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Irredundance and Maximum Degree in Graphs

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It is proved that the smallest cardinality among the maximal irredundant sets in an n−vertex graph with maximum degree Δ (≥ 2) is at least $\frac{2n}{3\Delta}$. This substantially improves a bound by Bollobás and Cockayne [1]. The class of graphs which attain this bound is characterised.

1. Introduction and main result

The *open neighbourhood* $N(v)$ of a vertex v of a graph $G = (V, E)$ is defined by $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighbourhood* $N[v]$ by $N[v] = N(v) \cup \{v\}$. The vertex v is said to *dominate* each vertex in $N[v]$. A set $X \subseteq V$ is *irredundant* if each $x \in X$ dominates at least one vertex of G which is not dominated by any other vertex of X. Equivalently, X is irredundant if for each $x \in X$, x is isolated in $G[X]$ (the subgraph induced by X) or x has at least one *private neighbour relative to* X in $V - X$ (which will be called an $X - PN$), *i.e.*, a vertex which is adjacent to x but not to any other vertex of X.

The concept of irredundance is closely related to independence and domination and an extensive bibliography concerning these topics appears in [3], updated in [2].

The (*lower*) *irredundance number* $\mathrm{ir}(G)$ of G is the smallest cardinality among the maximal irredundant sets of vertices of G. Bollobás and Cockayne [1] proved that if G has n vertices and maximum degree Δ, then $\mathrm{ir}(G) \geq \frac{n}{2\Delta - 1}$. In this paper we considerably improve this bound to $\frac{2n}{3\Delta}$:

Theorem 1.1. *If G has n vertices and maximum degree $\Delta \geq 2$, then $ir(G) \geq \frac{2n}{3\Delta}$.*

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2. Proof of the bound

Let G have n vertices, maximum degree Δ and a maximal irredundant set X such that Z is the set of vertices in X which are isolated in $G[X]$ and $Y = X - Z$. Each $x \in X$ has a set B_x of $X - PN$s and since X is irredundant, $B_y \neq \emptyset$ for each $y \in Y$. Also, the private neighbour property implies that the sets B_x $(x \in X)$ are pairwise disjoint. Denote by C the set of vertices of $V - X$ which are adjacent to at least two vertices of X, and let $B = \cup_{x \in X} B_x$ and $R = V - (X \cup B \cup C)$. Note that R is the set of vertices of G which are not dominated by any vertex of X.

For $y \in Y$, a vertex w *annihilates* y (or w is an *annihilator* of y) if $B_y \subseteq N[w]$. The maximality of X implies the following two facts, the proofs of which are given below.

Fact 1. For each $w \in R$ there exists at least one $y \in Y$ such that w annihilates y.

Fact 2. If $w \in B_{y'}$, where $y' \in Y$ and w is adjacent to some vertex $t \in R$, then there exists at least one $y \in Y$ (possibly $y = y'$) such that w annihilates y.

To establish Fact 1, observe that the vertices of $Z \cup \{w\}$ are isolates of $G[X \cup \{w\}]$. If Fact 1 is false, then for each $y \in Y$ there exists $t \in B_y - N(w)$ and t is an $(X \cup \{w\}) - PN$ of y. It follows that $X \cup \{w\}$ is irredundant, contrary to the maximality of X.

To prove Fact 2, note that the vertices of Z are isolates of $G[X \cup \{w\}]$. If Fact 2 is false, then for each $y \in Y$ there exists $y' \in B_y - N[w]$ and y' is an $(X \cup \{w\}) - PN$ of y. Moreover, t is an $(X \cup \{w\}) - PN$ of w and so $X \cup \{w\}$ is irredundant, contradicting the maximality of X.

For $y \in Y$, we define $R_y = \{w \in R \mid w \text{ annihilates } y\}$. It is possible that $R_y = \emptyset$ for some $y \in Y$, but Fact 1 implies that

$$R = \bigcup_{y \in Y} R_y. \tag{1}$$

For $x \in X$, let $b_x = |B_x|$ and for $y \in Y$, let $r_y = |R_y|$.

Theorem 2.1. *With the notation above,*

$$\sum_{y \in Y} (b_y + 2r_y) \leq (2\Delta - 1)|Y|.$$

Proof. Let

$$\begin{aligned} S_1 &= \{y \in Y \mid r_y = 0\} \\ S_2 &= \{y \in Y \mid r_y > 0 \quad \text{and} \quad b_y = 1\} \\ S_3 &= \{y \in Y \mid r_y > 0 \quad \text{and} \quad b_y > 1\} \end{aligned}$$

so that Y is the disjoint union of the S_i. Further, let $s_i = |S_i|$ for $i = 1, 2, 3$. Recall that each $w \in B_y$, where $y \in S_2 \cup S_3$, is an annihilator of some $y' \in Y$. For $y \in S_3$ let $P_y = \{w \in B_y \mid w \text{ annihilates a vertex of } S_1 \cup S_2\}$ and $Q_y = B_y - P_y$, $p_y = |P_y|$ and

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$q_y = |Q_y|$. Further, let $S_2 = \{y_1, \cdots, y_{s_2}\}$, for $i = 1, \cdots, s_2$, let $B_{y_i} = \{u_i\}$ and suppose that u_i is adjacent to precisely x_i vertices of S_3.

Each element of S_1 is annihilated by at most $\Delta - 1$ vertices and the elements of S_2 are annihilated by at most $\sum_{i=1}^{s_2} x_i$ elements of S_3. We conclude that

$$\sum_{y \in S_3} p_y \leq (\Delta - 1)\, s_1 + \sum_{i=1}^{s_2} x_i. \tag{2}$$

Each $w \in Q_y$ ($y \in S_3$) annihilates a vertex $y' \in S_3$ and each such y' is annihilated by at most $(\Delta - 1 - r_y)$ vertices. Therefore,

$$\sum_{y \in S_3} q_y \leq \sum_{y \in S_3} (\Delta - 1 - r_y). \tag{3}$$

Now

$$\sum_{y \in Y} (b_y + 2r_y) = \sum_{y \in S_1} b_y + \sum_{y \in S_2} b_y + \sum_{y \in S_3} p_y + \sum_{y \in S_3} q_y + 2 \sum_{y \in S_2} r_y + 2 \sum_{y \in S_3} r_y.$$

Applying (2) and (3), we deduce that

$$\begin{aligned} \sum_{y \in Y} (b_y + 2r_y) &\leq (\Delta - 1)\, s_1 + s_2 + \left((\Delta - 1)\, s_1 + \sum_{i=1}^{s_2} x_i \right) \\ &+ \left((\Delta - 1)\, s_3 - \sum_{y \in S_3} r_y \right) + 2 \left((\Delta - 1)\, s_2 - \sum_{i=1}^{s_2} x_i \right) + 2 \sum_{y \in S_3} r_y \\ &= s_1(2\Delta - 2) + s_2(2\Delta - 1) + s_3(\Delta - 1) + \sum_{y \in S_3} r_y - \sum_{i=1}^{s_2} x_i. \end{aligned}$$

But for $y \in S_3$, $r_y \leq \Delta - 2$, hence $\sum_{y \in S_3} r_y \leq s_3(\Delta - 2)$. Therefore,

$$\begin{aligned} \sum_{y \in Y} (b_y + 2r_y) &\leq s_1(2\Delta - 2) + s_2(2\Delta - 1) + s_3(2\Delta - 3) \\ &\leq (s_1 + s_2 + s_3)(2\Delta - 1) \\ &= (2\Delta - 1)|Y|. \end{aligned}$$

□

We now prove the main result of this paper.

Proof of Theorem 1.1. Define

$$Y_1 = \{y \in Y \mid |N(y) \cap Y| = 1\}$$

and

$$Y_2 = \{y \in Y \mid |N(y) \cap Y| \geq 2\}.$$

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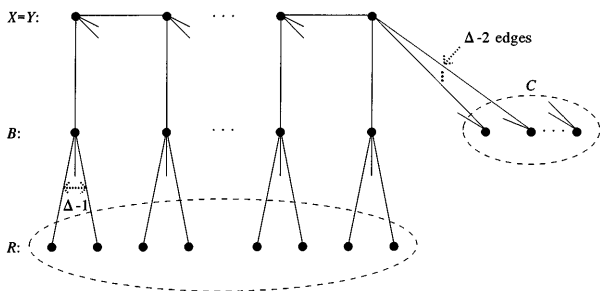


Figure 1 Irredundance and maximum degree in graphs.

We first find an upper bound for $|C|$. The number of edges joining C to X is at least $2|C|$ while the numbers of edges joining C to Z, Y_1, Y_2 are at most

$$\sum_{z\in Z}(\Delta - b_z), \quad \sum_{y\in Y_1}(\Delta - 1 - b_y), \quad \sum_{y\in Y_2}(\Delta - 2 - b_y),$$

respectively. Therefore

$$|C| \le \frac{1}{2}\left[\Delta|Z| + (\Delta-1)|Y_1| + (\Delta-2)|Y_2| - |B|\right]. \tag{4}$$

From (1) we have

$$|R| \le \sum_{y\in Y} r_y. \tag{5}$$

Since $n = |Z| + |Y_1| + |Y_2| + |B| + |C| + |R|$, (4) and (5) yield

$$\begin{aligned} n &\le \frac{1}{2}\left[(\Delta+2)|Z| + (\Delta+1)|Y_1| + \Delta|Y_2| + |B| + 2\sum_{y\in Y} r_y\right] \\ &= \frac{1}{2}\left[(\Delta+2)|Z| + (\Delta+1)|Y_1| + \Delta|Y_2| + \sum_{z\in Z} b_z + \sum_{y\in Y}(b_y + 2r_y)\right]. \end{aligned}$$

By Theorem 2.1, the last sum is at most $(2\Delta-1)(|Y_1| + |Y_2|)$. Also, $b_z \le \Delta$ for $z \in Z$, hence we obtain

$$\begin{aligned} n &\le (\Delta+1)|Z| + \frac{3}{2}\Delta|Y_1| + \left(\frac{3\Delta-1}{2}\right)|Y_2| \\ &\le \frac{3}{2}\Delta(|Z| + |Y_1| + |Y_2|) \\ &= \frac{3}{2}\Delta|X|. \end{aligned}$$

Hence $|X| \ge \frac{2n}{3\Delta}$ as required. □

Obviously, $\mathrm{ir}(G) = \frac{2n}{3\Delta}$ if and only if equality holds in all inequalities in the proofs of Theorems 1.1 and 2.1. It is easy to see that in this case, $S_1 = S_3 = Y_2 = \emptyset$ and $|R| = \sum_{y\in S_2} r_y = (\Delta - 1)\, s_2$, *i.e.*, each $y \in Y$ has exactly one $X - PN$. This vertex y' is

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adjacent to exactly $\Delta - 1$ vertices in R, while each $t \in R$ annihilates exactly one $y \in Y$. Furthermore, $G[Y]$ is 1-regular and either $Z = \emptyset$, in which case $|B| = |Y| = |X|$ and $|C| = \frac{1}{2}(\Delta - 2)|Y|$, or $Z \neq \emptyset$, in which case $\Delta = 2$, $b_z = 2$ for each $z \in Z$ and (by (4)) $C = \emptyset$. It follows that $\mathrm{ir}(G) = \frac{2n}{3\Delta}$ if and only if (i) $\Delta = 2$ and (after some reflection) G is the disjoint union of paths and cycles, where the number of vertices of each component of G is divisible by three, or (ii) $\Delta \geq 3$ and G belongs to the class of graphs illustrated in Figure 1, where broken lines indicate that the enclosed vertices may (or may not) be adjacent, provided no vertex is adjacent to more than Δ vertices. (Note that no $v \in C$ is adjacent to any $t \in R$, for otherwise $X \cup \{v\}$ is irredundant.)

If $\Delta = 2$, then $\frac{n}{2\Delta - 1} = \frac{2n}{3\Delta}$ and the extremal graphs in this case were also characterised in [4].

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