

Define

$$X_n' = \begin{cases} X_n, & n \in N_1, \\ 0, & \text{other}, \end{cases} \quad \alpha_n' = \begin{cases} \alpha_n, & n \in N_1, \\ 2, & \text{other}, \end{cases}$$

and

$$X_n'' = \begin{cases} X_n, & n \in N_2, \\ 0, & \text{other}, \end{cases} \quad \alpha_n'' = \begin{cases} \alpha_n, & n \in N_2, \\ 1, & \text{other}. \end{cases}$$

Then $\sum_{n=1}^{\infty} X_n = \sum_{n=1}^{\infty} X_n' + \sum_{n=1}^{\infty} X_n''$ converges a.s. by (i) and (ii). ■

PROOF OF THEOREM 2.6: For each $n \ge 1$, let $Z_n = \frac{X_n}{a_n}$, $Z_n^{(1)} = -I_{(Z_n < -1)} + Z_n I_{(|Z_n| \le 1)} + I_{(Z_n > 1)}$. Take $c = 1$ in Theorem 2.2. Since $g_n(x) \uparrow$ as $x > 0$, $g_n(X_n) \ge g_n(a_n)$ as $\{|Z_n| > 1\}$. So by (2.6),

$$\sum_{n=1}^{\infty} P(|Z_n| > 1) \le \sum_{n=1}^{\infty} P(g_n(X_n) \ge g_n(a_n)) \le \sum_{n=1}^{\infty} \frac{\mathbb{E} g_n(X_n)}{g_n(a_n)} < \infty. \tag{2.11}$$

We shall suppose that the function $g_n(x)$ satisfies condition (i); then in the interval $|x| \le a_n$, we have

$$\frac{|x|}{a_n} \le \frac{g_n(x)}{g_n(a_n)} \le 1.$$

Therefore,

$$\frac{|x|^r}{a_n^r} \le \frac{g_n^r(x)}{g_n^r(a_n)} \le \frac{g_n(x)}{g_n(a_n)} \quad \text{for} \quad 1 < r \le 2.$$

If, however, the function $g_n(x)$ satisfies condition (ii), then in the same interval $|x| \le a_n$, it follows from $\frac{g_n(x)}{|x|^r} \downarrow$ that $\frac{g_n(a_n)}{a_n^r} \le \frac{g_n(x)}{|x|^r}$, that is, $\frac{|x|^r}{a_n^r} \le \frac{g_n(x)}{g_n(a_n)}$. Therefore in both cases,

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we have

$$\frac{|x|^r}{a_n^r} \le \frac{g_n(x)}{g_n(a_n)}, \quad \text{for} \quad |x| \le a_n \quad \text{and} \quad 1 < r \le 2.$$

That, in conjunction with (2.6) and (2.11), yields

$$\sum_{n=1}^{\infty} \mathrm{Var} Z_n^{(1)} \le \sum_{n=1}^{\infty} \mathbb{E}(Z_n^{(1)})^2 \le \sum_{n=1}^{\infty} \left(P(|Z_n| > 1) + \mathbb{E}\frac{X_n^2}{a_n^2} I_{(|X_n| \le a_n)} \right)$$
$$\ll \sum_{n=1}^{\infty} \frac{\mathbb{E} g_n(X_n)}{g_n(a_n)} < \infty. \tag{2.12}$$

If condition (i) is satisfied, then

$$|\mathbb{E} Z_n^{(1)}| \le P(|Z_n| > 1) + \mathbb{E}|Z_n| I_{(|X_n| \le a_n)} \ll \frac{\mathbb{E} g_n(X_n)}{g_n(a_n)}.$$

If condition (ii) is satisfied, then since $\mathbb{E} X_n = 0$,

$$|\mathbb{E} Z_n^{(1)}| \le P(|Z_n| > 1) + |\mathbb{E} Z_n I_{(|X_n| > a_n)}| \le P(|Z_n| > 1) + \mathbb{E}|Z_n| I_{(|X_n| > a_n)} \ll \frac{\mathbb{E} g_n(X_n)}{g_n(a_n)}.$$

Therefore in both cases, by (2.6) we have

$$\sum_{n=1}^{\infty} |\mathbb{E} Z_n^{(1)}| \ll \sum_{n=1}^{\infty} \frac{\mathbb{E} g_n(X_n)}{g_n(a_n)} < \infty.$$

So, $\sum_{n=1}^{\infty} \mathbb{E} Z_n^{(1)}$ converges. Together with (2.11) and (2.12), from Theorem 2.2 we know that $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ converges a.s. ■

Proof of Theorem 2.8: Let $Y_n = -n^{1/p} I_{(X_n < -n^{1/p})} + X_n I_{(|X_n| < n^{1/p})} + n^{1/p} I_{(X_n > n^{1/p})}$, (2.7) implies that

$$\sum_{n=1}^{\infty} P(X_n \ne Y_n) = \sum_{n=1}^{\infty} P(|X_n| \ge n^{1/p}) = \sum_{n=1}^{\infty} P(|X_1| \ge n^{1/p}) \ll \mathbb{E}|X_1|^p < \infty. \tag{2.13}$$

Hence, $P(|X_n| \ne Y_n, \text{i.o.})=0$ by the Borel–Cantalli lemma. Thus, $\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_k \to 0$ a.s. if and only if $n^{-1/p} \sum_{k=1}^{n} Y_k \to 0$ a.s. So to prove (2.8) it suffices to show that

$$n^{-1/p} \sum_{k=1}^{n} (Y_k - \mathbb{E} Y_k) \to 0 \text{ a.s. } n \to \infty, \tag{2.14}$$

and

$$n^{-1/p} \sum_{k=1}^{n} \mathbb{E} Y_k \to 0 \qquad n \to \infty. \tag{2.15}$$

By Theorem 2.1, to prove (2.14), it suffices to show that

$$\sum_{n=1}^{\infty} \mathrm{Var}\left(\frac{Y_n}{n^{1/p}} \right) < \infty. \tag{2.16}$$

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Since $p < 2$, by (2.7) and (2.13),

$$\begin{aligned}\sum_{n=1}^{\infty} \mathrm{Var}\left(\frac{Y_n}{n^{1/p}}\right) &\le \sum_{n=1}^{\infty} \frac{\mathbb{E}Y_n^2}{n^{2/p}} \le \sum_{n=1}^{\infty} \frac{n^{2/p}P(|X_1| > n^{1/p}) + \mathbb{E}X_1^2 I_{(|X_1|<n^{1/p})}}{n^{2/p}} \\ &\ll \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \sum_{k=1}^{n} \mathbb{E}X_1^2 I_{(k-1\le|X_1|^p<k)} \\ &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} n^{-2/p}\mathbb{E}X_1^2 I_{(k-1\le|X_1|^p<k)} \\ &\ll \sum_{k=1}^{\infty} k^{-2/p+1}\mathbb{E}|X_1|^p k^{(2-p)/p} I_{(k-1\le|X_1|^p<k)} \\ &= \sum_{k=1}^{\infty} \mathbb{E}|X_1|^p I_{(k-1\le|X_1|^p<k)} \\ &< \infty.\end{aligned}$$

So (2.16) holds.

Now we prove (2.15).

(i) For $p = 1$, since $\mathbb{E}|X_1| < \infty$ implies $nP(|X_1| > n) \to 0$. Therefore, $\mathbb{E}Y_n = nP(|X_1| > n) + \mathbb{E}X_n I_{(|X_n|\le n)} = nP(|X_1| > n) + \mathbb{E}X_1 I_{(|X_1|\le n)} \to \mathbb{E}X_1 = 0$. By the Toeplitz lemma, (2.15) holds.

(ii) For $p \ne 1$, by the Kronecker lemma, to prove (2.15), it suffices to show that

$$\sum_{n=1}^{\infty} \frac{|\mathbb{E}Y_n|}{n^{1/p}} < \infty. \tag{2.17}$$

For $0 < p < 1$, by (2.7) and (2.13),

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|\mathbb{E}Y_n|}{n^{1/p}} &\le \sum_{n=1}^{\infty} \frac{n^{1/p}P(|X_1| > n^{1/p}) + \mathbb{E}|X_n| I_{(|X_n|<n^{1/p})}}{n^{1/p}} \\ &\ll \sum_{n=1}^{\infty}\sum_{j=1}^{n} n^{-1/p}\mathbb{E}|X_1| I_{(j-1\le|X_1|^p<j)} \\ &= \sum_{j=1}^{\infty}\sum_{n=j}^{\infty} n^{-1/p}\mathbb{E}|X_1| I_{(j-1\le|X_1|^p<j)} \\ &\ll \sum_{j=1}^{\infty} j^{-1/p+1}\mathbb{E}|X_1|^p j^{(1-p)/p} I_{(j-1\le|X_1|^p<j)} \\ &= \sum_{j=1}^{\infty} \mathbb{E}|X_1|^p I_{(j-1\le|X_1|^p<j)} \\ &< \infty.\end{aligned}$$

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For $1 < p < 2$, since $\mathbb{E}X_n = 0$ and from (2.7) and (2.13), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|\mathbb{E}Y_n|}{n^{1/p}} &\le \sum_{n=1}^{\infty}\frac{n^{1/p}P(|X_1|>n^{1/p})+|\mathbb{E}X_nI_{(|X_n|<n^{1/p})}|}{n^{1/p}} \ll \sum_{n=1}^{\infty}\frac{|\mathbb{E}X_nI_{(|X_n|\ge n^{1/p})}|}{n^{1/p}}\\ &\le \sum_{n=1}^{\infty}n^{-1/p}\mathbb{E}|X_n|I_{(|X_n|\ge n^{1/p})} = \sum_{n=1}^{\infty}\sum_{j=n}^{\infty}n^{-1/p}\mathbb{E}|X_1|I_{(j\le|X_1|^p<j+1)}\\ &= \sum_{j=1}^{\infty}\sum_{n=1}^{j}n^{-1/p}\mathbb{E}|X_1|I_{(j\le|X_1|^p<j+1)}\\ &\ll \sum_{j=1}^{\infty}j^{-1/p+1}\mathbb{E}|X_1|^pj^{(1-p)/p}I_{(j\le|X_1|^p<j+1)}\\ &< \infty.\end{aligned}$$

Thus, (2.17) holds. ■

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