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On elastic waves from a finite line-source

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Abstract. Hazebroek's problem of a finite line-source in an infinite elastic medium is reconsidered. Here the object is to present a simpler method of solution by treating the line-source as equivalent to an appropriate distribution of point sources over the surface of a prolate spheroid which degenerates into this line in the limit when its minor axis tends to zero. This is achieved by using the property of Green's functions. The solution thus obtained is verified to be identical with that derived by Hazebroek.

1. *Introduction.* In a recent paper Hazebroek (1) has studied the problem of a finite line-source in an infinite elastic medium. At the outset he points out that different solutions to this problem can exist depending upon the actual surface (and the conditions thereat) of which the line-source represents the model. In particular, he considered the line-source as the limit of a narrow elongated ellipsoid of revolution (i.e. a prolate spheroid) whose minor axis tends to zero. Further, he assumes that the interior of this surface is subjected to a pressure which is a given function of time. The final results of his work correctly account for the existence of both compressional and shear waves in the medium even though the above surface was assumed to be free from any shearing stress.

However, the method of solution adopted by Hazebroek to this boundary-value problem is extremely long and seems to be unduly complicated owing to his representation of the solution in terms of prolate spheroidal wave functions. At least for the degenerate surfaces of the type considered by him, a much more simple and straightforward analysis can be given by the use of an appropriate distribution of point-sources over such surfaces.

This is demonstrated in the course of the present paper where we consider the finite line-source problem under precisely the same conditions as were assumed by Hazebroek. The solution is obtained very easily by our new approach and its identity with that of Hazebroek also verified.

2. *The Green's function method.* Hazebroek's problem of the disturbance in an infinite elastic medium due to a finite line-source along the line-segment $x = y = 0, -1 \leqslant z \leqslant 1$ consists of determining two scalar functions ϕ and ω satisfying the equations

$$\left.\begin{aligned}(\nabla^2 + h^2)\,\phi &= 0,\\ (\nabla \wedge \nabla \wedge - k^2)\,(\mathbf{e}_\theta \omega) &= 0\end{aligned}\right\} \tag{2·1}$$

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and the boundary conditions (over a prolate spheroid which degenerates into the above line-segment)

$$\left.\begin{aligned} f\frac{\partial U}{\partial \xi}-\frac{\xi f U+\eta g V}{D^2}-\frac{\sigma}{1-2\sigma}h^2D^2\phi &= -\frac{D^2}{f_0^2} \\ g\frac{\partial U}{\partial \eta}+\frac{\eta g U-\xi f V}{D^2}+\tfrac{1}{2}k^2D^2\omega &= 0 \end{aligned}\right\} \quad (\xi=\xi_0), \tag{2·2}$$

for the limiting case when $\xi_0 \to 1$ (see (1) sections 2, 3). Here, $\mathbf{e}_\theta$ is a unit vector in the azimuthal direction, (ξ, η, θ) are the prolate spheroidal coordinates given by

$$\left.\begin{aligned} x &= fg\cos\theta \\ y &= fg\sin\theta \\ z &= \xi\eta \end{aligned}\right\} \quad (1 \leqslant \xi < \infty, -1 \leqslant \eta \leqslant 1, 0 \leqslant \theta \leqslant 2\pi), \tag{2·3}$$

$$f=(\xi^2-1)^{\frac{1}{2}}, \quad g=(1-\eta^2)^{\frac{1}{2}}, \quad D=(\xi^2-\eta^2)^{\frac{1}{2}}, \quad f_0=(\xi_0^2-1)^{\frac{1}{2}}, \quad h=\nu/a, \quad k=\nu/b,$$

$$\left.\begin{aligned} U &= f\frac{\partial\phi}{\partial\xi}+\frac{\partial}{\partial\eta}(g\omega), \\ V &= g\frac{\partial\phi}{\partial\eta}-\frac{\partial}{\partial\xi}(f\omega). \end{aligned}\right\} \tag{2·4}$$

ν is the frequency in the time-harmonic factor $e^{i\nu t}$, a and b are the longitudinal and transverse wave velocities, and σ is the Poisson's ratio. The displacement components in the medium are given by

$$d_\xi = \frac{U}{D}, \quad d_\eta = \frac{V}{D}. \tag{2·5}$$

We now proceed to solve (2·1) and (2·2) by means of an appropriate distribution of point sources over the surface of the prolate spheroid $\xi = \xi_0$ with an intensity which varies from point to point.

From (2·1) we note that ϕ satisfies a Helmholtz equation with wave-number h and further that if we write $\omega = \partial\psi/\partial r$, $r=(x^2+y^2)^{\frac{1}{2}}$, then ψ also satisfies a Helmholtz equation with wave-number k. That is,,

$$(\nabla^2+k^2)\psi = 0. \tag{2·6}$$

The free-space Green's functions of these equations are given by

$$\phi_G = \frac{e^{-ihR}}{4\pi R}, \quad \psi_G = \frac{e^{-ikR}}{4\pi R}, \tag{2·7}$$

where ϕ_G is the solution due to a compressional point source and ψ_G is that due to a shear point source, both being of unit strength, and represent outgoing waves at infinity. Here, R denotes the distance from the point source. Taking the coordinates of the point source as $(\xi_0, \eta_0, \theta_0)$ we have

$$R^2 = f^2g^2+f_0^2g_0^2-2fgf_0g_0\cos(\theta-\theta_0)+(\xi\eta-\xi_0\eta_0)^2. \tag{2·8}$$

The solutions, for $\xi > \xi_0$, of the ϕ-equation in (2·1) and ψ in (2·6) which correspond

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to our axisymmetric distribution of such sources over the surface of the prolate spheroid $\xi = \xi_0$ are then of the form

$$\left.\begin{aligned}\phi &= \int_{-1}^{1} \eta_0 A(\eta_0)\, d\eta_0 \int_0^{2\pi} \phi_G\, d\theta_0,\\ \psi &= \int_{-1}^{1} \eta_0 B(\eta_0)\, d\eta_0 \int_0^{2\pi} \psi_G\, d\theta_0,\end{aligned}\right\} \tag{2·9}$$

where the source-intensity functions $A(\eta_0)$ and $B(\eta_0)$ are to be determined from the boundary conditions (2·2).

First, consider ϕ and note that ϕ_G satisfies the Green's equation

$$(\nabla^2 + h^2)\,\phi_G = -\frac{\delta(\xi-\xi_0)\,\delta(\eta-\eta_0)\,\delta(\theta-\theta_0)}{h_\xi h_\eta h_\theta}, \tag{2·10}$$

where $\delta(z)$ is the Dirac delta function. The metric coefficients h_ξ, h_η, h_θ are given by

$$h_\xi = \left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{\frac{1}{2}}, \quad h_\eta = \left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{\frac{1}{2}}, \quad h_\theta = (\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}$$

and the operator ∇^2 by

$$\nabla^2 = \frac{1}{\xi^2-\eta^2}\left[\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} + \frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\theta^2}\right].$$

Noting the axisymmetry, and by virtue of the well-known properties of the delta functions, we find from (2·9) and (2·10) that ϕ satisfies the equation (which is identical with the Helmholtz equation except when $\xi = \xi_0$)

$$\frac{\partial}{\partial\xi}\left\{(\xi^2-1)\frac{\partial\phi}{\partial\xi}\right\} + \frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial\phi}{\partial\eta}\right\} + h^2(\xi^2-\eta^2)\,\phi = -\eta A(\eta)\,\delta(\xi-\xi_0). \tag{2·11}$$

Next integrate this equation with respect to ξ over a small interval (ξ_0, ξ). From the property of Green's functions we know that ϕ is continuous for all ξ including $\xi = \xi_0$ so that the contributions from the second and third terms on the left-hand side of this equation vanish as $\xi \to \xi_0$. Ignoring them, the remaining terms give, on integration, for $\xi \approx \xi_0$,

$$(\xi^2-1)\frac{\partial\phi}{\partial\xi}\Big|_{\xi_0}^{\xi} \approx -\eta A(\eta)\,p(\xi_0); \quad p(\xi_0) = \begin{cases}1, & \xi_0 = 1,\\ \frac{1}{2}, & \xi_0 > 1.\end{cases}$$

This means that

$$\frac{\partial\phi}{\partial\xi}\Big|_{\xi_0}^{\xi} \approx -\frac{\eta A(\eta)\,p(\xi_0)}{\xi_0^2-1},$$

or equivalently

$$\frac{\partial^2\phi}{\partial\xi^2} \approx \frac{\eta A(\eta)\,p(\xi_0)}{(\xi-\xi_0)(\xi^2-1)} \quad (\xi \approx \xi_0). \tag{2·12}$$

If we consider the limiting case when $\xi_0 = 1$, the point sources are distributed over the degenerate spheroid which coincides with the line-segment $x = y = 0$, $-1 \leqslant z \leqslant 1$ and leads to the line-source required. For this case, $p(1) = 1$ and (2·12) gives

$$\frac{\partial^2\phi}{\partial\xi^2} \approx \frac{2\eta A(\eta)}{f^4} \quad (\zeta \approx 1)$$

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where f is defined in (2·3). From this we easily obtain that

$$\phi \approx -\eta A(\eta)\log f \quad (\xi \approx 1). \tag{2·13}$$

An exactly similar argument for ψ in (2·9) gives

$$\psi \approx -\eta B(\eta)\log f \quad (\xi \approx 1). \tag{2·14}$$

Since $\omega = \partial\psi/\partial r$, we can make use of the relation

$$\frac{\partial}{\partial r} = \left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\Big/\left(\frac{\xi^2 g}{f} + \frac{\eta^2 f}{g}\right)$$

$$\approx \frac{f}{g}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) \quad (\xi \approx 1)$$

in (2·14) to obtain

$$\omega \approx -\frac{\eta B(\eta)}{fg} \quad (\xi \approx 1). \tag{2·15}$$

Using (2·13) and (2·15) in (2·4), we have

$$\left.\begin{aligned} U &\approx \frac{U_0(\eta)}{f} \\ V &\approx V_0(\eta)\log f \end{aligned}\right\} \quad (\xi \approx 1), \tag{2·16}$$

where

$$\left.\begin{aligned} U_0(\eta) &= -\left[\eta A(\eta) + \frac{d}{d\eta}\{\eta B(\eta)\}\right], \\ V_0(\eta) &= g\frac{dU_0(\eta)}{d\eta} - k^2\eta g B(\eta). \end{aligned}\right\} \tag{2·17}$$

The expression for $V_0(\eta)$ has been derived by integration once with respect to ξ of the relation†

$$\left(f\frac{\partial V}{\partial\xi} - g\frac{\partial U}{\partial\eta}\right) = k^2 D^2\omega$$

after substituting the approximate expressions for ω and U.

We now substitute from (2·13), (2·15) and (2·16) into our boundary conditions (2·2) for $\xi_0 \to 1$ and readily obtain the relations

$$\left.\begin{aligned} U_0(\eta) &= (1-\eta^2) \\ g\frac{dU_0}{d\eta} + \frac{\eta}{g}U_0 - \tfrac{1}{2}k^2\eta g B(\eta) &= 0 \end{aligned}\right\} \quad (-1 \leqslant \eta \leqslant 1). \tag{2·18}$$

From (2·17) and (2·18) we easily obtain

$$\left.\begin{aligned} A(\eta) &= \frac{1}{\eta}\left(\frac{2}{k^2} - 1\right) + \eta \\ B(\eta) &= -\frac{2}{k^2} \end{aligned}\right\} \quad (-1 \leqslant \eta \leqslant 1). \tag{2·19}$$

† This is easily derived from (2·4) by virtue of the ω equation in (2·1), see (1) p. 40.

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Putting the above values of $A(\eta)$ and $B(\eta)$ in (2·9), and further noting that when $\xi_0 = 1, f_0 \to 0$ and therefore from (2·7) and (2·8) also that

$$\left.\begin{aligned}\int_0^{2\pi} \phi_G d\theta_0 &= \frac{e^{-ik\hat{R}}}{2\hat{R}} \\ \int_0^{2\pi} \psi_G d\theta_0 &= \frac{e^{-ik\hat{R}}}{2\hat{R}}\end{aligned}\right\} \quad \hat{R} = [f^2g^2 + (\xi\eta - \eta_0)^2]^{\frac{1}{2}}, \tag{2·20}$$

we finally obtain

$$\left.\begin{aligned}\phi &= e^{ivt}\int_{-1}^{1}\left[\left(\frac{1}{k^2}-\frac{1}{2}\right)+\tfrac{1}{2}\eta_0^2\right]\frac{e^{-ik\hat{R}}}{\hat{R}}\,d\eta_0, \\ \omega &= \frac{\partial\psi}{\partial r} = -\frac{1}{k^2}e^{ivt}\int_{-1}^{1}\eta_0\frac{\partial}{\partial r}\left(\frac{e^{-ik\hat{R}}}{\hat{R}}\right)d\eta_0,\end{aligned}\right\} \tag{2·21}$$

where the harmonic time-factor e^{ivt} has been restored.

The solution to the finite line-source problem is therefore completely determined. It only remains to show that our solution (2·21) is precisely the same as that derived by Hazebroek. This is very easily done by writing

$$fg = r, \quad \xi\eta = z, \quad \xi = \coth u, \quad \eta = \tanh v$$

and transforming the variable of integration η_0 to ξ' where

$$\eta_0 = z - r\sinh\xi', \quad \hat{R} = r\cosh\xi'.$$

This leads, after a little simplification, to the following alternative form for (2·21):

$$\begin{aligned}\phi = e^{ivt}&\left(\frac{1}{k^2}+\frac{z^2}{2}-\frac{1}{2}\right)\int_{v-u}^{v+u} e^{-ihr\cosh\xi'}\,d\xi' \\ &+\tfrac{1}{2}r^2e^{ivt}\int_{v-u}^{v+u} e^{-ihr\cosh\xi'}\sinh^2\xi'\,d\xi' \\ &-\frac{iz}{h}e^{ivt}\left[e^{-ihr\cosh(u+v)} - e^{-ihr\cosh(u-v)}\right],\end{aligned} \tag{2·22}$$

$$\begin{aligned}\omega = e^{ivt}&\left[\frac{iz}{k}\int_{v-u}^{v+u} e^{-ikr\cosh\xi'}\cosh\xi'\,d\xi'\right. \\ &+\left\{\frac{1+\sinh(u+v)\cosh u\sinh v}{k^2\cosh(u+v)}\right\}e^{-ikr\cosh(u+v)} \\ &\left.-\left\{\frac{1+\sinh(v-u)\cosh u\sinh v}{k^2\cosh(u-v)}\right\}e^{-ikr\cosh(u-v)}\right],\end{aligned} \tag{2·23}$$

which are precisely the same as the expressions (4·38) and (5·19) of Hazebroek's solution. (In deriving (2·23) we first make use of the relation

$$\frac{\partial}{\partial r}\left(\frac{e^{-ik\hat{R}}}{\hat{R}}\right) = \frac{k}{ir}\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial\eta_0^2}\right)e^{-ik\hat{R}}$$

in the integral for ω in (2·21).)

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3. *Concluding remarks.* It has been shown that the finite line-source problem studied by Hazebroek using spheriodal wave functions is very easily solved by using a suitable distribution of point sources on this line. The idea of superposing such sources along the inter-focal line-segment for a boundary condition on a prolate spheroid is not in any way different from solving the problems of an infinite medium with a spherical (or cylindrical) cavity through solutions which are singular at the centre of the cavity (see, for instance, Jeffreys(2)). The principal consideration here is that the sphere contracts to its centre point when its radius tends to zero while the spheroid degenerates into the inter-focal line when its minor axis tends to zero, i.e. as $\xi_0 \to 1$. Finally, we hope that it should be of interest to extend this technique to include more general forms of surfaces and adopt also higher-order axisymmetric solutions of the wave equations in place of the spherically symmetric point sources for the purpose of superposition.

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