TPLP **26** (1-2): 130–146, 2026. © The Author(s), 2026. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

doi:10.1017/S1471068426100581 First published online 15 July 2026

*Reducing Arbitrary Metric Temporal Formulas into Logic Programs Under Answer Set Semantics**

MARTÍN DIÉGUEZ
LERIA, University of Angers, France
(*e-mail:* martin.dieguezlodeiro@univ-angers.fr)

SUSANA HAHN
Institute of Computer Science, University of Potsdam, Germany
(*e-mail:* hahnmartinlu@uni-potsdam.de)

TORSTEN SCHAUB
Institute of Computer Science, University of Potsdam, Germany
(*e-mail:* torsten@cs.uni-potsdam.de)

IGOR STÉPHAN
LERIA, University of Angers, France
(*e-mail:* igor.stephan@univ-angers.fr)

submitted 30 May 2026 UTC; revised 30 May 2026 UTC; accepted 1 June 2026 UTC

Abstract

Metric temporal equilibrium logic (MEL) extends temporal equilibrium logic (TEL) by incorporating quantitative timing constraints, enabling the specification and analysis of deadlines and durations. MEL is particularly suited for domains where time-bound properties are crucial, such as embedded systems, cyber-physical systems, and real-time software. It facilitates the precise expression of timing behaviors, such as the requirement that an event must occur within 5 milliseconds of a trigger, which often elude traditional qualitative temporal logics. In this paper, we present a Tseitin-like translation that maps any metric temporal formula into a logic programming fragment restricted to past operators. This translation provides a formal bridge to leverage existing answer set programming (ASP) solvers for reasoning about metric temporal constraints. By restricting the target fragment to past operators, we enable more effective evaluation and integration with current ASP-based toolchains for multi-shot solving.

KEYWORDS: logic programming methodology and applications, theory, knowledge representation and nonmonotonic reasoning

* We would like to sincerely thank the reviewers for their careful evaluation, insightful comments, and constructive suggestions, which greatly contributed to improving the quality and clarity of this work, as well as for identifying issues related to the size of the translation. We also extend our gratitude to David Fernández-Duque for his valuable advice, which helped us further improve the paper. This work has been partially supported by DFG grant SCHA 550/15.

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

1 Introduction

Reasoning about actions and change, or more generally about dynamic systems, is not only central to knowledge representation and reasoning but at the heart of computer science (Fisher *et al*. 2005). In practice, this kind of reasoning often requires both qualitative as well as quantitative dynamic constraints. For instance, when planning and scheduling at once, actions may have durations, and their effects may need to meet deadlines. On the other hand, any flexible formalism for actions and change must incorporate some form of non-monotonic reasoning to deal with inertia and other types of defaults.

Over the past years, we addressed qualitative dynamic constraints by combining traditional approaches, like Dynamic and Linear Temporal Logic (DL (Harel *et al*. 2000) and LTL (Pnueli 1977)), with the base logic of Answer Set Programming (ASP (Lifschitz 1999)), namely, the logic of Here-and-There (HT (Heyting 1930)) and its non-monotonic extension, called equilibrium logic (Pearce 1997). This resulted in (the non-monotonic formalisms) linear dynamic and temporal equilibrium logics (DEL (Bosser *et al*. 2018) and TEL (Aguado *et al*. 2023)) that gave rise to the temporal ASP system *telingo* (Cabalar *et al*. 2019) extending the ASP system *clingo* (Gebser *et al*. 2016).

A commonality of such dynamic and temporal logics is that they abstract from specific time points when capturing temporal relationships. For instance, in temporal logic, we can use the formula $\Box(\mathit{press} \rightarrow \mathit{moving}\ \mathbb{U}\ \mathit{arrives})$ to express that always after pressing the button, an elevator is moving until it arrives at the corresponding floor. However, nothing can be said about how long the elevator is moving until it arrives.

A key design decision was to base both aforementioned logics, TEL and DEL, on the same linear-time semantics. Becker *et al*. (2024b) introduce a variant of TEL that associates each state with a discrete time point. This variant, called *Metric Temporal Equilibrium Logic* (MEL), is defined in terms of a monotonic logic called *Metric Temporal logic of Here-and-There* (MHT), a metric temporal extension of the intermediate logic of Here-and-There, mentioned above, plus a criterion for selecting minimal MHT models called (metric) equilibrium models. We obtain in this way a non-monotonic entailment relation. For instance, in our example, we may thus express that whenever the button is pressed, the elevator will arrive at its destination in 0 to 3 seconds, by writing

$$\Box(\mathit{press} \rightarrow \mathit{moving}\ \mathbb{U}_{\leq 3}\ \mathit{arrives}). \tag{1}$$

Unlike the non-metric version, this stipulates that once *press* is true in a state, *arrives* must be true in some future state whose associated time is at most 3 time units after the time of *press*, and *moving* is true in all states in between.

An important step towards an implementation consists in reducing arbitrary input theories to normal forms[1] closer to logic programs. For instance, Becker *et al*. (2024a, 2026) propose a subclass of MEL that can be encoded in propositional ASP. This translation does not consider binary metric temporal operators such as *metric until* or *metric*

[1] We use the term *normal form* here because many different subclasses of MEL can be reduced to this form and, in addition, it enables the design of more efficient solving techniques focused solely on theories that satisfy our logic programming format. In propositional logic, conjunctive and disjunctive normal forms are the most common ones in classical logic. In ASP, however, the reduction of arbitrary theories results in disjunctive logic programs (Pearce 2006).

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

release. Moreover, the resulting logic program may contain pure future atoms in the rule heads and that leads to the introduction of future dependencies. It would be interesting to avoid such future dependendencies since, so and incremental solvers are more efficient when derivations of the current state can be directly done from information obtained at previous states (and already computed by the solver) rather than depending on "future" states that have not been still considered (Gabbay 1987).

In this paper, we introduce a Tseitin-like reduction from any arbitrary metric temporal formulas into metric temporal logic programs. Our translation generates a specific type of metric temporal logic program in which future operators affect only to implications. However, both the body and the head of each implication contain only past and present references. The advantage of this format lies in the use of incremental solving techniques already implemented in solvers such as *telingo* (Cabalar *et al*. 2019). In incremental solving, the use of past temporal operators in the body of the rules implies that these bodies can be readily evaluated at each iteration, since their truth values have already been determined in previous steps. Our translation requires that the intervals associated with binary temporal operators in the input formulas be of the form $[0..n]$, where $n < \omega$.

The rest of the paper is organized as follows. In Section 2, we present the syntax and semantics of MHT, the monotonic basis of MEL. In Section 3, we present an alternative (but equivalent) three-valued semantics that is used to prove the correctness of our Tseitin-like translation, which is itself presented in Section 4. We finish the paper with the conclusions and future work.

2 Metric logic of here-and-there

We begin by introducing the logic of Metric Here-and-There (MHT), a metric extension of HT which serves as the monotonic foundation for Metric Equilibrium Logic (MEL).

Given $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\omega\}$, we let $[m..n]$ stand for the set $\{i \in \mathbb{N} \mid m \leq i \leq n\}$, $[m..n)$ for $\{i \in \mathbb{N} \mid m \leq i < n\}$, $(m..n]$ for $\{i \in \mathbb{N} \mid m < i \leq n\}$ and $(m..n)$ stand for $\{i \in \mathbb{N} \mid m < i < n\}$. We use letters I, J to denote intervals and, since they stand for sets, we assume standard set relations like inclusion $I \subseteq J$ or membership $i \in I$.

Given a set $\mathcal{A}$ of propositional variables (called *alphabet*), a *metric formula* φ is defined by the grammar:

$$\begin{aligned}\varphi \quad ::= \quad & p \mid \bot \mid \varphi_1 \otimes \varphi_2 \mid \bullet_I \varphi \mid \varphi_1 \,\mathbf{S}_I\, \varphi_2 \mid \varphi_1 \,\mathbf{T}_I\, \varphi_2 \mid \circ_I \varphi \mid \varphi_1 \,\mathbb{U}_I\, \varphi_2 \mid \varphi_1 \,\mathbb{R}_I\, \varphi_2 \mid \\ & \bullet \varphi \mid \varphi_1 \,\mathbf{S}\, \varphi_2 \mid \varphi_1 \,\mathbf{T}\, \varphi_2 \mid \circ \varphi \mid \varphi_1 \,\mathbb{U}\, \varphi_2 \mid \varphi_1 \,\mathbb{R}\, \varphi_2\end{aligned}$$

where $p \in \mathcal{A}$ is an atom, $\otimes$ is any binary Boolean connective $\otimes \in \{\rightarrow, \wedge, \vee\}$. As usual in intuitionistic logic, we define $\neg\varphi \stackrel{def}{=} \varphi \rightarrow \bot$, $\varphi \leftrightarrow \psi \stackrel{def}{=} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\top \stackrel{def}{=} \neg\bot$.

The binary temporal operators $\varphi_1 \,\mathbf{S}_I\, \varphi_2$ and $\varphi_1 \,\mathbf{T}_I\, \varphi_2$ denote that φ_1 holds since or is triggered by φ_2, respectively. Similarly, $\varphi_1 \,\mathbb{U}_I\, \varphi_2$ represents the until operator, while $\varphi_1 \,\mathbb{R}_I\, \varphi_2$ signifies that φ_2 is released by φ_1. The non-metric counterparts of these operators share the same intuitive semantics but omit explicit reference to the interval I. The unary operators $\circ_I \varphi$ and $\bullet_I \varphi$ (along with their non-metric versions $\circ\varphi$ and $\bullet\varphi$) indicate that φ holds in the next or previous state, respectively. We distinguish between metric and non-metric variants based on the presence of interval constraints: in the metric

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

case, satisfaction is contingent upon conditions being met within the intervals augmenting the operators, whereas non-metric operators impose no such quantitative bounds. Additionally, the following derived operators can be defined:

$\blacksquare_I \varphi$	$\stackrel{def}{=}$	$\bot \mathbf{T}_I \varphi$	*always before*	$\Box_I \varphi$	$\stackrel{def}{=}$	$\bot \mathbb{R}_I \varphi$	*always afterward*
$\blacklozenge_I \varphi$	$\stackrel{def}{=}$	$\top \mathbf{S}_I \varphi$	*eventually before*	$\Diamond_I \varphi$	$\stackrel{def}{=}$	$\top \mathbb{U}_I \varphi$	*eventually afterward*
$\mathbf{I}$	$\stackrel{def}{=}$	$\neg\bullet\top$	*initial*	$\mathbb{F}$	$\stackrel{def}{=}$	$\neg\circ\top$	*final*
$\widehat{\bullet}_I \varphi$	$\stackrel{def}{=}$	$\bullet_I \top \rightarrow \bullet_I \varphi$	*weak previous*	$\widehat{\circ}_I \varphi$	$\stackrel{def}{=}$	$\circ_I \top \rightarrow \circ_I \varphi$	*weak next*

2.1 Semantics

A *Here-and-There trace* (for short HT-*trace*) of length $\lambda \in \mathbb{N} \cup \{\omega\}$ over alphabet $\mathcal{A}$ is a sequence of pairs $(\langle H_i, T_i \rangle)_{i \in [0..\lambda)}$ with $H_i \subseteq T_i \subseteq \mathcal{A}$ for any $i \in [0..\lambda)$. For convenience, we often represent an HT-trace as the pair $\langle \mathbf{H}, \mathbf{T} \rangle$ of sequences $\mathbf{H} = (H_i)_{i \in [0..\lambda)}$ and $\mathbf{T} = (T_i)_{i \in [0..\lambda)}$. Notice that, when $\lambda = \omega$, this covers traces of infinite length. We say that $\langle \mathbf{H}, \mathbf{T} \rangle$ is *total* whenever $\mathbf{H} = \mathbf{T}$, that is, $H_i = T_i$ for all $i \in [0..\lambda)$.

Definition 1 (Becker et al. 2024b)
A timed HT-trace $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ of length λ over $(\mathbb{N}, <)$ and alphabet $\mathcal{A}$ is a pair consisting of

- an HT-trace $\langle \mathbf{H}, \mathbf{T} \rangle$ of length λ over $\mathcal{A}$ and
- a function $\tau : [0..\lambda) \rightarrow \mathbb{N}$ such that $\tau(i) \leq \tau(i{+}1)$.

A timed HT-trace of length $\lambda > 1$ is called *strict* if $\tau(i) < \tau(i{+}1)$ for all $i{+}1 \in [0..\lambda)$ such that $i + 1 < \lambda$ and $non - strict$ otherwise. We assume w.l.o.g. that $\tau(0) = 0$.

Function τ assigns to each state index $i \in [0..\lambda)$ a time point $\tau(i) \in \mathbb{N}$ representing the number of time units (seconds, milliseconds, etc, depending on the chosen granularity) elapsed since time point $\tau(0) = 0$, chosen as the beginning of the trace. Given any timed HT-trace, satisfaction of formulas is defined as follows.

Definition 2 (MHT-satisfaction; Becker et al. 2024b)
A timed HT-trace $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ of length λ over alphabet $\mathcal{A}$ *satisfies* a metric formula φ over $\mathcal{A}$ at step $k \in [0..\lambda)$, written $\mathbf{M}, k \models \varphi$, if the following conditions hold:

1. $\mathbf{M}, k \not\models \bot$
2. $\mathbf{M}, k \models p$ if $p \in H_k$ for any atom $p \in \mathcal{A}$
3. $\mathbf{M}, k \models \varphi \wedge \psi$ iff $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \models \psi$
4. $\mathbf{M}, k \models \varphi \vee \psi$ iff $\mathbf{M}, k \models \varphi$ or $\mathbf{M}, k \models \psi$
5. $\mathbf{M}, k \models \varphi \rightarrow \psi$ iff $\mathbf{M}', k \not\models \varphi$ or $\mathbf{M}', k \models \psi$, for both $\mathbf{M}' = \mathbf{M}$ and $\mathbf{M}' = (\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$
6. $\mathbf{M}, k \models \bullet_I \varphi$ iff $k > 0$ and $\mathbf{M}, k{-}1 \models \varphi$ and $\tau(k) - \tau(k{-}1) \in I$
7. $\mathbf{M}, k \models \varphi \, \mathbf{S}_I \, \psi$ iff for some $j \in [0..k]$ with $\tau(k) - \tau(j) \in I$, we have $\mathbf{M}, j \models \psi$ and $\mathbf{M}, i \models \varphi$ for all $i \in (j..k]$
8. $\mathbf{M}, k \models \varphi \, \mathbf{T}_I \, \psi$ iff for all $j \in [0..k]$ with $\tau(k) - \tau(j) \in I$, we have $\mathbf{M}, j \models \psi$ or $\mathbf{M}, i \models \varphi$ for some $i \in (j..k]$
9. $\mathbf{M}, k \models \circ_I \varphi$ iff $k + 1 < \lambda$ and $\mathbf{M}, k{+}1 \models \varphi$ and $\tau(k{+}1) - \tau(k) \in I$

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

10. $\mathbf{M}, k \models \varphi \,\mathbb{U}_I\, \psi$ iff for some $j \in [k..\lambda)$ with $\tau(j) - \tau(k) \in I$, we have $\mathbf{M}, j \models \psi$ and $\mathbf{M}, i \models \varphi$ for all $i \in [k..j)$
11. $\mathbf{M}, k \models \varphi \,\mathbb{R}_I\, \psi$ iff for all $j \in [k..\lambda)$ with $\tau(j) - \tau(k) \in I$, we have $\mathbf{M}, j \models \psi$ or $\mathbf{M}, i \models \varphi$ for some $i \in [k..j)$

A formula φ is a *tautology* (or is valid), written $\models \varphi$, iff $\mathbf{M}, k \models \varphi$ for any timed HT-trace $\mathbf{M}$ and any $k \in [0..\lambda)$. MHT is the logic induced by the set of all such tautologies. For two formulas φ, ψ we write $\varphi \equiv \psi$, iff $\models \varphi \leftrightarrow \psi$, that is, $\mathbf{M}, k \models \varphi \leftrightarrow \psi$ for any timed HT-trace $\mathbf{M}$ of length λ and any $k \in [0..\lambda)$. A timed HT-trace $\mathbf{M}$ is an MHT *model* of a metric temporal formula φ if $\mathbf{M}, 0 \models \varphi$. Similarly, $\mathbf{M}$ is an MHT *model* of theory Γ if $\mathbf{M}, 0 \models \varphi$ for all $\varphi \in \Gamma$. The set of MHT models of Γ having length λ is denoted as $\mathrm{MHT}(\Gamma, \lambda)$, whereas $\mathrm{MHT}(\Gamma) \stackrel{def}{=} \bigcup_{\lambda=0}^{\omega} \mathrm{MHT}(\Gamma, \lambda)$ is the set of all MHT models of Γ of any length. We may obtain fragments of any metric logic by imposing restrictions on the timed HT-traces used for defining tautologies and models. That is, MHT_f stands for the restriction of MHT to traces of any finite length $\lambda \in \mathbb{N}$ and MHT_ω corresponds to the restriction to traces of infinite length $\lambda = \omega$.

An interesting subset of MHT is the one formed by total timed HT-traces like $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$. In the non-metric version of temporal HT, the restriction to total models corresponds to Linear Temporal Logic (LTL, Pnueli (1977)). In our case, the restriction to total traces defines a metric version of LTL, which we call *Metric Temporal Logic* (or MTL for short). Becker *et al.* (2024b) show that MHT satisfies the common persistence and negation properties, and that MHT_f is decidable.

2.2 Strict traces

Next, we consider a group of results that hold under the assumption of strict traces, namely, that $\tau(i) < \tau(i+1)$ for any pair of consecutive time points. We can enforce metric models to be traces with a strict timing function τ. This can be achieved with the addition of the formula $\Box\neg\bigcirc_0\top$ to the context, where 0 abbreviates [0..0].

Proposition 1.
For any MHT trace $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ of length $\lambda \in \mathbb{N} \cup \{\omega\}$, we have that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau) \models \Box\neg\bigcirc_0\top$ iff τ is a strict function.

In Lemma 1 below, the use of strict traces enables a recursive definition for operators of the form $Q_{\leq n}$, with $Q \in \{\mathbb{U}, \mathbb{R}, \mathbf{S}, \mathbf{T}\}$. This novel characterization, similar to the recursive definition of binary temporal operators in LTL (Pnueli 1977), is more suitable for the Tseitin-style translation developed in this paper. For this reason, we henceforth restrict our focus to strict traces, allowing us to assume the implicit inclusion of the aforementioned axiom. The following equivalences state that the interval [0..0] makes all binary metric operators collapse into their right-hand argument formula, whereas unary operators collapse to a truth constant. For metric formulas ψ and φ and for strict traces, we have:

$$\varphi \,\mathbb{U}_0\, \psi \equiv \varphi \,\mathbb{R}_0\, \psi \equiv \varphi \,\mathbf{S}_0\, \psi \equiv \varphi \,\mathbf{T}_0\, \psi \equiv \psi \tag{2}$$

$$\bigcirc_0 \varphi \equiv \bullet_0 \varphi \equiv \bot \tag{3}$$

$$\widehat{\bigcirc}_0 \varphi \equiv \widehat{\bullet}_0 \varphi \equiv \top \tag{4}$$

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

The last two lines are precisely an effect of dealing with strict traces. For instance, $\bigcirc_0 \varphi \equiv \bot$ tells us that it is always impossible to have a successor state with the same time as the current one, regardless of the formula φ at hand. Lemma 1 below allows unfolding *until*, *release*, *since,* and *trigger* for intervals of the form $[0..n]$.

Lemma 1 (Becker et al. 2024b).
For metric formulas ψ and φ and for $n \geq 0$, we have:

$$\varphi \,\mathbb{U}_{\leq n}\, \psi \equiv \psi \vee (\varphi \wedge \textstyle\bigvee_{x=1}^{n} \bigcirc_x(\varphi \,\mathbb{U}_{\leq (n-x)}\, \psi)) \tag{5}$$

$$\varphi \,\mathbb{R}_{\leq n}\, \psi \equiv \psi \wedge (\varphi \vee \textstyle\bigwedge_{x=1}^{n} \widehat{\bigcirc}_x(\varphi \,\mathbb{R}_{\leq (n-x)}\, \psi)) \tag{6}$$

$$\varphi \,\mathbf{S}_{\leq n}\, \psi \equiv \psi \vee (\varphi \wedge \textstyle\bigvee_{x=1}^{n} \bullet_x(\varphi \,\mathbf{S}_{\leq (n-x)}\, \psi)) \tag{7}$$

$$\varphi \,\mathbf{T}_{\leq n}\, \psi \equiv \psi \wedge (\varphi \vee \textstyle\bigwedge_{x=1}^{n} \widehat{\bullet}_x(\varphi \,\mathbf{T}_{\leq (n-x)}\, \psi)), \tag{8}$$

where intervals of the form x stand for $[x..x]$.

The equivalences presented above hold under the assumption of strict traces. To show it, let us consider the formula $p \,\mathbb{U}_{\leq 1}\, a$ and let us define the timed HT-trace $\mathbf{M} = (\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ of length $\lambda = 2$ as $T_0 = \{p\}$, $T_1 = \{a\}$, and $\tau(0) = \tau(1) = 0$. Since $\tau(0) \not< \tau(1)$, $\mathbf{M}$ is not a strict trace. Hence, $\mathbf{M}, 0 \models p \,\mathbb{U}_{\leq 1}\, a$ but $\mathbf{M}, 0 \not\models a \vee (p \wedge \bigcirc_1 (p \,\mathbb{U}_0\, a))$, which falsifies (5). Formulas of the form $\varphi \,\mathbb{U}_{[a..b)}\, \psi$, where $[a..b)$ is a general interval, accept more complicated recursive definitions (see (Becker *et al.* 2024b) for details) and they are left out of the scope of this paper.

2.3 Metric equilibrium models

As in (propositional) equilibrium logic (Pearce 2006), non-monotonicity is achieved by means of a selection among the timed HT-traces of a theory.

Definition 3 (Metric Equilibrium Model (Becker et al. 2024b))
Let $\mathfrak{G}$ be some set of timed HT-traces of a given theory. A total timed HT-trace $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ is a metric equilibrium model of $\mathfrak{G}$ iff there is no other $\mathbf{H}$ different from $\mathbf{T}$ such that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau) \in \mathfrak{G}$.

We talk about metric equilibrium models of a theory Γ when $\mathfrak{S} = \mathrm{MHT}(\Gamma)$, and we write $\mathrm{MEL}(\Gamma, \lambda)$ and $\mathrm{MEL}(\Gamma)$ to stand for the metric equilibrium models of $\mathrm{MHT}(\Gamma, \lambda)$ and $\mathrm{MHT}(\Gamma)$, respectively. *Metric Equilibrium Logic* (MEL) is the non-monotonic logic induced by the metric equilibrium models of metric temporal theories. Variants MEL_f and MEL_ω refer to MEL when restricted to traces of finite and infinite length, respectively.

3 Three-valued semantics

It is well known that HT semantics is equivalent to three-valued Gödel logic (Gödel 1932). The use of the latter semantics in the context of equilibrium logic and its temporal extensions facilitates correctness proofs of Tseitin reductions to logic programs and temporal logic programs, since it allows equivalences to be evaluated more conveniently. Three-valued Gödel logic was extended to the temporal setting in Cabalar (2010), within

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

the context of reducing temporal theories to temporal logic programs. Since, in this paper, we extend Cabalar's results to the metric case, we also provide a three-valued characterization of MHT. In our setting, a three-valued interpretation is a mapping

$$m : ([0..\lambda) \to \mathbb{N}) \to (([0..\lambda) \times \mathcal{A}) \to \{0, 1, 2\})$$

where, broadly speaking, the first input function $([0..\lambda) \to \mathbb{N})$ represents a timing function that gives the different time points associated with each state of the trace; we denote it by τ in the rest of this section. Given τ, we define

$$m(\tau) : [0..\lambda) \times \mathcal{A} \to \{0, 1, 2\}$$

as the associated evaluation for propositional variables, that is, a function that provides the truth value for each propositional variable $p \in \mathcal{A}$ at each time point $k \in [0..\lambda)$. Such $m(\tau)$[2] can be extended to $\boldsymbol{m}$ to cope with arbitrary formulas as shown below.

$$\boldsymbol{m}(k, \bot) = 0$$

$$\boldsymbol{m}(k, a) = m(\tau)(k, a) \qquad \text{for any atom } a$$

$$\boldsymbol{m}(k, \varphi \wedge \psi) = \min(\boldsymbol{m}(k, \varphi), \boldsymbol{m}(k, \psi))$$

$$\boldsymbol{m}(k, \varphi \vee \psi) = \max(\boldsymbol{m}(k, \varphi), \boldsymbol{m}(k, \psi))$$

$$\boldsymbol{m}(k, \varphi \to \psi) = \begin{cases} 2 & \text{if } \boldsymbol{m}(k, \varphi) \leq \boldsymbol{m}(k, \psi) \\ \boldsymbol{m}(k, \psi) & \text{otherwise} \end{cases}$$

$$\boldsymbol{m}(k, \bullet_I \varphi) = \begin{cases} 0 & \text{if } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin I \\ \boldsymbol{m}(k-1, \varphi) & \text{otherwise} \end{cases}$$

$$\boldsymbol{m}(k, \widehat{\bullet}_I \varphi) = \begin{cases} 2 & \text{if } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin I \\ \boldsymbol{m}(k-1, \varphi) & \text{otherwise} \end{cases}$$

$$\boldsymbol{m}(k, \varphi \, \mathbf{S}_I \, \psi) = \sup\{\min\{\boldsymbol{m}(i, \psi), \boldsymbol{m}(j, \varphi) \mid j \in (i..k]\} \mid i \in [0..k] \text{ and } \tau(k) - \tau(i) \in I\}^{3}$$

$$\boldsymbol{m}(k, \varphi \, \mathbf{T}_I \, \psi) = \inf\{\max\{\boldsymbol{m}(i, \psi), \boldsymbol{m}(j, \varphi) \mid j \in (i..k]\} \mid i \in [0..k] \text{ and } \tau(k) - \tau(i) \in I\}$$

$$\boldsymbol{m}(k, \bigcirc_I \varphi) = \begin{cases} 0 & \text{if } k+1 = \lambda \ (\neq \omega) \text{ or } \tau(k+1) - \tau(k) \notin I \\ \boldsymbol{m}(k+1, \varphi) & \text{otherwise} \end{cases}$$

$$\boldsymbol{m}(k, \widehat{\bigcirc}_I \varphi) = \begin{cases} 2 & \text{if } k+1 = \lambda \ (\neq \omega) \text{ or } \tau(k+1) - \tau(k) \notin I \\ \boldsymbol{m}(k+1, \varphi) & \text{otherwise} \end{cases}$$

$$\boldsymbol{m}(k, \varphi \, \mathbb{U}_I \, \psi) = \sup\{\min\{\boldsymbol{m}(i, \psi), \boldsymbol{m}(j, \varphi) \mid j \in [k..i)\} \mid i \in [k..\lambda) \text{ and } \tau(i) - \tau(k) \in I\}$$

$$\boldsymbol{m}(k, \varphi \, \mathbb{R}_I \, \psi) = \inf\{\max\{\boldsymbol{m}(i, \psi), \boldsymbol{m}(j, \varphi) \mid j \in [k..i)\} \mid i \in [k..\lambda) \text{ and } \tau(i) - \tau(k) \in I\}$$

The three-valued semantics defined above offers a significant advantage when establishing equivalences. For instance, demonstrating that $\boldsymbol{m}(k, \varphi \leftrightarrow \psi) = 2$ is equivalent to showing that $\boldsymbol{m}(k, \varphi) = \boldsymbol{m}(k, \psi)$ for all formulas φ and ψ. Similarly, proving that $\boldsymbol{m}(0, \Box(\varphi \leftrightarrow \psi))$ is equivalent to establishing that $\boldsymbol{m}(k, \varphi) = \boldsymbol{m}(k, \psi)$ for all $k \in [0..\lambda)$.

[2] Note that here τ is implicit.
[3] Here we consider that $\sup \emptyset = 0$ and $\inf \emptyset = 2$.

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

These properties streamline the formal verification process, as they allow complex logical equivalences to be treated as identities within our three-valued framework.

To establish the equivalence between the standard MHT semantics and our three-valued characterization, we define a function f from the set of timed HT-traces to the set of three-valued interpretations. For any timed HT-trace $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ of length λ, let $f(\mathbf{M}) = m(\tau)$ where $m(\tau) : [0 \cdot \cdot \lambda) \times \mathcal{A} \to \{0, 1, 2\}$ is defined for any $k \in [0..\lambda)$ and $a \in \mathcal{A}$ as follows:

$$m(\tau)(k, a) = \begin{cases} 0 & \text{if } a \notin T_k \\ 1 & \text{if } a \in T_k \setminus H_k \\ 2 & \text{if } a \in H_k \end{cases}$$

This function provides us with a 1-1 correspondence between timed HT-traces and three-valued interpretations. It can be checked that f is *bijective*. The following lemma shows that the satisfaction of arbitrary metric temporal formulas is preserved among the semantics.

Proposition 2.
For every timed HT-trace $\mathbf{M}$ of length λ, every $k \in [0..\lambda)$ and every formula φ, $\mathbf{M}, k \models \varphi$ iff $f(\mathbf{M})(k, \varphi) = 2$ and $(\langle \mathbf{T}, \mathbf{T} \rangle), k \models \varphi$ iff $f(\mathbf{M})(k, \varphi) \neq 0$.

Corollary 1.
The MHT semantics and its three-valued characterization are equivalent.

For our Tseitin translation, we impose the following restriction: the intervals associated to binary temporal operators $\mathbf{S}_I$, $\mathbf{T}_I$, $\mathbb{U}_I$ and $\mathbb{R}_I$ are of the form $I = [0..n]$, with $n < \omega$. By abbreviating the interval $[0..n]$ by '$\leq n$', the metric temporal operators are of the form $\mathbf{S}_{\leq n}$, $\mathbf{T}_{\leq n}$, $\mathbb{U}_{\leq n}$, $\mathbb{R}_{\leq n}$. The following proposition shows how equivalences (5)–(8) lead to an alternative three-valued interpretation for the metric operators.

Proposition 3.
For any interpretation $\boldsymbol{m}$ of length λ, any $k \in [0..\lambda)$ and any $n \in [0..\omega)$, we have:

$$\boldsymbol{m}(k, \varphi\, \mathbb{U}_{\leq n}\, \psi) = \max\{\boldsymbol{m}(k, \psi), \min\{\boldsymbol{m}(k, \varphi), \sup\{\boldsymbol{m}(k, \circ_x(\varphi\, \mathbb{U}_{\leq (n-x)}\, \psi)) \mid x \in [1..n]\}\}\}$$
$$\boldsymbol{m}(k, \varphi\, \mathbb{R}_{\leq n}\, \psi) = \min\{\boldsymbol{m}(k, \psi), \max\{\boldsymbol{m}(k, \varphi), \inf\{\boldsymbol{m}(k, \widehat{\circ}_x(\varphi\, \mathbb{R}_{\leq (n-x)}\, \psi)) \mid x \in [1..n]\}\}\}$$
$$\boldsymbol{m}(k, \varphi\, \mathbf{S}_{\leq n}\, \psi) = \max\{\boldsymbol{m}(k, \psi), \min\{\boldsymbol{m}(k, \varphi), \sup\{\boldsymbol{m}(k, \bullet_x(\varphi\, \mathbf{S}_{\leq (n-x)}\, \psi)) \mid x \in [1..n]\}\}\}$$
$$\boldsymbol{m}(k, \varphi\, \mathbf{T}_{\leq n}\, \psi) = \min\{\boldsymbol{m}(k, \psi), \max\{\boldsymbol{m}(k, \varphi), \inf\{\boldsymbol{m}(k, \widehat{\bullet}_x(\varphi\, \mathbf{T}_{\leq (n-x)}\, \psi)) \mid x \in [1..n]\}\}\}$$

Note that, by applying Proposition 3, we can conclude that $\boldsymbol{m}(k, \varphi \mathbb{U}_0 \psi) = \boldsymbol{m}(k, \varphi \mathbb{R}_0 \psi) = \boldsymbol{m}(k, \varphi\, \mathbf{S}_0\, \psi) = \boldsymbol{m}(k, \varphi\, \mathbf{T}_0\, \psi) = \boldsymbol{m}(k, \psi)$. In what follows, we use those semantics when dealing with metric modalities of the form $Q_{\leq n}$ with $Q \in \{\mathbb{U}, \mathbb{R}, \mathbf{S}, \mathbf{T}\}$.

4 Translation into metric temporal logic programs

Definition 4 (Metric temporal literal, rule and program).
Given an alphabet $\mathcal{A}$, we define the set of regular literals as $\{a, \neg a \mid a \in \mathcal{A}\}$, the set of metric temporal literals as $\{\bullet_I a, \neg \bullet_I a, \widehat{\bullet}_I a, \neg \widehat{\bullet}_I a, \bullet a, \neg \bullet a, \widehat{\bullet} a, \neg \widehat{\bullet} a \mid a \in \mathcal{A}\}$. A metric temporal rule is either

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

- an initial rule $B \to A$
- a dynamic rule of the form $\widehat{\bigcirc}\Box\,(B \to A)$, or
- a final rule of the form $\Box\,(\mathbb{F} \to (B \to A))$

where $B = \bigwedge_{i=1}^{n} b_i$ with $n \geq 0$, $A = \bigvee_{j=1}^{m} a_j$ with $m \geq 0$, and each b_i and a_j are metric temporal literals for dynamic rules, or regular literals for initial and final rules. Facts of the form p are equivalent to rules of the form $\to p$ while constraints of the form $\neg p$ are equivalent to rules of the type $p \to \bot$. A metric temporal logic program is a set of metric temporal rules.

In the propositional setting, the Tseitin transformation is typically performed by substituting subformulas with auxiliary atoms that represent their respective truth values. In the metric temporal case, the recursive nature of the metric temporal operators necessitates a more complex approach. For instance, if we follow Proposition 1, the satisfaction of the formula $m\ \mathbb{U}_{\leq 3}\ a$ requires the evaluation of $\bigcirc_x\,(m\ \mathbb{U}_{\leq n-x}\ a)$, for all $x \in [1..3]$, which are not subformulas of $m\ \mathbb{U}_{\leq 3}\ a$.

Inspired by the Fisher-Ladner closure used in dynamic logics (Fischer and Ladner 1979), the set of subformulas must be closed under the recursive definitions of the binary metric operators. This closure ensures that the auxiliary atoms correctly capture the temporal expansion of formulas across the discrete time points of the timed trace.

Definition 5 (Closure).
Let φ be a metric temporal formula. We define the closure of φ, in symbols $cl(\varphi)$, as the smallest theory satisfying the following properties:

1. $\varphi \in cl(\varphi)$
2. $cl(\varphi)$ is closed under subformulas
3. $\varphi\ \mathbb{U}_{\leq n}\ \psi \in cl(\varphi)$ and $n \geq 1$ imply $\bigcirc_i\,(\varphi\ \mathbb{U}_{\leq n-i}\ \psi) \in cl(\varphi)$, for all $i \in [1..n]$
4. $\varphi\ \mathbb{R}_{\leq n}\ \psi \in cl(\varphi)$ and $n \geq 1$ imply $\widehat{\bigcirc}_i\,(\varphi\ \mathbb{R}_{\leq n-i}\ \psi) \in cl(\varphi)$, for all $i \in [1..n]$
5. $\varphi\ \mathbf{S}_{\leq n}\ \psi \in cl(\varphi)$ and $n \geq 1$ imply $\bullet_i\,(\varphi\ \mathbf{S}_{\leq n-i}\ \psi) \in cl(\varphi)$, for all $i \in [1..n]$
6. $\varphi\ \mathbf{T}_{\leq n}\ \psi \in cl(\varphi)$ and $n \geq 1$ imply $\widehat{\bullet}_i\,(\varphi\ \mathbf{T}_{\leq n-i}\ \psi) \in cl(\varphi)$, for all $i \in [1..n]$

For instance, the closure of the formula (1), $\Box(press \to moving\ \mathbb{U}_{\leq 3}\ arrives)$, corresponds to the set cl ((1)) below. For brevity, we abbreviate *press*, *moving* and *arrives* as p, m and a, respectively.

$$cl((1)) = \left\{ \begin{array}{rrr} \Box(p \to m\ \mathbb{U}_{\leq 3}\ a)\ (\mathbf{F1}), & p \to m\ \mathbb{U}_{\leq 3}\ a\ (\mathbf{F2}), & m\ \mathbb{U}_{\leq 3}\ a\ (\mathbf{F3}), \\ \bigcirc_3(m\ \mathbb{U}_0\ a)\ (\mathbf{F4}), & \bigcirc_2(m\ \mathbb{U}_{\leq 1}\ a)\ (\mathbf{F5}), & \bigcirc_2(m\ \mathbb{U}_0\ a)\ (\mathbf{F6}), \\ \bigcirc_1(m\ \mathbb{U}_{\leq 2}\ a)\ (\mathbf{F7}), & \bigcirc_1(m\ \mathbb{U}_{\leq 1}\ a)\ (\mathbf{F8}), & \bigcirc_1(m\ \mathbb{U}_0\ a)\ (\mathbf{F9}), \\ m\ \mathbb{U}_{\leq 2}\ a\ (\mathbf{F10}), & m\ \mathbb{U}_{\leq 1}\ a\ (\mathbf{F11}), & m\ \mathbb{U}_0\ a, \\ m, & a, & p \end{array} \right\}$$

Each formula in $cl((1)) \setminus \{m\ \mathbb{U}_0\ a, m, p, a\}$ has been labeled in order to be referenced when displaying the translation of (1) into a metric temporal logic program shown in Table 4. Note that the formula (**F1**) corresponds to (1). The next proposition shows that the size of the closure of a formula φ, in symbols $|cl(\varphi)|$ can be bounded by a polynomial that depends on two parameters: the size of φ (denoted by $|\varphi|$) and the maximum number n used in a binary temporal operator occurring in φ.

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

Proposition 4.
For any metric formula φ, $cl(\varphi)$ is finite and the total size of all the formulas in $cl(\varphi)$, $|cl(\varphi)|$, is bounded by $|cl(\varphi)| \leq 2(k_\varphi^2 + 1)|\varphi|$, where $k_\varphi = \max\{1, n_\varphi\}$ and n_φ is the maximum n that occurs in the operators $\mathbb{U}_{\leq n}$, $\mathbb{R}_{\leq n}$, $\mathbf{S}_{\leq n}$ and $\mathbf{T}_{\leq n}$ in φ.

Note that the closure of a formula contains other formulas than subformulas, which are also translated into logical rules. To prove the correctness, we cannot only rely on structural induction since our translation depends on the notion of closure, which contains formulas that are not subformulas. For instance, let us consider the formula $m\ \mathbb{U}_{\leq 3}\ a \in cl((1))$. Following Proposition 3, the formulas $m\ \mathbb{U}_{\leq 3}$ and $(a \vee (m \wedge (\circ_1 (m\ \mathbb{U}_{\leq 2}\ a) \vee \circ_2 (m\ \mathbb{U}_{\leq 1}\ a) \vee \circ_3 (m\ \mathbb{U}_0\ a))))$ are MHT_f-equivalent. Our translation replaces $m\,\mathbb{U}_{\leq 3}\,a$, $\circ_1 (m\ \mathbb{U}_{\leq 2}\ a)$, $\circ_2 (m\ \mathbb{U}_{\leq 1}\ a)$ and $\circ_3 (m\ \mathbb{U}_0\ a)$ by fresh atoms and then adds extra formulas that ensure the relationship between those and the formulas they replace. Although the proof strategy is to proceed by structural induction, it is not directly possible here since, for instance, $\circ_1 (m\ \mathbb{U}_{\leq 2}\ a)$ is not a proper subformula of $m\ \mathbb{U}_{\leq 3}\ a$. We aim to define a well-founded strict partial order that takes into account not only the complexity of the input formula but also the unfolding of the $\mathbb{U}_{\leq n}$, $\mathbb{R}_{\leq n}$, $\mathbf{S}_{\leq n}$ and $\mathbf{T}_{\leq n}$ operators. We associate with each metric temporal formula φ a natural number $u(\varphi)$ that is used to define our well-founded strict partial order.

Definition 6.
For a given metric formula φ in the language, we define $u(\varphi)$, that captures the unfolding of the formulas $\varphi\ \mathbb{U}_{\leq n}\ \psi$, $\varphi\ \mathbb{R}_{\leq n}\ \psi$, $\varphi\ \mathbf{S}_{\leq n}\ \psi$ and $\varphi\ \mathbf{T}_{\leq n}\ \psi$, as follows:

- $u(\bot) = u(\top) = 1$;
- $u(a) = 1$, for all $a \in \mathcal{A}$;
- $u(\varphi \otimes \psi) = 1 + u(\varphi) + u(\psi)$, with $\otimes \in \{\wedge, \vee, \rightarrow, \mathbb{U}, \mathbb{R}, \mathbf{S}, \mathbf{T}\}$;
- $u(\oplus\varphi) = 1 + u(\varphi)$, with $\oplus \in \{\bigcirc, \bullet, \widehat{\bigcirc}, \widehat{\bullet}, \bigcirc_I, \bullet_I, \widehat{\bigcirc}_I, \widehat{\bullet}_I, \Diamond, \blacklozenge, \Box, \blacksquare\}$;
- $u(\varphi \otimes_{\leq n} \psi) = 2 * (n + 1) + u(\varphi) + u(\psi)$, with $\otimes \in \{\mathbb{U}, \mathbb{R}, \mathbf{S}, \mathbf{T}\}$;
- $u(\oplus_{\leq n}\varphi) = 2 * (n + 1) + u(\varphi)$, with $\oplus \in \{\Diamond, \blacklozenge, \Box, \blacksquare\}$.

Given two metric formulas, we say that $\varphi \prec_u \psi$ if and only if $u(\varphi) < u(\psi)$. Since the previous definition is given as a mapping into the natural numbers, we can readily prove the following result.

Proposition 5.
$\prec_u$ is a well-founded strict partial order.

Regarding our running example, we get $u(m\ \mathbb{U}_{\leq 3}\ a) = 10$, $u(\circ_1 (m\ \mathbb{U}_{\leq 2}\ a)) = 9$, $u(\circ_2 (m\ \mathbb{U}_{\leq 1}\ a)) = 7$ and $u(\circ_3 (m\ \mathbb{U}_0\ a)) = 5$. Consequently, $\circ_1 (m\ \mathbb{U}_{\leq 2}\ a) \prec_u m\ \mathbb{U}_{\leq 3}\ a$, $\circ_2 (m\ \mathbb{U}_{\leq 1}\ a) \prec_u m\ \mathbb{U}_{\leq 3}\ a$ and $\circ_3 (m\ \mathbb{U}_0\ a) \prec_u m\ \mathbb{U}_{\leq 3}\ a$, which allows us to apply the induction on the unfolding of binary metric temporal operators. Thanks to Proposition 5, we can use u to provide an induction hypothesis, which is not based on the typical structural induction.

The following theorem states that an arbitrary metric temporal formula φ[4] over an alphabet $\mathcal{A}$ can be compiled into a metric temporal logic program Γ over an extended

[4] We assume that the binary metric temporal operators occurring in φ are restricted to intervals of the form $[0..n]$.

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

alphabet $\mathcal{A}_\ell \supseteq \mathcal{A}$ in such a way that there exists a bijection between the timed HT-traces of length $\lambda \in [1..\omega)$ satisfying φ and the traces of length λ satisfying Γ projected onto $\mathcal{A}$. That is, we have *equivalence modulo auxiliary atoms*.

Theorem 1 (Metric temporal logic program reduction).
For every metric temporal formula φ[4] over $\mathcal{A}$ and for any $\lambda \in [1..\omega)$, we have that

$$\mathrm{MHT}(\varphi, \lambda) = \{\mathbf{M}|_{\mathcal{A}} \mid \mathbf{M} \in \mathrm{MHT}(\{\ell_\varphi\} \cup \eta^*(\varphi), \lambda)\}. \tag{9}$$

For transforming arbitrary temporal formulas into metric temporal logic programs, we use a Tseitin-style reduction (Tseitin 1968) that relies on an alphabet extended by new atoms for each formula in the original language. The equivalence result in Theorem 1 is then obtained after removing auxiliary atoms and, in fact, is still preserved inside the context of a larger theory for the original vocabulary. The results presented in Theorem 1 are preserved when selecting the minimal models. This means that there exists a bijection between the metric equilibrium models of φ and Γ projected onto $\mathcal{A}$. That is, we have *strong equivalence modulo auxiliary atoms*.

Corollary 2.
For every metric temporal formula φ[4] over $\mathcal{A}$, for any $\lambda \in [1..\omega)$ and any theory Γ over $\mathcal{A}$,

$$\mathrm{MEL}(\{\varphi\} \cup \Gamma, \lambda) = \{\langle(\mathbf{H}, \mathbf{T}), \tau\rangle|_{\mathcal{A}} \mid \langle(\mathbf{H}, \mathbf{T}), \tau\rangle \in \mathrm{MEL}(\{\ell_\varphi\} \cup \eta^*(\varphi) \cup \Gamma, \lambda)\}. \tag{10}$$

Theorem 1 and Corollary 2 can be extended to arbitrary theories by defining the translation σ as the metric temporal logic program:

$$\sigma(\Gamma) = \{\ell_\gamma \mid \gamma \in \Gamma\} \cup \{\eta^*(\mu) \mid \mu \in cl(\Gamma)\}$$

and then replacing $\{\ell_\varphi\} \cup \eta^*(\varphi)$ by $\sigma(\Gamma)$ in expressions (9) and (10). Table 4 shows the translation of the expression in (1), which is done by applying the translation $\eta(\mu)$ for each $\mu \in cl$ ((1)). A metric temporal logic program can be obtained by using $\eta^*(\mu)$ instead. The rest of this section is devoted to proving Theorem 1, for which we extend Cabalar's reduction for the case THT_ω (Cabalar 2010) to the case of MHT_f. The reduction uses an extended alphabet $\mathcal{A}_\ell \supseteq \mathcal{A}$ that additionally contains a new atom ℓ_φ (aka label) for each formula φ in the original language over $\mathcal{A}$. For convenience, we use $\ell_\varphi \stackrel{def}{=} \varphi$ if φ is $\top$, $\bot$ or an atom $a \in \mathcal{A}$. For any non-atomic formula γ over $\mathcal{A}$, we introduce the translation η given along Tables 1, 2, and 3, and call $\eta(\gamma)$ the *definition* of γ (the translation rules for Boolean and non-metric operators are provided in Tables A1 and A2). This definition is not a logic program yet, but it contains some double implication that can be easily transformed into a metric temporal logic program format by simple, non-modal transformations in the propositional logic of HT. This translation of $\eta(\mu)$ into a metric temporal logic program is shown in the tables as $\eta^*(\mu)$. In the rest of the section, we indistinctly use $\eta(\mu)$ and $\eta^*(\mu)$, as the former is more suitable for developing the proofs. As said before, their equivalence can be easily tested inside logic HT, and so, is not provided in this paper.

Given a trace $\mathbf{M} = \langle(\mathbf{H}, \mathbf{T}), \tau\rangle$ of length $\lambda \in \mathbb{N} \cup \{\omega\}$, we define its *projection* onto the alphabet $\mathcal{A}$ as $\mathbf{M}|_{\mathcal{A}} = \langle(\mathbf{H}', \mathbf{T}'), \tau\rangle$ where $H'_i = H_i \cap \mathcal{A}$ and $T'_i = T_i \cap \mathcal{A}$, for all $i \in [0..\lambda)$.

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

Table 1. *Translation for metric next, weak next, previous and weak previous*

μ	$\eta(\mu)$	$\eta^*(\mu)$
$\bigcirc_I \varphi$	$\widehat{\bigcirc}\Box\,(\bullet_I \top \rightarrow\ (\bullet \ell_\mu \leftrightarrow \ell_\varphi))$ $\widehat{\bigcirc}\Box\,(\neg \bullet_I \top \rightarrow \neg\, \bullet \ell_\mu)$ $\Box\,(\mathbb{F} \rightarrow \neg \ell_\mu)$	$\widehat{\bigcirc}\Box\,(\bullet_I \ell_\mu \rightarrow \ell_\varphi)$ $\widehat{\bigcirc}\Box\,(\bullet_I \top \wedge\ \ell_\varphi \rightarrow \bullet \ell_\mu)$ $\widehat{\bigcirc}\Box\,(\neg \bullet_I \top \rightarrow \neg\, \bullet \ell_\mu)$ $\Box\,(\mathbb{F} \rightarrow \neg \ell_\mu)$
$\widehat{\bigcirc}_I \varphi$	$\widehat{\bigcirc}\Box\,(\bullet_I \top \rightarrow\ (\bullet \ell_\mu \leftrightarrow \ell_\varphi))$ $\widehat{\bigcirc}\Box\,(\neg \bullet_I \top \rightarrow\ \bullet \ell_\mu)$ $\Box\,(\mathbb{F} \rightarrow \ell_\mu)$	$\widehat{\bigcirc}\Box\,(\bullet_I \ell_\mu \rightarrow \ell_\varphi)$ $\widehat{\bigcirc}\Box\,(\bullet_I \top \wedge\ \ell_\varphi \rightarrow \bullet \ell_\mu)$ $\widehat{\bigcirc}\Box\,(\neg \bullet_I \top \rightarrow\ \bullet \ell_\mu)$ $\Box\,(\mathbb{F} \rightarrow \ell_\mu)$
$\bullet_I \varphi$	$\widehat{\bigcirc}\Box\,(\ell_\mu \leftrightarrow \bullet_I \ell_\varphi)$ $\neg \ell_\mu$	$\widehat{\bigcirc}\Box\,(\ell_\mu \rightarrow \bullet_I \ell_\varphi)$ $\widehat{\bigcirc}\Box\,(\bullet_I \ell_\varphi \rightarrow \ell_\mu)$ $\neg \ell_\mu$
$\widehat{\bullet}_I \varphi$	$\widehat{\bigcirc}\Box\left(\ell_\mu \leftrightarrow \widehat{\bullet}_I \ell_\varphi\right)$ ℓ_μ	$\widehat{\bigcirc}\Box\left(\ell_\mu \wedge \widehat{\bullet}_I \top \rightarrow \bullet \ell_\varphi\right)$ $\widehat{\bigcirc}\Box\left(\widehat{\bullet}_I \ell_\varphi \rightarrow \ell_\mu\right)$ ℓ_μ

Table 2. *Translation of metric since and metric trigger*

μ	$\eta(\mu)$	$\eta^*(\mu)$
$\varphi\, \mathbf{S}_{\leq n}\, \psi$	$\Box\left(\ell_\mu \leftrightarrow \left(\ell_\psi \vee \left(\ell_\varphi \wedge \bigvee_{x=1}^{n} \ell_{\bullet_x(\varphi \mathbf{S}_{\leq (n-x)} \psi)}\right)\right)\right)$	$\ell_\mu \rightarrow \ell_\psi \vee \ell_\varphi$ $\ell_\mu \rightarrow \ell_\psi \vee \bigvee_{x=1}^{n} \ell_{\bullet_x(\varphi \mathbf{S}_{\leq (n-x)} \psi)}$ $\ell_\psi \rightarrow \ell_\mu$ $\ell_\varphi \wedge \ell_{\bullet_1(\varphi \mathbf{S}_{\leq (n-1)} \psi)} \rightarrow \ell_\mu$ $\ldots$ $\ell_\varphi \wedge \ell_{\bullet_n(\varphi \mathbf{S}_{\leq (n-n)} \psi)} \rightarrow \ell_\mu$ $\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\psi \vee \ell_\varphi)$ $\widehat{\bigcirc}\Box\left(\ell_\mu \rightarrow \ell_\psi \vee \bigvee_{x=1}^{n} \ell_{\bullet_x(\varphi \mathbf{S}_{\leq (n-x)} \psi)}\right)$ $\widehat{\bigcirc}\Box(\ell_\psi \rightarrow \ell_\mu)$ $\widehat{\bigcirc}\Box\left(\ell_\varphi \wedge \ell_{\bullet_1(\varphi \mathbf{S}_{\leq (n-1)} \psi)} \rightarrow \ell_\mu\right)$ $\ldots$ $\widehat{\bigcirc}\Box\left(\ell_\varphi \wedge \ell_{\bullet_n(\varphi \mathbf{S}_{\leq (n-n)} \psi)} \rightarrow \ell_\mu\right)$
$\varphi\, \mathbf{T}_{\leq n}\, \psi$	$\Box\left(\ell_\mu \leftrightarrow \left(\ell_\psi \wedge \left(\ell_\varphi \vee \bigwedge_{x=1}^{n} \ell_{\widehat{\bullet}_x(\varphi \mathbf{T}_{\leq (n-x)} \psi)}\right)\right)\right)$	$\ell_\mu \rightarrow \ell_\psi$ $\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\bullet}_1(\varphi \mathbf{T}_{\leq (n-1)} \psi)}$ $\ldots$ $\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\bullet}_n(\varphi \mathbf{T}_{\leq (n-n)} \psi)}$ $\ell_\psi \wedge \ell_\varphi \rightarrow \ell_\mu$ $\left(\ell_\psi \wedge \bigwedge_{x=1}^{n} \ell_{\widehat{\bullet}_x(\varphi \mathbf{T}_{\leq (n-x)} \psi)}\right) \rightarrow \ell_\mu$ $\widehat{\bigcirc}\Box\,(\ell_\mu \rightarrow \ell_\psi)$ $\widehat{\bigcirc}\Box\left(\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\bullet}_1(\varphi \mathbf{T}_{\leq (n-1)} \psi)}\right)$ $\ldots$ $\widehat{\bigcirc}\Box\left(\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\bullet}_n(\varphi \mathbf{T}_{\leq (n-n)} \psi)}\right)$ $\widehat{\bigcirc}\Box\,(\ell_\psi \wedge \ell_\varphi \rightarrow \ell_\mu)$ $\widehat{\bigcirc}\Box\left(\left(\ell_\psi \wedge \bigwedge_{x=1}^{n} \ell_{\widehat{\bullet}_x(\varphi \mathbf{T}_{\leq (n-x)} \psi)}\right) \rightarrow \ell_\mu\right)$

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

Table 3. *Translation of metric until and metric release*

μ	$\eta(\mu)$	$\eta^*(\mu)$
$\varphi\,\mathbb{U}_{\leq n}\,\psi$	$\Box\left(\ell_\mu \leftrightarrow \ell_\psi \vee \left(\ell_\varphi \wedge \bigvee_{x=1}^{n} \ell_{\mathsf{O}_x(\varphi\mathbb{U}_{\leq(n-x)}\psi)}\right)\right)$	$\ell_\mu \rightarrow \ell_\psi \vee \ell_\varphi$ $\ell_\mu \rightarrow \ell_\psi \vee \bigvee_{x=1}^{n} \ell_{\mathsf{O}_x(\varphi\mathbb{U}_{\leq(n-x)}\psi)}$ $\ell_\psi \rightarrow \ell_\mu$ $\ell_\varphi \wedge \ell_{\mathsf{O}_1(\varphi\mathbb{U}_{\leq(n-1)}\psi)} \rightarrow \ell_\mu$ $\cdots$ $\ell_\varphi \wedge \ell_{\mathsf{O}_n(\varphi\mathbb{U}_{\leq(n-n)}\psi)} \rightarrow \ell_\mu$ $\widehat{\mathsf{O}}\Box\,(\ell_\mu \rightarrow \ell_\psi \vee \ell_\varphi)$ $\widehat{\mathsf{O}}\Box\left(\ell_\mu \rightarrow \ell_\psi \vee \bigvee_{x=1}^{n} \ell_{\mathsf{O}_x(\varphi\mathbb{U}_{\leq(n-x)}\psi)}\right)$ $\widehat{\mathsf{O}}\Box\,(\ell_\psi \rightarrow \ell_\mu)$ $\widehat{\mathsf{O}}\Box\left(\ell_\varphi \wedge \ell_{\mathsf{O}_1(\varphi\mathbb{U}_{\leq(n-1)}\psi)} \rightarrow \ell_\mu\right)$ $\cdots$ $\widehat{\mathsf{O}}\Box\left(\ell_\varphi \wedge \ell_{\mathsf{O}_n(\varphi\mathbb{U}_{\leq(n-n)}\psi)} \rightarrow \ell_\mu\right)$
$\varphi\,\mathbb{R}_{\leq n}\,\psi$	$\Box\left(\ell_\mu \leftrightarrow \ell_\psi \wedge \left(\ell_\varphi \vee \bigwedge_{x=1}^{n} \ell_{\widehat{\mathsf{O}}_x(\varphi\mathbb{R}_{\leq(n-x)}\psi)}\right)\right)$	$\ell_\mu \rightarrow \ell_\psi$ $\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\mathsf{O}}_1(\varphi\mathbb{R}_{\leq(n-1)}\psi)}$ $\cdots$ $\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\mathsf{O}}_n(\varphi\mathbb{R}_{\leq(n-n)}\psi)}$ $\ell_\psi \wedge \ell_\varphi \rightarrow \ell_\mu$ $\left(\ell_\psi \wedge \bigwedge_{x=1}^{n} \ell_{\widehat{\mathsf{O}}_x(\varphi\mathbb{R}_{\leq(n-x)}\psi)}\right) \rightarrow \ell_\mu$ $\widehat{\mathsf{O}}\Box\,(\ell_\mu \rightarrow \ell_\psi)$ $\widehat{\mathsf{O}}\Box\left(\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\mathsf{O}}_1(\varphi\mathbb{R}_{\leq(n-1)}\psi)}\right)$ $\cdots$ $\widehat{\mathsf{O}}\Box\left(\ell_\mu \rightarrow \ell_\varphi \vee \ell_{\widehat{\mathsf{O}}_n(\varphi\mathbb{R}_{\leq(n-n)}\psi)}\right)$ $\widehat{\mathsf{O}}\Box\,(\ell_\psi \wedge \ell_\varphi \rightarrow \ell_\mu)$ $\widehat{\mathsf{O}}\Box\left(\left(\ell_\psi \wedge \bigwedge_{x=1}^{n} \ell_{\widehat{\mathsf{O}}_x(\varphi\mathbb{R}_{\leq(n-x)}\psi)}\right) \rightarrow \ell_\mu\right)$

Lemma 2.
Let $\mathbf{M} = \langle(\mathbf{H}, \mathbf{T}), \lambda\rangle$ be a MHT model of length $\lambda \in [1..\omega)$ of a formula φ over $\mathcal{A}$. Then, there exists some timed HT-trace $\mathbf{M}' = \langle(\mathbf{H}', \mathbf{T}'), \tau\rangle$ over alphabet $\mathcal{A}_\ell$ such that $\mathbf{M} = \mathbf{M}'|_{\mathcal{A}}$ and $\mathbf{M}', 0 \models \{\ell_\varphi\} \cup \eta^*(\varphi)$.

Lemma 3.
Let φ be a temporal formula over $\mathcal{A}$ and let $\langle(\mathbf{H}, \mathbf{T}), \tau\rangle$ be a MHT model, with length $\lambda \in [1..\omega)$, of $\{\ell_\varphi\} \cup \eta(\varphi)$. Let $\boldsymbol{m}$ be its associated three-valued interpretation. Then for any $\mu \in cl(\varphi)$ and any $k \in [0..\lambda)$, we have $\boldsymbol{m}(k, \ell_\mu) = \boldsymbol{m}(k, \mu)$.

Let us discuss now the proof of Theorem 1. We show that for any metric temporal formula φ over $\mathcal{A}$ and any $\lambda \in [1..\omega)$,

$$\mathrm{MHT}(\varphi, \lambda) = \{\mathbf{M}|_{\mathcal{A}} \mid \mathbf{M} \in \mathrm{MHT}(\{\ell_\varphi\} \cup \eta(\varphi), \lambda)\}.$$

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

Table 4. *Translation of (1) into a metric temporal logic program. Equivalences of the form* $\ell_\varphi \leftrightarrow p$*, with* $p \in \mathcal{A}$*, are not shown and* p *is used instead of* ℓ_p

$\mu \in cl((1))$	$\eta(\mu)$
(F1)	$\widehat{\mathsf{O}}\Box\left(\bullet\ell_{\mathbf{(F1)}} \leftrightarrow \bullet\ell_{\mathbf{(F2)}} \wedge \ell_{\mathbf{(F1)}}\right), \Box\left(\mathbb{F} \rightarrow \left(\ell_{\mathbf{(F1)}} \leftrightarrow \ell_{\mathbf{(F2)}}\right)\right)$
(F2)	$\Box\left(\ell_{\mathbf{(F2)}} \leftrightarrow \left(p \rightarrow \ell_{\mathbf{(F3)}}\right)\right)$
(F3)	$\Box\left(\ell_{\mathbf{(F3)}} \leftrightarrow a \vee \left(m \wedge \left(\ell_{\mathbf{(F7)}} \vee \ell_{\mathbf{(F5)}} \vee \ell_{\mathbf{(F4)}}\right)\right)\right)$
(F4)	$\widehat{\mathsf{O}}\Box\left(\bullet_3\top \rightarrow \left(\bullet\ell_{\mathbf{(F4)}} \leftrightarrow a\right)\right), \widehat{\mathsf{O}}\Box\left(\neg\bullet_3\top \rightarrow \neg\bullet\ell_{\mathbf{(F4)}}\right), \Box\left(\mathbb{F} \rightarrow \neg\ell_{\mathbf{(F4)}}\right)$
(F5)	$\widehat{\mathsf{O}}\Box\left(\bullet_2\top \rightarrow \left(\bullet\ell_{\mathbf{(F5)}} \leftrightarrow \ell_{\mathbf{(F11)}}\right)\right), \widehat{\mathsf{O}}\Box\left(\neg\bullet_2\top \rightarrow \neg\bullet\ell_{\mathbf{(F5)}}\right), \Box\left(\mathbb{F} \rightarrow \neg\ell_{\mathbf{(F5)}}\right)$
(F6)	$\widehat{\mathsf{O}}\Box\left(\bullet_2\top \rightarrow \left(\bullet\ell_{\mathbf{(F6)}} \leftrightarrow a\right)\right), \widehat{\mathsf{O}}\Box\left(\neg\bullet_2\top \rightarrow \neg\bullet\ell_{\mathbf{(F6)}}\right), \Box\left(\mathbb{F} \rightarrow \neg\ell_{\mathbf{(F6)}}\right)$
(F7)	$\widehat{\mathsf{O}}\Box\left(\bullet_1\top \rightarrow \left(\bullet\ell_{\mathbf{(F7)}} \leftrightarrow \ell_{\mathbf{(F10)}}\right)\right), \widehat{\mathsf{O}}\Box\left(\neg\bullet_1\top \rightarrow \neg\bullet\ell_{\mathbf{(F7)}}\right), \Box\left(\mathbb{F} \rightarrow \neg\ell_{\mathbf{(F7)}}\right)$
(F8)	$\widehat{\mathsf{O}}\Box\left(\bullet_1\top \rightarrow \left(\bullet\ell_{\mathbf{(F8)}} \leftrightarrow \ell_{\mathbf{(F11)}}\right)\right), \widehat{\mathsf{O}}\Box\left(\neg\bullet_1\top \rightarrow \neg\bullet\ell_{\mathbf{(F8)}}\right), \Box\left(\mathbb{F} \rightarrow \neg\ell_{\mathbf{(F8)}}\right)$
(F9)	$\widehat{\mathsf{O}}\Box\left(\bullet_1\top \rightarrow \left(\bullet\ell_{\mathbf{(F9)}} \leftrightarrow a\right)\right), \widehat{\mathsf{O}}\Box\left(\neg\bullet_1\top \rightarrow \neg\bullet\ell_{\mathbf{(F9)}}\right), \Box\left(\mathbb{F} \rightarrow \neg\ell_{\mathbf{(F9)}}\right)$
(F10)	$\Box\left(\ell_{\mathbf{(F10)}} \leftrightarrow a \vee \left(m \wedge \left(\ell_{\mathbf{(F8)}} \vee \ell_{\mathbf{(F6)}}\right)\right)\right)$
(F11)	$\Box\left(\ell_{\mathbf{(F11)}} \leftrightarrow a \vee \left(m \wedge \ell_{\mathbf{(F9)}}\right)\right)$

The '$\subseteq$' direction follows from Lemma 2. For the '$\supseteq$' direction, take some MHT model $\mathbf{M}$ of $\eta(\varphi)$ of length $\lambda \in [1..\omega)$. This implies that its associated 3-valuation satisfies $\boldsymbol{m'}(0, \ell_\varphi) = 2$, which is included in $\eta(\varphi)$. Since $\varphi \in cl(\varphi)$, we can apply Lemma 3 to conclude $\boldsymbol{m'}(k, \ell_\varphi) = \boldsymbol{m'}(k, \varphi)$ for all $k \in [0..\lambda)$ but then, in particular, $2 = \boldsymbol{m'}(0, \ell_\varphi) = \boldsymbol{m'}(0, \varphi)$ and so, $\mathbf{M}$ is also a model of φ. Finally, $\mathbf{M}|_{\mathcal{A}}$ is still a model of φ because the latter is a theory over vocabulary $\mathcal{A}$. The theorem below provides a parametric polynomial bound of the size of $\eta^*(\varphi)$.

Theorem 2.
The number of rules in $\eta^*(\varphi)$ has a (parametric) polynomial bound that depends on n and $|\varphi|$, where n is the maximum interval value occurring in a subformula of φ, which is of the form $Q_{\leq n}$, where $Q \in \{\mathbb{U}, \mathbb{R}, \mathbf{S}, \mathbf{T}\}$.

5 Conclusion

In this paper, we have described a Tseitin-like reduction from metric temporal logic formulas into a logic programming format under metric equilibrium logic semantics. Our current translation only considers intervals of the form $[0..n]$ in the binary metric operators *since* (resp. *trigger*) and *until* (resp. *release*). In the future, we expect to extend our translation in order to cope with arbitrary intervals. If that is possible, we would have an input format to which any arbitrary metric temporal theory could be reduced, making it easily implementable in ASP.

Our contribution sheds light on the concept of metric temporal logic programs and its use as a "normal form" for interpreting metric temporal theories under answer set

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

semantics. As future work, we expect to implement this translation in the solver *telingo* (Cabalar *et al*. 2019) and compare it with other solvers proposed by Becker *et al*. (2024a, 2026). In addition, we would like to explore the notion of splittable metric temporal logic programs as done by Aguado *et al*. (2023) for TEL. In the temporal case, those programs possess nice computational methods via the use of splitting and loop formulas. An interesting research line could be to extend those results to the metric case. Finally, we would like to extend our reduction to metric extensions of *dynamic equilibrium logic* (Becker *et al*. 2023)

References

Aguado, F., Cabalar, P., Diéguez, M., Pérez, G., Schaub, T., Schuhmann, A. and Vidal, C. 2023. Linear-time temporal answer set programming. *Theory and Practice of Logic Programming* 23, 1, 2–56. https://doi.org/10.1017/S1471068421000557.

Becker, A., Cabalar, P., Diéguez, M., del Cerro, L. F., Schaub, T. and Schuhmann, A. 2023. Metric dynamic equilibrium logic. *Journal of Applied Non-Classical Logics* 33, 3–4, 495–519. https://doi.org/10.1080/11663081.2023.2244365.

Becker, A., Cabalar, P., Diéguez, M., Hahn, S., Romero, J. and Schaub, T. 2024a. Compiling metric temporal answer set programming. In *Proceedings of the Seventeenth International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR'24), Lecture Notes in Artificial Intelligence*, vol. 15245, 15–29. https://doi.org/10.1007/978-3-031-74209-5_2.

Becker, A., Cabalar, P., Diéguez, M., Schaub, T. and Schuhmann, A. 2024b. Metric temporal answer set programming over timed traces. *Theory and Practice of Logic Programming* 24, 3, 425–452. https://doi.org/10.1017/S1471068424000139.

Becker, A., Cabalar, P., Diéguez, M., Hahn, S., Romero, J. and Schaub, T. 2026. Implementing metric temporal answer set programming. https://arxiv.org/abs/2601.20735.

Bosser, A., Cabalar, P., Diéguez, M. and Schaub, T. 2018. Introducing temporal stable models for linear dynamic logic. In *Proceedings of the Sixteenth International Conference on Principles of Knowledge Representation and Reasoning (KR'18)*, M. Thielscher, F. Toni and F. Wolter, Eds. AAAI Press, Palo Alto, CA, USA, 12–21.

Cabalar, P. 2010. A normal form for linear temporal equilibrium logic. In *Proceedings of the Twelfth European Conference on Logics in Artificial Intelligence (JELIA'10)*, Lecture Notes in Artificial Intelligence, vol. 6341, T. Janhunen and I. Niemelä, Eds. Springer-Verlag, Berlin, Heidelberg, 74–76.

Cabalar, P., Kaminski, R., Morkisch, P. and Schaub, T. 2019. Telingo = ASP + time. In *Proceedings of the Fifteenth International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR'19)*, Lecture Notes in Artificial Intelligence, vol.11481, M. Balduccini, Y. Lierler and S. Woltran, Eds. Springer-Verlag, Cham, Switzerland, 256–269. https://doi.org/10.1007/978-3-030-20528-7_19.

Fischer, M. and Ladner, R. 1979. Propositional dynamic logic of regular programs. *Journal of Computer and System Sciences* 18, 2, 194–211.

Fisher M., Gabbay D. and Vila L.(eds). 2005. *Handbook of Temporal Reasoning in Artificial Intelligence*. Foundations of Artificial Intelligence, Vol. 1. Elsevier Science, Amsterdam, The Netherlands.

Gabbay, D. 1987. The declarative past and imperative future: Executable temporal logic for interactive systems. In *Proceedings of the Conference on Temporal Logic in Specification*,

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

Lecture Notes in Computer Science, vol. 398, B. Banieqbal, H. Barringer and A. Pnueli, Eds. Springer-Verlag, Berlin, Heidelberg, 409–448.

Gebser, M., Kaminski, R., Kaufmann, B., Ostrowski, M., Schaub, T. and Wanko, P. 2016. Theory solving made easy with clingo 5. In *Technical Communications of the Thirty-Second International Conference on Logic Programming (ICLP'16)*, Open Access Series in Informatics (OASIcs), vol. 52, M. Carro, A. King, N. Saeedloei and M. De Vos, Eds. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2:2–2:15.

Gödel, K. 1932. Zum intuitionistischen Aussagenkalkül. *Anzeiger der Akademie der Wissenschaften in Wien* 69, 65–66.

Harel, D., Tiuryn, J. and Kozen, D. 2000. *Dynamic Logic*. MIT Press, Cambridge, MA, USA. https://doi.org/10.1145/568438.568456.

Heyting, A. 1930. Die formalen Regeln der intuitionistischen Logik. In *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, Deutsche Akademie der Wissenschaften zu Berlin, Berlin, Germany, 42–71, 158–169.

Lifschitz, V. 1999. Answer set planning. In *Proceedings of the International Conference on Logic Programming (ICLP'99)*, D. de Schreye, Ed. MIT Press, Cambridge, MA, USA, 23–37.

Pearce, D. 1997. A new logical characterisation of stable models and answer sets. In *Proceedings of the Sixth International Workshop on Non-Monotonic Extensions of Logic Programming (NMELP'96)*, Lecture Notes in Computer Science, vol. 1216, 57–70. https://doi.org/10.1007/BFb0023801.

Pearce, D. 2006. Equilibrium logic. *Annals of Mathematics and Artificial Intelligence* 47, 1–2, 3–41. https://doi.org/10.1007/s10472-006-9028-z.

Pnueli, A. 1977. The temporal logic of programs. In *Proceedings of the Eight-Teenth Symposium on Foundations of Computer Science (FOCS'77)*, 46–57. https://doi.org/10.1109/SFCS.1977.32.

Tseitin, G. 1968. On the complexity of derivation in the propositional calculus. *Zapiski Nauchnykh Seminarov LOMI* 8, 234–259.

Appendix A Translation of Non-Metric Connectives

Table A1. *Translation of the propositional connectives*

μ	$\eta(\mu)$	$\eta^*(\mu)$		
$\varphi \wedge \psi$	$\Box(\ell_\mu \leftrightarrow \ell_\varphi \wedge \ell_\psi)$	$\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\varphi)$ $\ell_\mu \rightarrow \ell_\varphi$	$\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\psi)$ $\ell_\mu \rightarrow \ell_\psi$	$\widehat{\bigcirc}\Box(\ell_\varphi \wedge \ell_\psi \rightarrow \ell_\mu)$ $\ell_\varphi \wedge \ell_\psi \rightarrow \ell_\mu$
$\varphi \vee \psi$	$\Box(\ell_\mu \leftrightarrow \ell_\varphi \vee \ell_\psi)$	$\widehat{\bigcirc}\Box(\ell_\varphi \rightarrow \ell_\mu)$ $\ell_\varphi \rightarrow \ell_\mu$	$\widehat{\bigcirc}\Box(\ell_\psi \rightarrow \ell_\mu)$ $\ell_\psi \rightarrow \ell_\mu$	$\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\varphi \vee \ell_\psi)$ $\ell_\mu \rightarrow \ell_\varphi \vee \ell_\psi$
$\varphi \rightarrow \psi$	$\Box(\ell_\mu \leftrightarrow (\ell_\varphi \rightarrow \ell_\psi))$	$\widehat{\bigcirc}\Box(\ell_\mu \wedge \ell_\varphi \rightarrow \ell_\psi)$ $\ell_\mu \wedge \ell_\varphi \rightarrow \ell_\psi$ $\widehat{\bigcirc}\Box(\ell_\varphi \vee \neg\ell_\psi \vee \ell_\mu)$	$\widehat{\bigcirc}\Box(\neg\ell_\varphi \rightarrow \ell_\mu)$ $\neg\ell_\varphi \rightarrow \ell_\mu$ $\ell_\varphi \vee \neg\ell_\psi \vee \ell_\mu$	$\widehat{\bigcirc}\Box(\ell_\psi \rightarrow \ell_\mu)$ $\ell_\psi \rightarrow \ell_\mu$

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press

Table A2. *Translation of the non-metric temporal operators*

μ	$\eta(\mu)$	$\eta^*(\mu)$
$\bullet\varphi$	$\widehat{\bigcirc}\Box(\ell_\mu \leftrightarrow \bullet\ell_\varphi) \quad \neg\ell_\mu$	$\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \bullet\ell_\varphi) \quad \widehat{\bigcirc}\Box(\bullet\ell_\varphi \rightarrow \ell_\mu) \quad \neg\ell_\mu$
$\widehat{\bullet}\varphi$	$\widehat{\bigcirc}\Box(\ell_\mu \leftrightarrow \bullet\ell_\varphi) \quad \ell_\mu$	$\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \bullet\ell_\varphi) \quad \widehat{\bigcirc}\Box(\bullet\ell_\varphi \rightarrow \ell_\mu) \quad \ell_\mu$
$\bigcirc\varphi$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \leftrightarrow \ell_\varphi) \quad \Box(\mathbb{F} \rightarrow \neg\ell_\mu)$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \rightarrow \ell_\varphi) \quad \widehat{\mathsf{O}}\Box(\ell_\varphi \rightarrow \bullet\ell_\mu) \quad \Box(\mathbb{F} \rightarrow \neg\ell_\mu)$
$\widehat{\bigcirc}\varphi$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \leftrightarrow \ell_\varphi) \quad \Box(\mathbb{F} \rightarrow \ell_\mu)$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \rightarrow \ell_\varphi) \quad \widehat{\mathsf{O}}\Box(\ell_\varphi \rightarrow \bullet\ell_\mu) \quad \Box(\mathbb{F} \rightarrow \ell_\mu)$
$\varphi\,\mathbf{S}\,\psi$	$\widehat{\bigcirc}\Box(\ell_\mu \leftrightarrow \ell_\psi \vee (\ell_\varphi \wedge \bullet\ell_\mu))$ $\ell_\mu \leftrightarrow \ell_\psi$	$\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\psi \vee \ell_\varphi)$ $\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\psi \vee \bullet\ell_\mu)$ $\widehat{\bigcirc}\Box(\ell_\psi \rightarrow \ell_\mu)$ $\widehat{\bigcirc}\Box(\ell_\varphi \wedge \bullet\ell_\mu \rightarrow \ell_\mu)$ $\ell_\mu \rightarrow \ell_\psi$ $\ell_\psi \rightarrow \ell_\mu$
$\varphi\,\mathbf{T}\,\psi$	$\widehat{\bigcirc}\Box(\ell_\mu \leftrightarrow \ell_\psi \wedge (\ell_\varphi \vee \bullet\ell_\mu))$ $\ell_\mu \leftrightarrow \ell_\psi$	$\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\psi)$ $\widehat{\bigcirc}\Box(\ell_\mu \rightarrow \ell_\varphi \vee \bullet\ell_\mu)$ $\widehat{\bigcirc}\Box(\ell_\psi \wedge \ell_\varphi \rightarrow \ell_\mu)$ $\widehat{\bigcirc}\Box(\ell_\psi \wedge \bullet\ell_\mu \rightarrow \ell_\mu)$ $\ell_\mu \rightarrow \ell_\psi$ $\ell_\psi \rightarrow \ell_\mu$
$\varphi\,\mathbb{U}\,\psi$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \leftrightarrow \bullet\ell_\psi \vee (\bullet\ell_\varphi \wedge \ell_\mu))$ $\Box(\mathbb{F} \rightarrow (\ell_\mu \leftrightarrow \ell_\psi))$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \rightarrow \bullet\ell_\psi \vee \bullet\ell_\varphi)$ $\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \rightarrow \bullet\ell_\psi \vee \ell_\mu)$ $\widehat{\mathsf{O}}\Box(\bullet\ell_\varphi \wedge \ell_\mu \rightarrow \bullet\ell_\mu)$ $\widehat{\mathsf{O}}\Box(\bullet\ell_\psi \rightarrow \bullet\ell_\mu)$ $\Box(\mathbb{F} \rightarrow (\ell_\mu \rightarrow \ell_\psi))$ $\Box(\mathbb{F} \rightarrow (\ell_\psi \rightarrow \ell_\mu))$
$\varphi\,\mathbb{R}\,\psi$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \leftrightarrow \bullet\ell_\psi \wedge (\bullet\ell_\varphi \vee \ell_\mu))$ $\Box(\mathbb{F} \rightarrow (\ell_\mu \leftrightarrow \ell_\psi))$	$\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \rightarrow \bullet\ell_\psi)$ $\widehat{\mathsf{O}}\Box(\bullet\ell_\mu \rightarrow \bullet\ell_\varphi \vee \ell_\mu)$ $\widehat{\mathsf{O}}\Box(\bullet\ell_\psi \wedge \ell_\mu \rightarrow \bullet\ell_\mu)$ $\widehat{\mathsf{O}}\Box(\bullet\ell_\psi \wedge \bullet\ell_\varphi \rightarrow \bullet\ell_\mu)$ $\Box(\mathbb{F} \rightarrow (\ell_\mu \rightarrow \ell_\psi))$ $\Box(\mathbb{F} \rightarrow (\ell_\psi \rightarrow \ell_\mu))$

https://doi.org/10.1017/S1471068426100581 Published online by Cambridge University Press