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ON MINIMAL ADDITIVE COMPLEMENTS

GUO WANYAN and MIN TANG✉

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Abstract

Let C and W be two integer sets. If $C + W = \mathbb{Z}$, then we say that C is an additive complement to W. If no proper subset of C is an additive complement to W, then we say that C is a minimal additive complement to W. We study the existence of a minimal additive complement to $W = \{w_i\}_{i=1}^{\infty}$ when W is not eventually periodic and $w_{i+1} - w_i \in \{2, 3\}$ for all i.

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1. Introduction

Let $\mathbb{Z}$ be the set of all integers. For $C, W \subseteq \mathbb{Z}$ and an integer a, we define

$$C + W = \{c + w : c \in C,\ w \in W\},$$

$$a + W = \{a + w : w \in W\}, \quad aW = \{aw : w \in W\}.$$

For integers a and b, $[a, b]$ denotes the set of all integers between a and b. For a positive integer m and a set A, we define

$$A \pmod m = \{i \in [0, m-1] : i \equiv a \pmod m, a \in A\}.$$

If there exists a positive integer T such that $x + T \in X$ for all sufficiently large integers $x \in X$, then we call X eventually periodic with period T. The set C is called an additive complement to W if $C + W = \mathbb{Z}$. If no proper subset of C is an additive complement to W, then the set C is called a minimal additive complement to W.

In 2011, Nathanson [8] initiated the study of minimal complements in infinite abelian groups. He proved the following theorem.

THEOREM 1.1 [8, Theorem 8]. *Let W be a nonempty finite set of integers. In $\mathbb{Z}$, every complement to W contains a minimal complement to W.*

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The study of minimal complements is related to problems in coding theory (see, for example, [5]). The question of determining which subsets W of $\mathbb{Z}$ admit minimal complements has received considerable attention. In 2012, Chen and Yang [4] showed that $W \subseteq \mathbb{Z}$ has a minimal complement if it is unbounded both above and below.

THEOREM 1.2 [4, Theorem 1]. *Let W be a set of integers with $\inf W = -\infty$ and $\sup W = +\infty$. Then, there exists a minimal complement to W.*

There is a basic result that $\mathbb{N}$ does not have a minimal additive complement. Chen and Yang [4] showed that removing an extremely sparse set from $\mathbb{N}$ can still yield a set that does not have a minimal additive complement.

THEOREM 1.3 [4, Theorem 2]. *Let $W = \{1 = w_1 < w_2 < \cdots\}$ be a set of integers and $\overline{W} = (\mathbb{Z} \cap (0, +\infty)) \setminus W = \{\overline{w_1} < \overline{w_2} < \cdots\}$.*

(a) *If $\limsup_{i\to+\infty}(w_{i+1} - w_i) = +\infty$, then there exists a minimal complement to W.*
(b) *If $\lim_{i\to+\infty}(\overline{w_{i+1}} - \overline{w_i}) = +\infty$, then there does not exist a minimal complement to W.*

In 2019, Kiss *et al.* [6] studied the existence of minimal complements for eventually periodic subsets of $\mathbb{Z}$. Suppose that W is an eventually periodic set and m is a positive period. By shifting a number, we may assume that W has the structure

$$W = (m\mathbb{N} + X_m) \cup Y^{(0)} \cup Y^{(1)}, \tag{1.1}$$

where $X_m \subseteq \{0, 1, \ldots, m-1\}$, $Y^{(0)} \subseteq \mathbb{Z}^-$ and $Y^{(1)}$ are finite sets with $Y^{(0)} \pmod m \subseteq X_m$ and $(Y^{(1)} \pmod m) \cap X_m = \emptyset$. They gave conditions to ensure the existence of a minimal complement to W.

THEOREM 1.4 [6, Theorem 1]. *Let W be defined as in (1.1). If there exists a minimal complement to W, then there exists $C \subseteq \{0, 1, \ldots, m-1\}$ such that the following two conditions hold:*

(a) $C + (X_m \cup Y^{(1)}) \bmod m = \{0, 1, \ldots, m-1\}$;
(b) *for any $c \in C$, there exists $y \in Y^{(1)}$ such that $c + y \not\equiv c' + x \pmod m$, where $c' \in C$, $x \in X_m$.*

Recently, there has been some progress on these problems in infinite abelian groups other than $\mathbb{Z}$. For related problems, see [1–3, 7, 9].

With these results established, the next problem to consider concerns sets of integers that are bounded below and have bounded gaps between consecutive elements. In 2019, Kiss, Sándor and Yang obtained the following result.

THEOREM 1.5 [6, Theorem 4]. *There exists an infinite, not eventually periodic set $W \subseteq \mathbb{N}$ such that $w_{i+1} - w_i \in \{1, 2\}$ for all i and such that there exists a minimal complement to W.*

We study the existence of a minimal additive complement to W when W is not eventually periodic and $w_{i+1} - w_i \in \{2, 3\}$ for all i. We obtain the following result.

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THEOREM 1.6. *Let $\{b_i\}_{i=1}^{\infty}, \{V_i\}_{i=1}^{\infty}$ and $C = \{c_i\}_{i=1}^{\infty}$ be such that:*

(i) $b_1 = -1, V_1 = \{1, 2, \ldots, 12\}, c_1 = -6$;
(ii) b_i *is the largest negative integer* $\notin V_{i-1} + \{c_1, \ldots, c_{i-1}\}$ *for* $i \geq 2$;
(iii) $c_i < b_i + 2c_{i-1} - 3$ *for all* $i \geq 2$;
(iv) $V_i = V_{i-1} \cup ([-2c_{i-1}, -2c_i - 1] \setminus \bigcup_{j=1}^{i-1}\{-c_i + b_j\})$ *for* $i \geq 2$.

There exists a not eventually periodic set $W = \{w_i\}_{i=1}^{\infty}$ such that $b_i - c_i \in W$ and $w_{i+1} - w_i \in \{2, 3\}$ for all i. In particular, if $|\mathbb{Z} \setminus (C + W)| < +\infty$, then there exists a minimal complement to W.

2. A lemma

LEMMA 2.1. *Let $I \subseteq [a, b]$ be an integer set with $a, b \in I$ and $b - a \geq 8$. Write $U := [a, b] \setminus I$. If $|U| \geq 2$ and the distance of any adjacent integers of U is no less than 5, or if $|U| \leq 1$, then there exists a set $J = \{j_i : i \geq 1\} \subset I$ with $a, b \in J$ and $j_{i+1} - j_i \in \{2, 3\}$.*

PROOF. We divide the proof into three cases.

Case 1: $|U| = \emptyset$. Choose

$$J = \left\{a + 2i : i = 0, 1, \ldots, \left\lfloor \frac{b-a}{2} \right\rfloor - 1\right\} \cup \{b\}.$$

Case 2: $|U| = 1$. Write $U = \{u_1\}$. If $u_1 - a$ is odd, choose

$$J = \left\{a + 2i : i = 0, 1, \ldots, \left\lfloor \frac{b-a}{2} \right\rfloor - 1\right\} \cup \{b\}.$$

If $b - a$ and $u_1 - a$ are even, choose

$$J = \{a\} \cup \left\{a + 3 + 2i : i = 0, 1, \ldots, \left\lfloor \frac{b-a}{2} \right\rfloor - 3\right\} \cup \{b\}.$$

If $b - a$ is odd and $u_1 - a$ is even, choose

$$J = \{a\} \cup \left\{a + 3 + 2i : i = 0, 1, \ldots, \left\lfloor \frac{b-a}{2} \right\rfloor - 2\right\} \cup \{b\}.$$

Case 3: $|U| \geq 2$ and $u_{i+1} - u_i \geq 5$ for any adjacent integers $u_i, u_{i+1} \in U$. If $b - a = 8, 9$ or 10, then $|U| = 2$. Write $U = \{u_1, u_2\}$.

Case 3.1: $b - a = 8$. Then, $\{u_1, u_2\} = \{a + 1, a + 6\}, \{a + 1, a + 7\}$ or $\{a + 2, a + 7\}$. We can choose $J = \{a, a + 3, a + 5, a + 8\}$.

Case 3.2: $b - a = 9$. Then, $\{u_1, u_2\}$ is one of $\{a + 1, a + 6\}, \{a + 1, a + 7\}, \{a + 1, a + 8\}, \{a + 2, a + 7\}, \{a + 2, a + 8\}$ or $\{a + 3, a + 8\}$. We can choose

$$J = \{a, a + 3, a + 5, a + 7, a + 9\}, \{a, a + 2, a + 4, a + 6, a + 9\} \text{ or } \{a, a + 3, a + 6, a + 9\}.$$

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Case 3.3: $b-a=10$. Then, $\{u_1,u_2\}$ is one of

$$\{a+1,a+6\},\{a+1,a+7\},\{a+1,a+8\},\{a+1,a+9\},\{a+2,a+7\},$$

$$\{a+2,a+8\},\{a+2,a+9\},\{a+3,a+8\},\{a+3,a+9\},\{a+4,a+9\}.$$

We can choose

$$J=\{a,a+2,a+5,a+7,a+10\},\quad \{a,a+2,a+4,a+6,a+8,a+10\},$$

$$\{a,a+3,a+6,a+8,a+10\}\quad\text{or}\quad\{a,a+3,a+5,a+7,a+10\}.$$

From this point on, we can use induction on $b-a$. If $b-a=11$, then $|U|\geq 2$. If $a+2\in U$, then $a+3\notin U$. By Case 3.1, we can choose

$$\widetilde{J}=\{j_i : i\geq 1\}\subset[a+3,a+11]$$

with $a+3,a+11\in\widetilde{J}$ and $\widetilde{j}_{i+1}-\widetilde{j}_i\in\{2,3\}$, and we take $J=\{a\}\cup\widetilde{J}$. If $a+2\notin U$, then by Case 3.2, we can choose

$$\widetilde{J}=\{j_i : i\geq 1\}\subset[a+2,a+11]$$

with $a+2,a+11\in\widetilde{J}$ and $\widetilde{j}_{i+1}-\widetilde{j}_i\in\{2,3\}$, and we take $J=\{a\}\cup\widetilde{J}$.

Assume that the result holds whenever $8\leq b-a\leq m$, where $m\geq 11$. Now, we consider the case $b-a=m+1$. If $a+2\in U$, then by the hypothesis, we can choose

$$\widetilde{J}=\{j_i : i\geq 1\}\subset[a+3,a+m+1]$$

with $a+3,a+m+1\in\widetilde{J}$ and $\widetilde{j}_{i+1}-\widetilde{j}_i\in\{2,3\}$, and we take $J=\{a\}\cup\widetilde{J}$. If $a+2\notin U$, then by the hypothesis, we can choose

$$\widetilde{J}=\{j_i : i\geq 1\}\subset[a+2,a+m+1]$$

with $a+2,a+m+1\in\widetilde{J}$ and $\widetilde{j}_{i+1}-\widetilde{j}_i\in\{2,3\}$, and we take $J=\{a\}\cup\widetilde{J}$.

By induction, we see that there exists a set $J=\{j_i : i\geq 1\}\subset I$ with $a,b\in J$ and $j_{i+1}-j_i\in\{2,3\}$. This completes the proof of Lemma 2.1. □

3. Proof of Theorem 1.6

For $i\geq 2$, write

$$\mathrm{Gap}(V_i)=\{-c_i+b_j : j=1,\dots,i-1\}:=\{h_j^{(i)} : j=1,\dots,i-1\}.$$

Let

$$V=\bigcup_{i=1}^{\infty}V_i,\quad \mathrm{Gap}(V)=\bigcup_{i=1}^{\infty}\mathrm{Gap}(V_i).$$

Then, we have the following three basic facts.

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Fact A: $b_{i+1} - b_i \le -5$ for all integers $i \ge 1$. Clearly, $b_2 = -6, b_2 - b_1 = -5$. Suppose that $b_{i+1} - b_i \le -5$ for all integers $i < k$, where $k \ge 2$. By item (ii), $\{b_i\}_{i=1}^{\infty}$ is a decreasing sequence. By item (iv), for any $k \ge 2$,

$$[-2c_{k-1}, -c_k + b_{k-1}) \subseteq V_k.$$

By item (iii),

$$-2c_{k-1} \le b_k - 4 - c_k < b_k - c_k < -c_k + b_{k-1}.$$

Thus,

$$b_k - c_k - i \in [-2c_{k-1}, -c_k + b_{k-1}) \quad \text{for } 0 \le i \le 4,$$

and so

$$b_k - i = (b_k - c_k - i) + c_k \in c_k + V_k \quad \text{for } 0 \le i \le 4.$$

Again, by the definition of b_{k+1}, we have $b_{k+1} - b_k \le -5$.

Fact B: $C + V = \mathbb{Z}$. By item (iv), $v_{j+1} - v_j \in \{1, 2\}$. Since $b_k \to -\infty$, by item (ii), we have

$$(-\infty, 6] \in C + V.$$

For any integer $n \ge 7$, there exists an i such that $-c_{i-1} \le n < -c_i$. Hence,

$$-c_i + b_1 < -c_{i-1} - c_i \le n - c_i < -2c_i,$$

and so $n - c_i \in V_i$, that is, $n \in c_i + V_i$. Therefore, $C + V = \mathbb{Z}$.

Fact C: if we write $b_s = c + v$ for $s \ge 1, c \in C$ and $v \in V$, then $c = c_s$. For any $k > s$, by item (iv), $b_s - c_k \notin V$. Thus, $c \ne c_k$ for all $k > s$. For any $k \le s - 1$, noting that $-2c_{s-1} = \min(V \backslash V_{s-1})$ and

$$b_s - c_k \le b_s - c_{s-1} < -2c_{s-1},$$

we have $b_s - c_k \notin V \backslash V_{s-1}$. Moreover, by item (ii),

$$b_s \notin V_{s-1} + \{c_1, \ldots, c_{s-1}\}.$$

Thus, $c \ne c_k$ for all $k \le s - 1$.

It follows that C is a minimal complement to V. Write

$$d_i = b_i - c_i \quad \text{for } i = 1, 2, \ldots.$$

By Fact C, $d_i \in V$ for all $i \ge 1$. We prove that $d_{i+1} - d_i \ge 8$ for all $i \ge 1$. Since $c_{i+1} < b_{i+1} + 2c_i - 3$ and $b_{i+1} < 0, c_{i+1} < 0$, we have

$$-d_{i+1} < 2c_i \le c_i + c_1 = c_i - 6 < c_i < c_i - b_1 \le c_i - b_i = -d_i.$$

Hence,

$$d_{i+1} - d_i \ge -d_i - (c_i - 6) + 1 = -b_i + 7 \ge 8.$$

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Fix $i \geq 2$. By Fact A, for $1 \leq j \leq i-1$,

$$h_j^{(i)} - h_{j+1}^{(i)} = -c_i + b_j - (-c_i + b_{j+1}) = b_j - b_{j+1} \geq 5. \tag{3.1}$$

Moreover, by item (iii),

$$\begin{aligned} h_i^{(i+1)} - h_1^{(i)} &= -c_{i+1} + b_i - (-c_i + b_1) = c_i - c_{i+1} + b_i + 1 \\ &> c_i - b_{i+1} - 2c_i + 3 + b_i + 1 \\ &= -c_i + (b_i - b_{i+1}) + 4 \geq 15. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), the distance of all adjacent integers of $\mathrm{Gap}(V)$ is no less than 5. For $i \geq 1$, write

$$I_i = [d_i, d_{i+1}] \cap V.$$

By Lemma 2.1, there exists a set

$$W^{(i)} = \{w_j^{(i)} : j \geq 1\} \subset I_i$$

such that $w_{j+1}^{(i)} - w_j^{(i)} \in \{2, 3\}$ and $d_i, d_{i+1} \in W^{(i)}$. Put

$$W = \bigcup_{i=1}^{\infty} W^{(i)}.$$

Then, $W = \{w_i\}_{i=1}^{\infty}$ satisfies $w_{i+1} - w_i \in \{2, 3\}$ for all i.

Furthermore, by item (iii), we can choose suitable c_i such that W is infinite and not eventually periodic. Since $|\mathbb{Z} \setminus (C + W)| < +\infty$, we may assume that

$$\mathbb{Z} \setminus (C + W) = \{n_i : i = 1, \ldots, k\}.$$

Since $C + V = \mathbb{Z}$, for $i = 1, \ldots, k$, there exist $c_{s(n_i)} \in C$ and $v_{s(n_i)} \in V$ such that

$$n_i = c_{s(n_i)} + v_{s(n_i)}.$$

Hence, $v_{s(n_i)} \notin W$ for all $i = 1, \ldots, k$. By the construction of W, $v_{s(n_i)} + 1 \in W$ or $v_{s(n_i)} - 1 \in W$. Put

$$l_i = \begin{cases} c_{s(n_i)} - 1 & \text{if } v_{s(n_i)} + 1 \in W, \\ c_{s(n_i)} + 1 & \text{otherwise.} \end{cases}$$

The l_i may appear repeatedly. Without loss of generality, we may assume that

$$L = \{l_1, \ldots, l_m\}.$$

Then, $m \leq k$. By item (iii), $l_i \notin C$ for all $1 \leq i \leq m$. Write

$$C_0 = C \cup L.$$

Since $\{b_i\}_{i=1}^{\infty}$, $\{c_i\}_{i=1}^{\infty}$ are decreasing sequences and $b_1 = -1$, $c_1 = b_2 = -6$, there exists an integer $h(n_i) \geq 2$ such that

$$b_{h(n_i)} < c_{s(n_i)} \leq b_{h(n_i)-1}. \tag{3.3}$$

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Write

$$t(k) = \max\{h(n_1), \ldots, h(n_k)\} - 1.$$

For $j = 1, \ldots, t(k)$, let

$$C_j = \begin{cases} C_{j-1} \setminus \{c_j\} & \text{if } C_{j-1} \setminus \{c_j\} + W = \mathbb{Z}, \\ C_{j-1} & \text{otherwise.} \end{cases} \tag{3.4}$$

For $\mu = 1, \ldots, m$, let

$$C_{t(k)+\mu} = \begin{cases} C_{t(k)+\mu-1} \setminus \{l_\mu\} & \text{if } C_{t(k)+\mu-1} \setminus \{l_\mu\} + W = \mathbb{Z}, \\ C_{t(k)+\mu-1} & \text{otherwise.} \end{cases} \tag{3.5}$$

Write

$$C^* = C_{t(k)+m}.$$

Then, $C^* + W = \mathbb{Z}$. Now we shall prove C^* is a minimal complement to W.

If there exists an integer $c_j \in C^*$ with $1 \le j \le t(k)$, then by the construction of C_j in (3.4),

$$C_{j-1} \setminus \{c_j\} + W \neq \mathbb{Z}.$$

Thus, $C^* \setminus \{c_j\} + W \neq \mathbb{Z}$. If there exists an integer $l_\mu \in C^*$ for some $1 \le \mu \le m$, then by the construction of $C_{t(k)+\mu}$ in (3.5),

$$C_{t(k)+\mu-1} \setminus \{l_\mu\} + W \neq \mathbb{Z}.$$

Thus, $C^* \setminus \{l_\mu\} + W \neq \mathbb{Z}$.

Next, we shall show that for all $j > t(k)$,

$$b_j \notin (C^* \cap L) + W.$$

In fact, if $b_j = (c_{s(n_i)} - 1) + w$ for some $c_{s(n_i)} - 1 \in C^*$ and $w \in W$, then

$$b_j - c_{s(n_i)} = w - 1 \ge 0.$$

By (3.3) and the fact that $\{b_i\}_{i=1}^{\infty}$ is a decreasing sequence, we see that the above inequality cannot hold. Similarly,

$$b_j \neq (c_{s(n_i)} + 1) + w'$$

for any $c_{s(n_i)} + 1 \in C^*$ and $w' \in W$. Since $C^* + W = \mathbb{Z}$, there exist $c \in C, w \in W$ such that

$$b_j = c + w.$$

By Fact B, $c = c_j$. Thus, $C^* \setminus \{c_j\} + W \neq \mathbb{Z}$ for all $j > t(k)$. Therefore, C^* is a minimal complement to W. This completes the proof of Theorem 1.6.

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GUO WANYAN, School of Mathematics and Statistics,
Anhui Normal University, Wuhu 241002, PR China
e-mail: gwy202411@163.com

MIN TANG, School of Mathematics and Statistics,
Anhui Normal University, Wuhu 241002, PR China
e-mail: tmzzz2000@163.com

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