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On Waring's problem for cubes

BY JÖRG BRÜDERN

Mathematisches Institut, Bunsenstrasse 3–5, *D*-3400 *Göttingen, Germany*

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1. *Introduction*

A classical conjecture in the additive theory of numbers is that all sufficiently large natural numbers may be written as the sum of four positive cubes of integers. This is known as the Four Cubes Problem, and since the pioneering work of Hardy and Littlewood one expects a much more precise quantitative form of the conjecture to hold. Let $\nu(n)$ be the number of representations of n in the proposed manner. Then the expected formula takes the shape

$$\nu(n) \sim \Gamma(\tfrac{4}{3})^3 \mathfrak{S}(n)\, n^{\frac{1}{3}} \tag{1·1}$$

where $\mathfrak{S}(n)$ is the singular series associated with four cubes as familiar in the Hardy–Littlewood theory.

The first noteworthy contribution to the subject we owe to Davenport [**4**]. Let $E(N)$ be the number of all natural numbers not exceeding n and not being the sum of four cubes. Then Davenport proved that

$$E(N) \ll N^{\frac{29}{30}+\epsilon} \tag{1·2}$$

so that in particular almost all integers admit a representation as sums of four cubes. Since then there have been improvements on the exponent in (1·2). The best currently known result is due to the author [**1**] who obtained (1·2) with $\frac{131}{147}$ in place of $\frac{29}{30}$, thereby improving slightly on important work of Vaughan [**14**]. In the present memoir we shall show

THEOREM 1. *Let $E(N)$ be the number of all natural numbers not exceeding N and not being the sum of four cubes. Then $E(N) \ll N^{\frac{37}{42}+\epsilon}$.*

The asymptotic formula (1·1) is of course out of reach of existing methods, but it has been verified by Vaughan [**14**] for almost all n. Subject to the truth of the Riemann hypothesis for the Hasse–Weil L-functions of certain cubic three-folds Hooley [**9**] established a stronger result of this type, namely that the number of n not exceeding N for which (1·1) fails to hold is $O(N^{\frac{101}{114}+\epsilon})$. If in (1·1) emphasis is not on the asymptotic formula but only on the lower bound derived therefrom then our

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method can be adapted to this problem and produces an unconditional result which should be compared with the aforementioned bound due to Hooley.

THEOREM 2. *Let $E_A^*(N)$ be the number of all n not exceeding N for which $\nu(n) > n^{\frac{1}{3}}(\log n)^{-A}$ fails to hold. Then for any $\epsilon > 0$ there is a positive number A such that $E_A^*(N) \ll N^{\frac{11}{12}+\epsilon}$.*

It is known that

$$(\log\log n)^{-C} \ll \mathfrak{S}(n) \ll (\log\log n)^{C}$$

for some constant $C > 0$. Hence for any n not counted by $E_A^*(N)$ we see that $\nu(n)$ is close to the expected order of magnitude.

In general our approach has much in common with earlier work in this field. We use Bessel's inequality to bound $E(N)$ and $E_A^*(N)$ in terms of certain Fourier integrals which may then be estimated by the Hardy–Littlewood method. We then follow Vaughan [**14**] and apply his p-adic iteration method as refined in [**1**]. In both [**14**] and [**1**] Theorem A of Vaughan [**14**] plays a crucial role, but from recent fundamental work of Vaughan [**15**] a more efficient mean value theorem is available. Let

$$\mathscr{A}(P,R) = \{n : n \leqslant P ; p \mid n \Rightarrow p \leqslant R\}. \tag{1·3}$$

Then Vaughan's result reads

$$\int_0^1 \Big| \sum_{x \in \mathscr{A}(P,\, P^{\eta})} e^{2\pi i \alpha x^3} \Big|^6 d\alpha \ll P^{\frac{13}{4}+\epsilon}$$

provided η is sufficiently small in terms of ϵ. Such mean values can be brought in easily on the minor arcs in our application of the circle method, but on the major arcs one is faced with the problem of approximating exponential sums like

$$\sum_{x \in \mathscr{A}(P,\, P^{\eta})} e^{2\pi i \alpha x^3} \tag{1·4}$$

when α is close to a rational point a/q, say. Vaughan [**15**] got around this problem by using a straightforward argument which is available when q is bounded by a small power of $\log P$, and combined this with a kind of pruning technique to show that the contribution from larger values of q is negligible. This technique is of no use in the present context; it will be clear from arguments to be given later that one has to find a good approximation to (1·4) when q runs up to a not too small power of P. In view of our relatively poor knowledge of the distribution of the sequence $\mathscr{A}(P,R)$ in arithmetic progressions this appears to be a difficult problem if a neighbourhood of a single rational point a/q is considered. We shall therefore consider mean values over both a and q, and are able to express such means in terms of character sums in order to prepare for an application of the large sieve. The approach is not dissimilar to the usual proof of the Barban–Davenport–Halberstam theorem as generalized to sequences other than primes by Hooley [**8**]. Various complications arise from large common factors of $x \in \mathscr{A}$ and q, and to control this more precisely we shall not work with $\mathscr{A}(P,R)$ but with a sufficiently dense subsequence. This enables us to take (in the present set-up) the upper bound for the denominators on the major arcs as large as $P^{\frac{2}{3}}$ which is just enough to press the method home. We postpone further comments on details of the new technique to later sections but conclude these introductory

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remarks with expressing our hope that there will be further applications to additive number theory.

The plan of the paper is as follows. The method of attack to Theorem 1 is explained in Section 2. In Section 3 we establish the comparatively easy minor arc estimate. The major arcs are treated in Sections 4–8. This forms the bulk of the paper, and more information on the content of the individual sections is to be found at the beginnings of these parts. In a final section we shall briefly indicate the modifications required to establish Theorem 2.

Notation. Small italics and δ usually denote positive integers. A prime number will be denoted by π whereas we reserve p, with or without suffices, for primes $\equiv 2 \pmod 3$ throughout. Capitals as N, P, Q, X, Y, Z, Ξ are real parameters tending to infinity in most cases. Real variables are denoted by α, β, γ, ξ, or ϑ. Throughout $\epsilon > 0$ is fixed but arbitrarily small, and η satisfying $0 < \eta < \epsilon^2$ will be fixed in Section 2. The numbers A, B, C are positive real constants not necessarily the same at each occurrence. Gothic characters denote sets of real numbers, and script letters denote sets of integers. For a sequence $\mathscr{A}$ of integers and a natural number d we use $d^{-1}\mathscr{A}$ to denote the set of all integers x with $dx \in \mathscr{A}$.

As usual, $\phi(n)$ is Euler's function, $\mu(n)$ is the Möbius function, and $[\alpha]$ is the largest integer not exceeding α. Also $\omega(n)$ is the number of different prime factors of n. Throughout χ is a Dirichlet character modulo r, with χ_0 denoting the principal character; and χ^* is a primitive character modulo r^*. Finally we shall use the abbreviation $e(\alpha) = \exp(2\pi i\alpha)$.

2. *Bringing in the circle method*

We would like to count the solutions of the equation

$$n = x^3 + p^3(y^3 + z_1^3 + z_2^3) \tag{2·1}$$

with $p \nmid x$, and z_i lying in some sequence $\mathscr{A}$. However the precise definition of this sequence depends on parameters stemming from an auxiliary estimate which we need later. Therefore we begin with stating this result.

LEMMA 1 (Vaughan [16]). *Let S denote the number of solutions of*

$$x_1^3 + p_1^3 y_1^3 + p_2^3 y_2^3 = x_2^3 + p_3^3 y_3^3 + p_4^3 y_4^3$$

subject to

$$x_i \leqslant Q; \quad Q^{\frac{1}{8}} < p_i < 2Q^{\frac{1}{8}}; \quad y_i \in \mathscr{A}(Q^{\frac{7}{8}}, Q^{\eta})$$

where $\mathscr{A}(P, R)$ is the sequence introduced in (1·3). *Then, for all $\epsilon > 0$ there exists $\eta_0 > 0$ such that for $0 < \eta < \eta_0$ one has*

$$S \ll Q^{\frac{13}{4}+\epsilon}.$$

Note that we have modified Vaughan's lemma slightly, but that the proof applies equally well.

Given $\epsilon > 0$ we choose η_0 in accordance with Lemma 1, and fix an η which satisfies $0 < \eta < \min(\epsilon^2, \eta_0)$ and is of the shape $\eta = 7/(8s)$ where s is an integer. Now introduce the sequence

$$\mathscr{A} = \{p_0 p_1 \dots p_s : Q^{\frac{1}{8}} < p_0 < 2Q^{\frac{1}{8}}; 2^{-i-1}Q^{\eta} < p_i < 2^{-i}Q^{\eta} \ (1 \leqslant i \leqslant s)\}.$$

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Note that any $x \in \mathscr{A}$ is of the form $x = p_0 y$ with $y \in \mathscr{A}(Q^{\frac{7}{8}}, Q^{\eta})$. Therefore the number of solutions of

$$x_1^3 + z_1^3 + z_2^3 = x_2^3 + z_3^3 + z_4^3 \tag{2·2}$$

subject to $x_i \leqslant Q$ and $z_i \in \mathscr{A}$ is $O(Q^{\frac{13}{4}+\epsilon})$, by Lemma 1.

We now fix parameters in much the same way as in [1]. Let

$$P = [(\tfrac{1}{3}N)^{\frac{1}{3}}]; \quad Q = P^{\frac{6}{7}}; \quad Y = P^{\frac{1}{7}}. \tag{2·3}$$

The exponential sums at the centre of the argument are

$$f_p(\alpha) = \sum_{\substack{P < x \leqslant 2P \\ p \nmid x}} e(\alpha x^3); \quad g(\alpha) = \sum_{Q < x \leqslant 2Q} e(\alpha x^3); \quad h(\alpha) = \sum_{x \in \mathscr{A}} e(\alpha x^3). \tag{2·4}$$

Let $\mathfrak{B}$ be a measurable set of finite measure. We shall consider

$$\nu(n, \mathfrak{B}) = \nu_N(n, \mathfrak{B}) = \sum_{Y < p \leqslant 2Y} \int_{\mathfrak{B}} f_p(\alpha) g(p^3\alpha) h(p^3\alpha)^2 e(-\alpha n)\, d\alpha. \tag{2·5}$$

Let $\mathfrak{U}$ denote the unit interval $[XP^{-3}, 1 + XP^{-3}]$. Then $\nu(n, \mathfrak{U})$ equals the number of solutions to (2·1) subject to $p \nmid x$ and

$$P < x \leqslant 2P; \quad Y < p \leqslant 2Y; \quad Q < y \leqslant 2Q; \quad z_i \in \mathscr{A}.$$

Therefore, in order to prove Theorem 1 it suffices to show that

$$\sum_{\substack{N < n \leqslant 2N \\ \nu(n, \mathfrak{U}) = 0}} 1 \ll N^{\frac{37}{42}+\epsilon}. \tag{2·6}$$

We split $\mathfrak{U}$ into major and minor arcs as follows. When $1 \leqslant a \leqslant q \leqslant P$, $(a, q) = 1$, let $\mathfrak{M}(q, a)$ denote the interval $|q\alpha - a| \leqslant P^{-2}$. Let $\mathfrak{M}$ denote the union of all $\mathfrak{M}(q, a)$ and write $\mathfrak{m} = \mathfrak{U} \backslash \mathfrak{M}$. The principal estimates required are:

PROPOSITION 1. *One has*

$$\sum_{N < n \leqslant 2N} |\nu(n, \mathfrak{m})|^2 \ll (YPQ^3)^2 N^{-1-\frac{5}{42}+\epsilon}.$$

Of course this implies that for all but $O(N^{\frac{37}{42}+2\epsilon})$ values of n in $N < n \leqslant 2N$ one has $\nu(n, \mathfrak{m}) \ll YPQ^3 N^{-1-\eta}$. In the complementary direction we have

PROPOSITION 2. *For all but $O(N^{\frac{37}{42}+\epsilon})$ values of n in $N < n \leqslant 2N$,*

$$\nu(n, \mathfrak{M}) \gg YPQ^3 N^{-1} (\log N)^{-2s-4}.$$

This gives (2·6). We shall prove Proposition 1 in the next section, and then come to the proof of Proposition 2.

3. *A minor arc estimate*

For brevity we shall write (in this section only)

$$S(\alpha) = |g(\alpha) h(\alpha)^2|^2. \tag{3·1}$$

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Then, from (2·5), Bessel's inequality and Cauchy's inequality,

$$\sum_{N<n\leqslant 2N} |\nu(n,\mathfrak{m})|^2 \leqslant \int_{\mathfrak{m}} \Big| \sum_{Y<p\leqslant 2Y} f_p(\alpha)\, g(p^3\alpha)\, h(p^3\alpha)^2 \Big|^2 d\alpha$$

$$\leqslant Y \int_{\mathfrak{m}} \sum_{Y<p\leqslant 2Y} |f_p(\alpha)|^2 S(p^3\alpha)\, d\alpha = YW, \quad \text{say.} \tag{3·2}$$

In estimating W we shall follow Vaughan [**14**], Sections 5–8, quite closely. This technique is somewhat complicated in nature but it has been used recently in a number of contexts (Brüdern [**1**, **2**, **3**]). In view of this, and since there are only minor modifications required compared with Vaughan [**14**] and Brüdern [**1**], large parts are only given in outline.

Let

$$\Phi_p(\alpha) = \sum_{\substack{P<y\leqslant 2P \\ p\nmid y}} 1 + 2\,\mathrm{Re} \sum_{h\leqslant H} \sum_{\substack{2P+hp^3<y\leqslant 4P-hp^3 \\ p\nmid y;\, y\equiv h \,(\mathrm{mod}\, 2)}} e(\tfrac{3}{4}\alpha h y^2 + \tfrac{1}{4}\alpha h^3 p^6)$$

where $H = CPY^{-3} = CP^{\frac{4}{7}}$ and C is sufficiently large. This coincides with Vaughan [**14**], (5·7) save that as in Brüdern [**1**] our value of Y is somewhat smaller than Vaughan's.

Put $M = \frac{1}{16}P^{\frac{4}{7}}$. Let $\mathfrak{n}$ denote the set of all real numbers $\alpha \in (0, 1]$ such that whenever $|q\alpha - a| \leqslant MQ^{-3}$ with $(a, q) = 1$ we have $q > M$. Then the proof of lemma 10 of Vaughan [**14**] gives

$$W \ll \int_{\mathfrak{n}} \sum_{Y<p\leqslant 2Y} \Phi_p(\alpha)\, S(\alpha)\, d\alpha \tag{3·3}$$

(recall that in our notation $p \equiv 2 \pmod 3$).

It is now necessary to imitate Sections 5–7 of Vaughan [**14**]. We remark that

$$\int_0^1 S(\alpha)\, d\alpha \ll Q^{\frac{13}{4}+\epsilon}. \tag{3·4}$$

This is our analogue of [**14**], (5·16) and (5·17). For a proof, note that by (3·1) the left-hand side of (3·4) equals the number of solutions of (2·2). This gives (3·4).

In Sections 5–7 of [**14**] Vaughan applies a complicated method of splitting up $\Phi_p(\alpha)$ into various exponential sums which are then estimated via lemmas 7 and 8 of [**14**] (essentially). After all, this method shows that

$$\int_{\mathfrak{n}} \sum_{Y<p\leqslant 2Y} \Phi_p(\alpha)\, S(\alpha)\, d\alpha \ll (YP + YH^{\frac{7}{8}}P^{\frac{1}{2}+\epsilon}) \int_0^1 S(\alpha)\, d\alpha + P^{\epsilon}V \tag{3·5}$$

where

$$V = YH^{\frac{7}{8}}P \int_{\mathfrak{L}} E(\alpha)\, S(\alpha)\, d\alpha, \tag{3·6}$$

$$\mathfrak{L} = \mathfrak{N}(H^{\frac{7}{4}}) \backslash \mathfrak{N}(P^{\frac{4}{7}}), \tag{3·7}$$

$\mathfrak{N}(\Xi)$ is the union of all intervals

$$\mathfrak{N}(q, a; \Xi) = \{\alpha : |q\alpha - a| \leqslant \Xi Q^{-3}\},$$

and $E(\alpha)$ is the function defined on $\mathfrak{N}(H^{\frac{7}{4}})$ by

$$E(\alpha) = q^{-\frac{1}{2}}\Big(1 + Q^3\Big|\alpha - \frac{a}{q}\Big|\Big)^{-\frac{1}{2}} \quad \text{when} \quad \alpha \in \mathfrak{N}(q, a; H^{\frac{7}{4}}).$$

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Two comments are required here. First of all, the first term on the right of (3·5) arises naturally from Vaughan's treatment providing one may take $Y = P^{\frac{1}{7}}$ in place of Vaughan's slightly larger choice for Y. That this is true has been explained in [1] already. Secondly, we remark that (3·6) is the analogue of [**14**], (6·23) or [1], (7). In both [**14**] and [1] another integral needs to be considered, namely [**14**], (6·27) or [1], (8). In the present context such an integral does not arise. In fact [1], (8) takes care of those α which lie in $\mathfrak{N}(H^{\frac{3}{4}}) \cap \mathfrak{n}$ (in the present notation). This set, however, is empty since $P^{\frac{4}{7}} > H^{\frac{3}{4}}$.

The next step is to establish the bound

$$\int_{\mathfrak{L}} E(\alpha) S(\alpha)\, d\alpha \ll Q^{3+\epsilon} P^{-\frac{2}{7}}. \tag{3·8}$$

Taking this for granted we deduce from (3·2), (3·3), (3·4), (3·5), (3·6), (3·8) and (2·3) that

$$\begin{aligned} W &\ll Y(P + H^{\frac{7}{8}} P^{\frac{1}{2}+\epsilon}) Q^{\frac{13}{4}+\epsilon} + YH^{\frac{7}{8}} P^{1+\epsilon} Q^{3+\epsilon} P^{-\frac{2}{7}} \\ &\ll YP^2 Q^6 N^{-1-\frac{5}{42}+\epsilon}. \end{aligned}$$

Proposition 1 now follows from (3·2).

To prepare for our treatment of the integral in (3·8) introduce the exponential sum

$$S(q, a) = \sum_{x=1}^{q} e\left(\frac{ax^3}{q}\right) \tag{3·9}$$

and let

$$w(\beta) = \int_{Q}^{2Q} e(\beta\gamma^3)\, d\gamma. \tag{3·10}$$

Then define

$$g^*(\alpha) = g^*(\alpha; q, a) = q^{-1} S(q, a)\, w\left(\alpha - \frac{a}{q}\right). \tag{3·11}$$

By theorem 2 of Vaughan [**13**],

$$g(\alpha) - g^*(\alpha) \ll q^{\frac{1}{2}+\epsilon}\left(1 + Q^3\left|\alpha - \frac{a}{q}\right|\right)^{\frac{1}{2}}. \tag{3·12}$$

Let $\kappa(q)$ and $\kappa^*(q)$ be the multiplicative functions defined by

$$\kappa(\pi^{3k}) = \pi^{-k}, \quad \kappa(\pi^{3k+1}) = 2\pi^{-k-\frac{1}{2}}, \quad \kappa(\pi^{k+2}) = \pi^{-k-1}$$

$$\kappa^*(\pi) = 2\pi^{-\frac{1}{2}}, \quad \kappa^*(\pi^2) = \pi^{-\frac{3}{4}}, \quad \kappa^*(\pi^l) = \pi^{-l/3} \quad (l \geqslant 3).$$

Then $\kappa(q) \leqslant \kappa^*(q)$, and by lemmata 4·3, 4·4 and theorem 4·2 of Vaughan [**12**],

$$q^{-1} S(q, a) \ll \kappa(q) \quad \text{when} \quad (a, q) = 1. \tag{3·13}$$

A partial integration applied to (3·10) now shows that

$$g^*(\alpha) \ll \kappa(q)\, Q\left(1 + Q^3\left|\alpha - \frac{a}{q}\right|\right)^{-1} \tag{3·14}$$

providing $(a, q) = 1$.

The functions κ and κ^* play a crucial role in our subsequent arguments. Although $\kappa^*(q)$ is a less precise bound for the left-hand side of (3·13) it is sometimes the more useful one since for any two numbers u, v, whether coprime or not, one has

$$\kappa^*(uv) \leqslant u^{-\frac{1}{4}} \kappa^*(v). \tag{3·15}$$

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This is not true for κ but will be used in Section 7. Another useful result which is easy to check is

$$\kappa(u^3v) \ll u^{-1}\kappa(v). \tag{3·16}$$

In mean κ and κ^* do not differ very much. Indeed one has

$$\sum_{q\leqslant\Xi} \kappa^*(q)^2 \leqslant \prod_{\nu\leqslant\Xi}\left(1+4\pi^{-1}+\pi^{-\frac{3}{2}}+\sum_{l=3}^{\infty}\pi^{-2l/3}\right) \ll (\log\Xi)^5. \tag{3·17}$$

Similarly,

$$\sum_{q\leqslant\Xi} q\kappa^*(q)^4 \ll (\log\Xi)^{18}. \tag{3·18}$$

Corresponding bounds for κ are no better than this.

We now prove (3·8). Recall the definition of $E(\alpha)$. By (3·14),

$$|g(\alpha)|^2 \ll |g^*(\alpha)|^2 + P^{\epsilon}E(\alpha)^{-2}$$

when $\alpha\in\mathfrak{N}(H^{\frac{7}{4}})$. Also note that for these α one has $E(\alpha)^{-1} \ll P^{\frac{1}{2}}$ (since $H^{\frac{7}{4}} \ll P$). From (3·1) and (3·8) we now infer that

$$\int_{\mathfrak{L}} E(\alpha)\,S(\alpha)\,d\alpha \ll \int_{\mathfrak{L}} E(\alpha)\,|g^*(\alpha)|^2|h(\alpha)|^4\,d\alpha + P^{\frac{1}{2}+\epsilon}\int_0^1 |h(\alpha)|^4\,d\alpha. \tag{3·19}$$

From (3·14) and (3·18) it is readily seen that

$$\int_{\mathfrak{N}(H^{\frac{7}{4}})} |g^*(\alpha)|^4\,d\alpha \ll Q^{1+\epsilon}. \tag{3·20}$$

We also have

$$\int_0^1 |h(\alpha)|^4\,d\alpha \ll Q^{2+\epsilon}.$$

To see this, note that the integral here is bounded by the number of solutions to $y_1^3+y_2^3 = y_3^3+y_4^3$ subject to $y_i \leqslant Q$. By Hua's Lemma ([12], lemma 2·5) this number in turn is bounded by $O(Q^{2+\epsilon})$. The same argument leads to the bound

$$\int_0^1 |h(\alpha)|^8\,d\alpha \ll Q^{5+\epsilon}. \tag{3·21}$$

From (3·7) we have

$$\sup_{\alpha\in\mathfrak{L}} E(\alpha) \ll P^{-\frac{2}{7}}.$$

The last four estimates combined with Schwarz's inequality show that the right-hand side of (3·19) is

$$\ll P^{-\frac{2}{7}}(Q^{1+\epsilon})^{\frac{1}{2}}(Q^{5+\epsilon})^{\frac{1}{2}} + P^{\frac{1}{2}+\epsilon}Q^{2+\epsilon} \ll Q^{3+\epsilon}P^{-\frac{2}{7}}.$$

This gives (3·8).

4. *First steps on the major arcs*

In this section we show that $\nu(n,\mathfrak{M})$ is well approximated by integrals similar to (2·5) but with $f_p(\alpha)$ and $g(p^3\alpha)$ replaced by suitable approximations. Use (3·9) to define

$$S(q,a,p) = S(q,a) - \frac{1}{p}S(q,ap^3). \tag{4·1}$$

Also write

$$v(\beta) = \int_P^{2P} e(\beta\gamma^3)\,d\gamma \tag{4·2}$$

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and put
$$f_p^*(\alpha) = f_p^*(\alpha; q, a) = q^{-1}S(q, a, p)\, v\Big(\alpha - \frac{a}{q}\Big). \tag{4·3}$$

In the notation introduced in (3·11) we also write
$$g_p^*(\alpha) = g^*(p^3\alpha; q, ap^3).$$

For $\alpha \in \mathfrak{M}(q, a)$ we write
$$D(\alpha) = q^{\frac{1}{2}}\Big(1 + P^3\Big|\alpha - \frac{a}{q}\Big|\Big)^{\frac{1}{2}}. \tag{4·4}$$

Then, for $Y < p \leqslant 2Y$ we have
$$f_p(\alpha) - f_p^*(\alpha) \ll q^{\epsilon}D(\alpha),$$
$$g(p^3\alpha) - g_p^*(\alpha) \ll q^{\epsilon}D(\alpha).$$

The first bound here follows from theorem 2 of Vaughan [3], and the second is a consequence of (3·12). It is obvious from (4·1) that for $(a, q) = 1$ one has
$$q^{-1}S(q, a, p) \ll \kappa(q).$$

As in (3·14) we deduce that
$$f_p^*(\alpha) \ll \kappa(q) P\Big(1 + P^3\Big|\alpha - \frac{a}{q}\Big|\Big)^{-1}. \tag{4·5}$$

We now introduce the numbers
$$\nu_1(n) = \int_{\mathfrak{M}} \sum_{Y < p \leqslant 2Y} f_p^*(\alpha)\, g(p^3\alpha)\, h(p^3\alpha)^2 e(-\alpha n)\, d\alpha,$$
$$\nu_2(n) = \int_{\mathfrak{M}} \sum_{Y < p \leqslant 2Y} f_p^*(\alpha)\, g_p^*(\alpha)\, h(p^3\alpha)^2 e(-\alpha n)\, d\alpha,$$

and proceed to show that $\nu_1(n)$ is a good approximation to $\nu(n, \mathfrak{M})$, at least almost always, and that similarly $\nu_2(n)$ approximates $\nu_1(n)$ almost always.

Observe that $\nu(n, \mathfrak{M}) - \nu_1(n)$ are the Fourier coefficients of the function with period one which is
$$\sum_{Y < p \leqslant 2Y} (f_p(\alpha) - f_p^*(\alpha))\, g(p^3\alpha)\, h(p^3\alpha)^2$$
when $\alpha \in \mathfrak{M}$, and zero elsewhere. Hence, by Bessel's inequality,
$$\sum_{N < n \leqslant 2N} |\nu(n, \mathfrak{M}) - \nu_1(n)|^2 \leqslant \int_{\mathfrak{M}} \Big| \sum_{Y < p \leqslant 2Y} (f_p(\alpha) - f_p^*(\alpha))\, g(p^3\alpha)\, h(p^3\alpha)^2 \Big|^2 d\alpha.$$

We use Cauchy's inequality and note that $D(\alpha) \ll P^{\frac{1}{2}}$ for $\alpha \in \mathfrak{M}$ to see that the above is
$$\ll YP^{1+\epsilon} \sum_{Y < p \leqslant 2Y} \int_0^1 |g(p^3\alpha)\, h(p^3\alpha)^2|^2\, d\alpha$$
$$\ll Y^2P^{1+\epsilon} \int_0^1 |g(\alpha)\, h(\alpha)^2|^2\, d\alpha$$
$$\ll Y^2P^{1+\epsilon}Q^{\frac{13}{4}+\epsilon}.$$

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Here a change of variable, the fact that $g(\alpha)\,h(\alpha)^2$ has period one, and (3·4) have been used. Recalling (2·3) we rewrite this as

$$\sum_{N<n\leqslant 2N} |\nu(n,\mathfrak{M})-\nu_1(n)|^2 \ll Y^2P^{2+\epsilon}Q^6N^{-1-\frac{5}{42}}. \tag{4·6}$$

We apply a similar reasoning via Bessel's, Cauchy's, and Schwarz's inequality to deduce that

$$\begin{aligned}\sum_{N<n\leqslant 2N} |\nu_1(n)-\nu_2(n)|^2 &\leqslant \int_{\mathfrak{M}} \Big|\sum_{Y<p\leqslant 2Y} f_p^*(\alpha)\,(g(p^3\alpha)-g_p^*(\alpha))\,h(p^3\alpha)^2\Big|^2 d\alpha\\ &\ll YP^{\epsilon}\int_{\mathfrak{M}}\sum_{Y<p\leqslant 2Y}|f_p^*(\alpha)\,D(\alpha)\,h(p^3\alpha)^2|^2\,d\alpha\\ &\ll YP^{\epsilon}(I_1I_2)^{\frac{1}{2}}\end{aligned}$$

where

$$I_1=\int_{\mathfrak{M}}\sum_{Y<p\leqslant 2Y}|f_p(\alpha)\,D(\alpha)|^4\,d\alpha,$$

$$I_2=\int_0^1\sum_{Y<p\leqslant 2Y}|h(p^3\alpha)|^8\,d\alpha.$$

Changing variables and using (3·21) one gets $I_2\ll YQ^{5+\epsilon}$. Also, by (4·4), (4·5) and (3·18), one has $I_1\ll YP^{3+\epsilon}$. This shows

$$\sum_{N<n\leqslant 2N}|\nu_1(n)-\nu_2(n)|^2\ll Y^2P^{\frac{3}{2}+\epsilon}Q^{\frac{5}{2}+\epsilon}\ll Y^2P^2Q^6N^{-1-\frac{5}{42}}. \tag{4·7}$$

The next step will be a kind of pruning argument to remove major arcs with large denominator. Let $\mathfrak{M}_1$ denote the set of all major arcs $\mathfrak{M}(q,a)$ subject to $1\leqslant a\leqslant q\leqslant X,(a,q)=1$ where

$$X=P^{\frac{4}{7}-\eta}. \tag{4·8}$$

Put

$$\nu^*(n)=\int_{\mathfrak{M}_1}\sum_{Y<p\leqslant 2Y} f_p^*(\alpha)\,g_p^*(\alpha)\,h(p^3\alpha)^2\,e(-\alpha n)\,d\alpha. \tag{4·9}$$

We postpone a proof of the estimate

$$\int_{\mathfrak{M}\setminus\mathfrak{M}_1}\Big|\sum_{Y<p\leqslant 2Y} f_p^*(\alpha)\,g_p^*(\alpha)\,h(p^3\alpha)^2\Big|^2 d\alpha\ll Y^2P^2Q^6N^{-1-\frac{5}{42}+2\eta} \tag{4·10}$$

to Section 6. Taking this for granted one has of course that

$$\sum_{N<n\leqslant 2N}|\nu_2(n)-\nu^*(n)|^2\ll Y^2P^2Q^6N^{-1-\frac{5}{42}+2\eta}$$

by a now familiar argument. Combining this with (4·6) and (4·7) we see that

$$\sum_{N<n\leqslant 2N}|\nu(n,\mathfrak{M})-\nu^*(n)|^2\ll Y^2P^2Q^6N^{-1-\frac{5}{42}+2\eta}.$$

This may be reformulated as follows.

LEMMA 2. *For all but $O(N^{\frac{37}{42}+2\epsilon})$ values of n in the range $N<n\leqslant 2N$ one has*

$$\nu(n,\mathfrak{M})-\nu^*(n)\ll YPQ^3N^{-1-\epsilon}.$$

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It remains to show that $\nu^*(n)$ is large, for all n. This will be the theme of Sections 5–8.

5. *Approximating* $h(\alpha)$

We now come to the more original part of this memoir. A natural way to proceed would be to replace $h(\alpha)$ by an approximating function, very much like we did with $f_p(\alpha)$ and $g(p^3\alpha)$ in the last section. However a necessary prerequisite would then be to have good knowledge of the distribution of the somewhat complicated set $\mathscr{A}$ in arithmetic progressions. Only weakish results are available in this direction which are of no use when dealing with major arcs as large as our $\mathfrak{M}_1$, but very strong information is available on average, in view of the generalized Barban–Davenport–Halberstam theorem due to Hooley[8]. This section is devoted to preparing the ground to bring in the latter.

We suppose that α is close to a rational point a/q and write $\alpha = a/q+\beta$. Now the sum defining $h(\alpha)$ is split into parts according to common factors like

$$h(\alpha) = \sum_{x\in\mathscr{A}} e(\alpha x^3) = \sum_{\substack{x\in\mathscr{A}\\(x,\,q)=1}} e(\alpha x^3) + \sum_{d>2^{-s-1}Q^\eta} \sum_{\substack{x\in\mathscr{A}\\(x,\,q)=d}} e(\alpha x^3),$$

by basic properties of $\mathscr{A}$. One might expect that terms with $d>1$ make a negligible contribution to $\nu^*(n)$ but this is difficult to show if q has certain cubic divisors. Many of the complications arising will reflect this particular problem.

It is convenient to introduce the sequence

$$\mathscr{D} = \{d\leqslant Q : \pi\,|\,d \text{ implies } \pi\equiv 2\,(\mathrm{mod}\,3),\ \pi^2\nmid d \text{ and } 2^{-s-1}Q^\eta<\pi\leqslant \tfrac{1}{2}Q^\eta \text{ or } Q^{\frac{1}{3}}<\pi\leqslant 2Q^{\frac{1}{3}}\}. \quad (5{\cdot}1)$$

It is clear that whenever $x\in\mathscr{A}$ and $d\,|\,x$ then $d\in\mathscr{D}$. Our basic formula for $h(\alpha)$ can now be written as

$$h\left(\frac{a}{q}+\beta\right) = \sum_{\substack{d\,|\,q\\ d\in\mathscr{D}}} T_d(q,a;\beta) \quad (5{\cdot}2)$$

where

$$T_d = \sum_{\substack{x\in\mathscr{A}\\(x,\,q)=d}} e\left(\left(\frac{a}{q}+\beta\right)x^3\right) = \sum_{\substack{y\in d^{-1}\mathscr{A}\\(y,\,q/d)=1}} e\left(\left(\frac{a}{q}+\beta\right)d^3y^3\right). \quad (5{\cdot}3)$$

For any such sum there is a corresponding main term,

$$T_d^* = T_d^*(q,a;\beta) = \phi\left(\frac{q}{d}\right)^{-1} S^*\left(\frac{q}{d},ad^2\right) t_d(\beta) \quad (5{\cdot}4)$$

where

$$S^*(r,b) = \sum_{\substack{y=1\\(y,\,r)=1}}^{r} e\left(\frac{by^3}{r}\right); \quad (5{\cdot}5)$$

$$t_d(\beta) = \tfrac{1}{3}\sum_{C_d(Q/d)^3<y\leqslant(Q/d)^3} y^{-\frac{2}{3}}(\log y)^{\omega(d)-s-1}\,e(\beta y d^3) \quad (5{\cdot}6)$$

with some constant C_d. We put

$$\Delta_d = \Delta_d(q,a;\beta) = T_d - T_d^*,\quad \Delta(q,a,\beta) = \sum_{d\,|\,q}\Delta_d,\quad h^*(\alpha) = \sum_{d\,|\,q} T_d^*(q,a;\beta). \quad (5{\cdot}7)$$

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Then, by (5·2), $h(\alpha) = h^*(\alpha) + \Delta(q, a, \beta)$ which gives

$$h(\alpha)^2 = h^*(\alpha)^2 + O(|\Delta(q, a, \beta)|(|h^*(\alpha)| + |\Delta(q, a, \beta)|)).$$

Now replace α by $p^3\alpha$ (that is, replace a by ap^3 and β by $p^3\beta$) and put this formula into (4.9) to deduce that

$$\nu^*(n) = \nu^{**}(n) + O(\Upsilon_1 + \Upsilon_2)$$

where
$$\nu^{**}(n) = \int_{\mathfrak{M}_1} \sum_{Y < p \leqslant 2Y} f_p^*(\alpha)\, g_p^*(\alpha)\, h^*(p^3\alpha)^2\, e(-\alpha n)\, d\alpha, \tag{5.8}$$

$$\Upsilon_1 = \int_{\mathfrak{M}_1} \sum_{Y < p \leqslant 2Y} |f_p^*(\alpha)\, g_p^*(\alpha)\, h^*(p^3\alpha)\, \Delta(q, ap^3; p^3\beta)|\, d\alpha,$$

$$\Upsilon_2 = \int_{\mathfrak{M}_1} \sum_{Y < p \leqslant 2Y} |f_p^*(\alpha)\, g_p^*(\alpha)\, \Delta(q, ap^3; p^3\beta)^2|\, d\alpha. \tag{5·9}$$

By Cauchy's inequality and Schwarz's inequality,

$$\Upsilon_1 \leqslant (\Upsilon_2 \Upsilon_3)^{\frac{1}{2}}$$

where
$$\Upsilon_3 = \int_{\mathfrak{M}_1} \sum_{Y < p \leqslant 2Y} |f_p^*(\alpha)\, g_p^*(\alpha)\, h^*(p^3\alpha)^2|\, d\alpha. \tag{5·10}$$

LEMMA 3. *Let* Υ_3 *be given by* (5·10). *Then* $\Upsilon_3 \ll YP^{-2}Q^3(\log P)^{15}$.

LEMMA 4. *Let* Υ_2 *be given by* (5·9). *Then* $\Upsilon_2 \ll YP^{-2}Q^3(\log P)^{-A}$ *holds for all* $A > 0$.

In particular we see that

$$\nu^*(n) = \nu^{**}(n) + O_A(YP^{-2}Q^3(\log P)^{-A}). \tag{5·11}$$

We prove Lemma 3 in the next section, and Lemma 4 in Section 7. A lower bound for $\nu^{**}(n)$ is given in Section 8.

6. *Auxiliary integrals*

In this section we prove Lemma 3 and (4·10). These estimates are based in part on a common principle. We begin with various properties of the exponential sum $S(q, a)$. By lemmata 4·3 and 4·4 of Vaughan [12] when $\pi \nmid a$ one has

$$S(\pi^l, a) = \begin{cases} \pi^{[2l/3]} & l \not\equiv 1 \pmod 3, \\ 0 & l \equiv 1 \pmod 3, \pi \equiv 2 \pmod 3, \\ \pi^{2(l-1)/3} S(\pi, a) & l \equiv 1 \pmod 3, \pi \equiv 1 \pmod 3. \end{cases} \tag{6·1}$$

Also, standard methods show that

$$S(q, ab^3) = S(q, a) \quad \text{when } (b, q) = 1; \tag{6·2}$$

$$S(qr, a) = S(q, ar^2)\, S(r, aq^2) \quad \text{when } (q, r) = (qr, a) = 1. \tag{6·3}$$

Now suppose that $Y < p \leqslant 2Y$ (and recall $p \equiv 2 \pmod 3$). When $p \nmid q$ we have from (6·2) and (4·1)

$$S(q, a, p) = \left(1 - \frac{1}{p}\right) S(q, a) \quad \text{if } (q, a) = 1.$$

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When $p \| q$ write $q = pr$ with $p \nmid r$. We find that

$$S(pr, a, p) = S(pr, a) - \frac{1}{p} S(pr, ap^3).$$

If $(pr, a) = 1$ then by (6·3) and (6·1) the first term on the right here vanishes, and $S(pr, ap^3) = pS(r, ap^2)$ by an obvious argument. This gives

$$S(pr, a, p) = -S(r, ap^2).$$

In particular this implies via (3·13) that for $q = pr, p \nmid r$ and $(a, q) = 1$ one has

$$q^{-1}|S(q, a, p)| \ll p^{-1}\kappa(r).$$

We argue similarly when $q = p^l r$, $p \nmid r$ and $l \geqslant 2$. In this case, a now familiar argument gives

$$\begin{aligned} S(p^l r, a, p) &= S(p^l r, a) - p^{-1} S(p^l r, ap^3) \\ &= S(r, ap^{2l})\,(S(p^l, ar^2) - p^{-1} S(p^l, ap^3 r^2)) = 0. \end{aligned} \tag{6·4}$$

That the second factor vanishes is easily seen directly when $l = 2$, and when $l \geqslant 3$ this follows from the observation that $S(p^l, ap^3 r^2) = p^3 S(p^{l-3}, ar^2)$ together with (6·1). We may combine these results with (3·11) and (4·3) to deduce that for $(a, q) = 1$ we have

$$f_p^*(\alpha, q, a)\, g_p^*(\alpha, q, a) \begin{cases} = 0 & \text{if } p^2 \mid q \\ \ll p^{-1}\kappa(r)^2 PQ\left(1 + P^3\left|\alpha - \frac{a}{q}\right|\right)^{-2} & \text{if } q = pr, p \nmid r \\ \ll \kappa(q)^2 PQ\left(1 + P^3\left|\alpha - \frac{a}{q}\right|\right)^{-2} & \text{if } p \nmid q. \end{cases} \tag{6·5}$$

This bound will be crucial in the proofs of both estimates. We add here one further result which shows another vanishing property of the left-hand side of (6·5) when q has a divisor from the sequence $\mathscr{D}$.

LEMMA 5. *Let $(a, q) = 1$ and $Y < p \leqslant 2Y$. Suppose $d \in \mathscr{D}$ satisfies $d \mid q, d^2 \nmid q$. Then $S(q, ap^3) = 0$.*

In particular, on the hypotheses of this Lemma, we have $g_p^*(\alpha, q, a) = 0$.

Proof. Any $d \in \mathscr{D}$ is squarefree. Thus from (5·1) we see that there is a prime $p' \leqslant 2Q^{\frac{1}{8}} < Y$ such that $p' \| q$, and of course $p' \neq p$. Hence, by (6·3) and (6·2), $S(q, ap^3)$ contains a factor $S(p', b)$ with $p' \nmid b$, which vanishes by (6·1).

Proof of (4·10). To get a simpler formula, we note that $4p^{-1}\kappa(r)^2 = \kappa(pr)^2$ for $p \nmid r$ so that the two lines in (6·5) can actually be taken together. Recalling (4·8) we find that

$$\begin{aligned} &\sum_{Y < p \leqslant 2Y} \int_{\mathfrak{M} \setminus \mathfrak{M}_1} |f_p^*(\alpha)\, g_p^*(\alpha)\, h(p^3\alpha)^2|^2\, d\alpha \\ &\ll P^2 Q^2 \sum_{Y < p \leqslant 2Y} \sum_{\substack{X < q \leqslant P \\ p^2 \nmid q}} \kappa(q)^4 \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}(q,a)} \left(1 + P^3\left|\alpha - \frac{a}{q}\right|\right)^{-4} |h(p^3\alpha)|^4\, d\alpha \\ &\ll P^2 Q^2 \sum_{Y < p \leqslant 2Y} \sum_{\substack{X < q \leqslant P \\ p^2 \nmid q}} \kappa(q)^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 + P^3|\beta|)^{-4} \sum_{a=1}^{q} \left|h\left(p^3\left(\frac{a}{q} + \beta\right)\right)\right|^4 d\beta. \end{aligned}$$

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Now

$$\sum_{a=1}^{q}\left|h\left(p^3\left(\frac{a}{q}+\beta\right)\right)\right|^4 = \sum_{x_1,\dots,x_4\in\mathscr{A}}\sum_{a=1}^{q} e\left(p^3\left(\frac{a}{q}+\beta\right)(x_1^3+x_2^3-x_3^3-x_4^3)\right)$$

$$= q \sum_{\substack{x_i\in\mathscr{A}\\ p^3(x_1^3+x_2^3-x_3^3-x_4^3)\equiv 0 \,(\mathrm{mod}\, q)}} e(p^3\beta(x_1^3+x_2^3-x_3^3-x_4^3))$$

$$\leqslant q\psi(q,p)$$

where $\psi(q,p)$ denotes the number of solutions of the congruence

$$p^3(x_1^3+x_2^3-x_3^3-x_4^3)\equiv 0\,(\mathrm{mod}\, q) \tag{6·6}$$

subject to $x_i \leqslant Q$. We now have

$$\sum_{Y<p\leqslant 2Y}\int_{\mathfrak{M}\setminus\mathfrak{M}_1} |f_p^*(\alpha)\, g_p^*(\alpha)\, h(p^3\alpha)^2|^2\, d\alpha$$

$$\ll P^{-1}Q^2 \sum_{Y<p\leqslant 2Y}\sum_{\substack{X<q\leqslant P\\ p^2\nmid q}} q\kappa(q)^4\psi(q,p) = P^{-1}Q^2(\Sigma_0+\Sigma_1), \quad \text{say,} \tag{6·7}$$

where Σ_j is the part of the remaining sum with $p^j \parallel q$.

By (6·6) when $p\nmid q$ we have $\psi(q,p)=\psi(q,1)=\psi(q)$, say. Thus, by a simple splitting up argument and Cauchy's inequality,

$$\Sigma_0 \ll Y\log P \sum_{\Xi<q\leqslant 2\Xi} q\kappa(q)^4\psi(q) \ll Y\log P\Big(\sum_{\Xi<q\leqslant 2\Xi} q\kappa(q)^8\Big)^{\frac{1}{2}}\Big(\sum_{q\leqslant 2\Xi} q\psi(q)^2\Big)^{\frac{1}{2}} \tag{6·8}$$

for some Ξ in the range $X<\Xi\leqslant P$.

Let $\rho(h)$ denote the number of solutions to

$$x_1^3+x_2^3-x_3^3-x_4^3 = h$$

subject to $x_i\leqslant Q$. Then

$$\psi(q) = \sum_{|l|\leqslant Q^3q^{-1}} \rho(lq),$$

and by Cauchy's inequality,

$$\psi(q)^2 \leqslant (2Q^3q^{-1}+1)\sum_{|l|\leqslant Q^3q^{-1}}\rho(lq)^2.$$

Of course, $\rho(0)\ll Q^{2+\eta}$ so that

$$\sum_{q\leqslant 2\Xi} q\psi(q)^2 \ll \Xi Q^{7+2\eta} + Q^3 \sum_{0<lq\leqslant Q^3}\rho(lq)^2.$$

Now, by a divisor argument,

$$\sum_{0<lq\leqslant Q^3}\rho(lq)^2 \ll Q^{\eta}\sum_{h\leqslant Q^3}\rho(h)^2;$$

and the final sum is bounded by the number of solutions of

$$x_1^3+x_2^3+x_3^3+x_4^3 = x_5^3+x_6^3+x_7^3+x_8^3$$

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subject to $x_i \leqslant Q$. By Hua's Lemma (lemma 2·5 of Vaughan [12]) this number is bounded by $O(Q^{5+\eta})$. We now have

$$\sum_{q \leqslant 2\Xi} q\psi(q)^2 \ll Q^{2\eta}(\Xi Q^7 + Q^8).$$

Note that $\kappa(q) \ll 2^{\omega(q)} q^{-\frac{1}{3}}$. Hence, from (3·18),

$$\sum_{\Xi < q \leqslant 2\Xi} q\kappa(q)^8 \ll \Xi^{-\frac{4}{3}} (\log \Xi)^{18}.$$

In particular, from (6·8) we have

$$\Sigma_0 \ll YQ^{4+2\eta} X^{-\frac{2}{3}} \ll YQ^4 P^{-\frac{15}{42}}.$$

An estimate for Σ_1 is obtained along similar lines. We write $q = pr$ with $p \nmid r$. Then, from (6·6), $\psi(pr, p) = \psi(r, 1) = \psi(r)$. Therefore, since κ is multiplicative,

$$\Sigma_1 \ll \sum_{Y < p \leqslant 2Y} p^{-1} \sum_{X/p < r \leqslant P/p} r\kappa(r)^4 \psi(r).$$

By the same argument as before, the inner sum is $\ll Q^{4+2\eta}(X/p)^{-\frac{2}{3}}$. Summing over p we see that the bound for Σ_1 is even stronger than the bound for Σ_0. From (6·7) we now deduce that

$$\sum_{Y < p \leqslant 2Y} \int_{\mathfrak{M} \setminus \mathfrak{M}_1} |f_p^*(\alpha)\, g_p^*(\alpha)\, h(p^3\alpha)^2|^2 \, d\alpha \ll YP^{-1-\frac{5}{42}} Q^6. \tag{6·9}$$

By another application of Cauchy's inequality this implies (4·10).

Proof of Lemma 3. We use (6·5), the remark made after Lemma 5 and (5·7) to deduce, in much the same way as in the beginning of the proof of (4·10), that

$$\Upsilon_3 = \int_{\mathfrak{M}_1} \sum_{Y < p \leqslant 2Y} |f_p^*(\alpha)\, g_p^*(\alpha)\, h(p^3\alpha)^2| \, d\alpha$$

$$\ll PQ \sum_{Y < p \leqslant 2Y} \sum_{\substack{q \leqslant X \\ p^2 \nmid q}} \kappa(q)^2 \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 + P^3|\beta|)^{-2} \left(\sum_{\substack{d \in \mathscr{D} \\ d^2 | q}} |T_d^*(q, ap^3, p^3\beta)| \right)^2 d\beta$$

Now a number $q \leqslant X$ has only $O(1)$ divisors $d \in \mathscr{D}$. We use Cauchy's inequality and the bound $t_d(\beta) \ll Qd^{-1}$ (which is obvious from (5·6)) to find that

$$\Upsilon_3 \ll P^{-2}Q^3 \sum_{Y < p \leqslant 2Y} \sum_{\substack{q \leqslant X \\ p^2 \nmid q}} \sum_{\substack{d \in \mathscr{D} \\ d^2 | q}} d^{-2}\kappa(q)^2 \phi(q/d)^{-2} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| S^*\left(\frac{q}{d}, ad^2p^3\right) \right|^2. \tag{6·10}$$

From (5·5), when $d^2 | q$,

$$S^*\left(\frac{q}{d}, ap^3d^2\right) = \sum_{\substack{x \leqslant q/d \\ (x, q/d)=1}} e\left(\frac{ax^3d^2p^3}{q/d}\right) = \frac{1}{d} \sum_{\substack{x \leqslant q \\ (x, q/d)=1}} e\left(\frac{a(xpd)^3}{q}\right) = \frac{1}{d} S^*(q, ap^3d^3) \tag{6·11}$$

(observe that $(x, q/d) = 1$ if and only if $(x, q) = 1$). Neglecting the condition $(a, q) = 1$ in the innermost sum in (6·10) we obtain

$$\Upsilon_3 \ll P^{-2}Q^3 \sum_{Y < p \leqslant 2Y} \sum_{\substack{q \leqslant X \\ p^2 \nmid q}} \sum_{\substack{d \in \mathscr{D} \\ d^2 | q}} \left(\frac{\kappa(q)}{d^2 \phi(q/d)} \right)^2 qW(q, d^3p^3) \tag{6·12}$$

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where $W(q,h)$ is the number of solutions of the congruence

$$h(x^3-y^3) \equiv 0 \,(\mathrm{mod}\, q)$$

subject to $1 \leqslant x, y \leqslant q$ and $(x,q) = (y,q) = 1$.

Again we must consider terms with $p \nmid q$ and $p \| q$ separately. If $p \nmid q$ then $W(q, d^3p^3) = W(q, d^3)$. Interchanging the order of summation and writing $q = rd^2$ we see that terms with $p \nmid q$ in (6·12) contribute

$$\ll P^{-2}Q^3Y \sum_{\substack{d \leqslant X^{\frac{1}{2}} \\ d \in \mathscr{D}}} \sum_{r \leqslant Xd^{-2}} \frac{r\kappa(rd^2)^2}{d^2\phi(rd^2)^2} W(rd^2, d^3). \tag{6·13}$$

Now $W(rd^2, d^3)$ is the number of solutions to the congruence $d(x^3-y^3) \equiv 0 \,(\mathrm{mod}\, r)$ subject to $(x, d^2r) = (y, d^2r) = 1$ and $1 \leqslant x, y \leqslant rd^2$. This gives $W(rd^2, d^3) \leqslant d^4 W(r,d)$. Let $\delta = (r,d)$ and write $r = r_1\delta$, $d = d_1\delta$. Repeating the previous argument we get $W(r_1\delta, d_1\delta) = \delta^2 W(r_1, d_1) = \delta^2 W(r_1, 1)$ since $(r_1, d_1) = 1$. We estimate $W(k,1)$. There are $\phi(k)$ choices for y, and for any such choice of y there are $\ll 3^{\omega(k)}$ choices for x. Hence $W(k,1) \ll 3^{\omega(k)}\phi(k)$. Collecting together, we obtain the bound

$$W(rd^2, d^3) \ll d_1^4 \delta^6 3^{\omega(r_1)} \phi(r_1).$$

Summing over δ first we find that (6·13) is

$$\ll P^{-2}Q^3Y \sum_{\substack{\delta \leqslant X^{\frac{1}{2}} \\ \delta \in \mathscr{D}}} \sum_{\substack{d_1 \leqslant X^{\frac{1}{2}}\delta^{-1} \\ d_1\delta \in \mathscr{D}}} \sum_{\substack{r_1 \leqslant Xd_1^{-2}\delta^{-3} \\ (r_1, d_1) = 1}} \ll 3^{\omega(r_1)} r_1 d_1^2 \delta^5 \frac{\kappa(r_1 d_1^2 \delta^3)^2 \phi(r_1)}{\phi(r_1 d_1 \delta^2)^2}.$$

Now use (3.16) and the well known fact that $\phi(uv) \geqslant \phi(u)\phi(v)$ first, and then recall $(r_1, d_1) = 1$ to see that it is

$$\ll P^{-2}Q^3Y \sum_{\substack{\delta \leqslant X^{\frac{1}{2}} \\ \delta \in \mathscr{D}}} \frac{\delta^3}{\phi(\delta^2)^2} \sum_{\substack{d_1 \leqslant X^{\frac{1}{2}} \\ d_1 \in \mathscr{D}}} \frac{d_1^2}{\phi(d_1)^2} \kappa(d_1^2)^2 \sum_{r_1 \leqslant X} 3^{\omega(r_1)} \frac{r_1}{\phi(r_1)} \kappa(r_1)^2.$$

Both δ and d_1 are squarefree so that the sum over δ is $O((\log X)^2)$ and the sum over d_1 is $O(1)$ by straightforward estimates. The sum over r_1 is dealt with by the same technique as the sum in (3·17) was treated, and is seen to be $\ll (\log X)^{13}$ by this argument. After all, the contribution of terms with $p \nmid q$ to (6·12) is

$$\ll P^{-2}Q^3Y(\log X)^{15}$$

which is acceptable.

Now consider terms with $p \| q$. Write $q = pq'$ with $p \nmid q'$. As at earlier stages of the present estimation we see that $W(pq', d^3p^3) \leqslant p^2 W(q', d^3)$. Also, by (5·1), when $d \in \mathscr{D}$ and $Y < p \leqslant 2Y$ we always have $(p,d) = 1$. Therefore $d^2 \mid q$ if and only if $d^2 \mid q'$. Hence, terms with $p \| q$ in (6·12) contribute at most

$$\ll P^{-2}Q^3 \sum_{Y < p \leqslant 2Y} \sum_{\substack{q' \leqslant Xp^{-1} \\ p \nmid q'}} \sum_{\substack{d \in \mathscr{D} \\ d^2 \mid q'}} p^3 q' \left(\frac{\kappa(pq')}{d\phi(pq'/d)}\right)^2 W(q', d^3)$$

$$\ll P^{-2}Q^3 \sum_{Y < p \leqslant 2Y} \frac{p^2}{\phi(p)^2} \sum_{q \leqslant X} \sum_{\substack{d \in \mathscr{D} \\ d^2 \mid q}} q \left(\frac{\kappa(q)}{d\phi(q/d)}\right)^2 W(q, d^3)$$

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on omitting the dash from q, and enlarging the range of summation for q. The outer sum over p is $O(Y)$, and the remaining sum is exactly the sum we already estimated while discussing the case $p \nmid q$. We are therefore led to the same bound for the distribution from terms with $p \| q$ to (6·12). This establishes Lemma 3.

7. *Estimating the error term*

In this section we prove Lemma 4. We begin very much as in the early stages of the proof of Lemma 3. In fact, recalling (5·9), (6·5) and the remark following Lemma 5 we obtain the basic estimate

$$\Upsilon_2 \ll PQ \sum_{Y<p\leqslant 2Y} \sum_{\substack{q\leqslant X\\ p^2\nmid q}} \kappa^*(q)^2 \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{\substack{d\in\mathscr{D}\\ d^2\mid q}} \int_{-XP^{-3}}^{XP^{-3}} (1+P^3|\beta|)^{-2} |\Delta_d(q, ap^3, p^3\beta)|^2 \, d\beta; \quad (7\cdot1)$$

compare (6·10).

It is now important to take advantage from cancellations by summing non-trivially over both a and q here. We do this via the large sieve, and therefore wish to transform Δ_d into a suitable character sum. Let q be fixed and $d\in\mathscr{D}$ with $d^2\mid q$. Then, writing $q = d^2r$ we deduce from (5·3) that

$$T_d(q,a;\beta) = \sum_{\substack{y\in d^{-1}\mathscr{A}\\(y,q/d)=1}} e\left(\left(\frac{a}{q}+\beta\right)y^3d^3\right) = \sum_{\substack{y\in d^{-1}\mathscr{A}\\(y,r)=1}} e\left(\left(\frac{a}{d^2r}+\beta\right)y^3d^3\right)$$

(note that for $y\in d^{-1}\mathscr{A}$ one has $(y,d)=1$ so that in this case $(y,dr)=1$ is equivalent to $(y,r)=1$). Splitting of into residue classes we infer that

$$\begin{aligned} T_d(q,a;\beta) &= \sum_{\substack{b=1\\(b,r)=1}}^{r} \sum_{\substack{y\in d^{-1}\mathscr{A}\\ y\equiv b \,(\mathrm{mod}\, r)}} e\left(\frac{adb^3}{r}\right) e(\beta d^3y^3) \\ &= \frac{1}{\phi(r)} \sum_{\substack{b=1\\(b,r)=1}}^{r} e\left(\frac{adb^3}{r}\right) \sum_{\chi} \bar{\chi}(b) \sum_{y\in d^{-1}\mathscr{A}} \chi(y)\, e(\beta d^3y^3). \end{aligned}$$

We subtract T_d^* on both sides and obtain from (5·4), (5·5) and (5·6) that

$$\Delta_d(q,a,\beta) = \frac{1}{\phi(r)} \sum_{\chi} \sum_{b=1}^{r} \bar{\chi}(b)\, e\left(\frac{adb^3}{r}\right) J_d(\chi,\beta) \quad (7\cdot2)$$

where

$$J_d(\chi_0,\beta) = \sum_{\substack{y\in d^{-1}\mathscr{A}\\(y,r)=1}} e(\beta d^3y^3) - \tfrac{1}{3} \sum_{C_d(Q/d)^3 < x \leqslant (Q/d)^3} x^{-\frac{2}{3}} (\log x)^{\omega(d)-s-1}\, e(\beta d^3 x), \quad (7\cdot3)$$

$$J_d(\chi,\beta) = \sum_{y\in d^{-1}\mathscr{A}} \chi(y)\, e(\beta d^3y^3) \quad (\chi \neq \chi_0). \quad (7\cdot4)$$

We consider terms with $p\nmid q$ in (7·1) first. By (7·2) and an obvious transformation,

$$\begin{aligned} \sum_{\substack{a=1\\(a,q)=1}}^{q} |\Delta_d(q,ap^3,p^3\beta)|^2 &\leqslant \sum_{a=1}^{q} |\Delta_d(q,a,p^3\beta)|^2 \\ &= \phi(r)^{-2} \sum_{\chi_1,\chi_2} J_d(\chi_1,p^3\beta)\, J_d(\overline{\chi_2},-p^3\beta) \sum_{a=1}^{q} \sum_{b_1,b_2=1}^{r} \overline{\chi_1}(b_1)\chi_2(b_2)\, e\left(\frac{ad}{r}\left(b_1^3-b_2^3\right)\right) \end{aligned}$$

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(we still write $q = rd^2$). But $ad/r = ad^3/q$ so that

$$\sum_{a=1}^{q} \sum_{b_1, b_2=1}^{r} \overline{\chi_1}(b_1)\chi_2(b_2)\, e\left(\frac{ad}{r}\left(b_1^3 - b_2^3\right)\right)$$

$$= q \sum_{\substack{b_1, b_2=1 \\ d^3(b_1^3 - b_2^3) \equiv 0 \pmod q}}^{r} \overline{\chi_1}(b_1)\chi_2(b_2) = q \sum_{\substack{b_1, b_2=1 \\ d(b_1^3 - b_2^3) \equiv 0 \pmod r}}^{r} \overline{\chi_1}(b_1)\chi_2(b_2).$$

Let $e = (d, r), d = d_1 e, r = r_1 e$. Then the above is

$$q \sum_{\substack{b_1, b_2=1 \\ b_1^3 \equiv b_2^3 \pmod{r_1}}}^{r} \overline{\chi_1}(b_1)\chi_2(b_2).$$

Here the sum is only over $(b_i, r) = 1$ of course. Thus we may write $b_2 = b_1 b_3$ and sum over b_3 in place of b_2. The previous expression then equals

$$q \sum_{\substack{b_1=1 \\ (b_1, r)=1}}^{r} \overline{\chi}_1(b_1)\chi_2(b_1) \sum_{\substack{b_3=1 \\ b_3^3 \equiv 1 \pmod{r_1}}}^{r} \chi_2(b_3).$$

The sum over b_1 vanishes unless $\chi_1 = \chi_2$ in which case it is $\phi(r)$. Also the congruence $x^3 \equiv 1 \pmod k$ has $O(3^{\omega(k)})$ solutions in reduced residues modulo k. This shows that the sum over b_3 is $O(3^{\omega(r_1)} e)$. Collecting together we find that

$$\sum_{\substack{a=1 \\ (a, q)=1}}^{q} |\Delta_d(q, ap^3, p^3\beta)|^2 \ll \frac{q}{\phi(r)}(r, d)\, 3^{\omega(r)} \sum_{\chi} |J_d(\chi, p^3\beta)|^2. \tag{7·5}$$

Using this estimate we see that terms with $p \nmid q$ in (7·1) contribute at most

$$\ll PQY \max_{Y < p \leqslant 2Y} \sum_{\substack{d \leqslant X^{\frac{1}{3}} \\ d \in \mathscr{D}}} \sum_{r \leqslant Xd^{-2}} \frac{rd^2}{\phi(r)} 3^{\omega(r)}(r, d)\, \kappa^*(rd^2)^2$$

$$\times \int_{-XP^{-3}}^{XP^{-3}} (1 + P^3|\beta|)^{-2} \sum_{\chi} |J_d(\chi, p^3\beta)|^2\, d\beta. \tag{7·6}$$

We single out all principal characters first. Here we have

$$\sum_{\substack{x \in d^{-1}\mathscr{A} \\ (x, r)=1}} e(\gamma x^3) = \sum_{x \in d^{-1}\mathscr{A}} e(\gamma x^3) + O\left(\sum_{\substack{\delta \mid r \\ \delta > 1}} \sum_{\substack{x \in d^{-1}\mathscr{A} \\ (x, r) = \delta}} 1\right).$$

Now $x \in d^{-1}\mathscr{A}$ and $\delta \mid x$ implies $\delta \in \mathscr{D}$, thus $\delta > 1$ implies $\delta \gg Q^{\eta}$. Since any $r \leqslant X$ can have at most $O(1)$ such divisors $\delta \in \mathscr{D}$ we see that the error term is $O(Q^{1-\eta}d^{-1})$. From (7·3) we now get

$$J_d(\chi_0, p^3\beta) = \sum_{l \leqslant P^3} u(l, d, pd)\, e(\beta l) + O(Q^{1-\eta}d^{-1})$$

where

$$u(l, d, k) = \begin{cases} 1 - \frac{1}{3}(\log x)^{\omega(d)-s-1} x^{-\frac{2}{3}} & \text{if } l = k^3y^3, y^3 = x, y \in d^{-1}\mathscr{A} \\ -\frac{1}{3}(\log x)^{\omega(d)-s-1} x^{-\frac{2}{3}} & \text{if } l = k^3x, C_d(Q/d)^3 < x \leqslant (Q/d)^3, \quad \text{otherwise,} \\ 0 & \text{in all other cases.} \end{cases}$$

The integral occurring in (7·6) can be estimated by a familiar method due to Gallagher.

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LEMMA 6 (Gallagher [6]). *Let $u_1, u_2, \dots, u_L$ be arbitrary complex numbers. Then, for any $\Theta > 0$,*

$$\int_{-\Theta}^{\Theta} \Big| \sum_{l \leqslant L} u_l e(\gamma l) \Big|^2 d\gamma \ll \Theta^2 \int_{-\infty}^{\infty} \Big| \sum_{\vartheta < l \leqslant \vartheta + (2\Theta)^{-1}} u_l \Big|^2 d\vartheta.$$

Applying this to our formula just obtained for $J_d(\chi_0, p^3\beta)$ we find for $1 \leqslant \Xi \leqslant X$ that

$$\int_{-\Xi P^{-3}}^{\Xi P^{-3}} |J_d(\chi_0, p^3\beta)|^2 \, d\beta$$

$$\ll \Xi^2 P^{-6} \int_{-\infty}^{\infty} \Big| \sum_{\substack{\vartheta < l \leqslant \vartheta + \frac{1}{2}\Xi^{-1}P^3 \\ 0 < l \leqslant P^3}} u(l, d, pd) \Big|^2 d\vartheta + O(\Xi P^{-3} Q^{2-2\eta} d^{-2})$$

$$\ll \Xi^2 P^{-6} \int_0^{P^3/(pd)^3} p^3 d^3 \Big| \sum_{\substack{\xi < m \leqslant \xi + \frac{1}{2}\Xi^{-1}(P/pd)^3 \\ 0 < l \leqslant P^3/(pd)^3}} u(p^3 d^3 m, d, 1) \Big|^2 d\xi + O(\Xi P^{-3} Q^{2-2\eta} d^{-2}).$$

In the ranges we have to consider in (7·6) we have $Q \ll P/p \ll Q$ and $d \leqslant X^{\frac{1}{2}}$. By a standard result in prime number theory the single sum over m thus is $\ll_A (Q/d)(\log Q/d)^{-A} \ll (Q/d)(\log P)^{-A}$, for all $A > 0$. It follows that

$$\int_{-\Xi P^{-3}}^{\Xi P^{-3}} |J_d(\chi_0, p^3\beta)|^2 \, d\beta \ll \Xi^2 P^{-3} Q^2 d^{-2} (\log P)^{-2A} + \Xi P^{-3} Q^{2-2\eta} d^{-2}.$$

Now split the range for $|\beta|$ into the interval $|\beta| \leqslant P^{-3}$ and $O(\log X)$ parts of the shape $\frac{1}{2}\Xi P^{-3} < |\beta| \leqslant \Xi P^{-3}$ where $2 \leqslant \Xi \leqslant X$. Then

$$\int_{-XP^{-3}}^{XP^{-3}} (1 + P^3|\beta|)^{-2} \sum_{\chi} |J_d(\chi, p^3\beta)|^2 \, d\beta \ll (\log P)^{1-2A} P^{-3} Q^2 d^{-2}.$$

Therefore the contribution from all principal characters to (7·6) is

$$\ll (\log P)^{1-2A} P^{-2} Q^3 Y \sum_{\substack{d \leqslant X^{\frac{1}{2}} \\ d \in \mathscr{D}}} \sum_{r \leqslant X d^{-2}} \frac{r}{\phi(r)} 3^{\omega(r)} (r, d) \kappa^*(rd^2)^2$$

$$\ll (\log P)^{1-2A} P^{-2} Q^3 Y \sum_{\substack{d \leqslant X^{\frac{1}{2}} \\ d \in \mathscr{D}}} \sum_{r \leqslant X} \frac{r}{\phi(r)} 3^{\omega(r)} \frac{(r, d)}{d} \kappa^*(r)^2.$$

Here we use (3·15). Now recall that $d \in \mathscr{D}$ is squarefree. Therefore the same argument as in (3·17) shows that

$$\sum_{r \leqslant X} \frac{r}{\phi(r)} 3^{\omega(r)} (r, d) \kappa^*(r)^2 \leqslant \prod_{\substack{\pi \leqslant X \\ \pi \nmid d}} \Big(1 + 3 \sum_{l=1}^{\infty} \frac{\pi^l}{\phi(\pi^l)} \kappa^*(\pi^l)^2 \Big) \prod_{\substack{p \leqslant X \\ p \mid d}} \Big(1 + 3p \sum_{l=1}^{\infty} \frac{p^l}{\phi(p^l)} \kappa^*(p^l)^2 \Big).$$

The sums over l are $\leqslant 10\pi^{-1}$ and $\leqslant 10p^{-1}$ respectively. Since the second product has only $O(1)$ factors it is $O(1)$ itself. Thus

$$\sum_{r \leqslant X} \frac{r}{\phi(r)} 3^{\omega(r)} (r, d) \kappa^*(r)^2 \ll (\log X)^{30}. \tag{7·7}$$

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Collecting together, we deduce that all principal characters contribute to (7·6) at most

$$\ll (\log P)^{-A} P^{-2} Q^3 Y \tag{7·8}$$

for all $A > 0$, which is acceptable.

We now turn our attention to the contribution to (7·5) arising from non-principal characters. We start in much the same way. Let $1 \leqslant \Xi \leqslant X$ as before. Then, by (7·4) and Lemma 6, and then applying the change of variable $\vartheta = (dp\xi)^3$,

$$\int_{-\Xi P^{-3}}^{\Xi P^{-3}} |J_d(\chi, p^3\beta)|^2 \, d\beta \ll \Xi^2 P^{-6} \int_{-\infty}^{\infty} \left| \sum_{\substack{\vartheta < p^3 d^3 x^3 \leqslant \vartheta + \frac{1}{2}\Xi^{-1} P^3 \\ x \in d^{-1}\mathscr{A}}} \chi(x) \right|^2 d\vartheta$$

$$\ll \Xi^2 P^{-4} pd \int_0^{2P/(dp)} \left| \sum_{\substack{\xi < x \leqslant \xi_1 \\ x \in d^{-1}\mathscr{A}}} \chi(x) \right|^2 d\xi$$

where $$\xi_1 = (\xi^3 + \tfrac{1}{2}\Xi^{-1} P^3 (pd)^{-3})^{\frac{1}{3}}.$$

As before we split the range for $|\beta|$ in (7·6) and see that non-principal characters contribute to (7·6) at most

$$\ll (\log P) P^{-3} QY \max_{Y < p \leqslant 2Y} \sum_{\substack{d \leqslant X^{\frac{1}{2}} \\ d \in \mathscr{D}}} \sum_{r \leqslant Xd^{-2}} \frac{prd^3}{\phi(r)} 3^{\omega(r)} (r, d) \kappa^*(rd^2)^2 \int_0^{2P/(pd)} \sum_{\chi \neq \chi_0} \left| \sum_{\substack{\xi < x \leqslant \xi_1 \\ x \in d^{-1}\mathscr{A}}} \chi(x) \right|^2 d\xi$$

$$\ll (\log P)^2 P^{-3} QY^2 D \max_{Y < p \leqslant 2Y} \int_0^{2P/(YD)} \sum_{\substack{D < d \leqslant 2D \\ d \in \mathscr{D}}} \sum_{r \leqslant Xd^{-2}} \frac{rd^2}{\phi(r)} 3^{\omega(r)} (r, d) \kappa^*(rd^2)^2$$

$$\times \sum_{\chi \neq \chi_0} \left| \sum_{\substack{\xi < x \leqslant \xi_1 \\ x \in d^{-1}\mathscr{A}}} \chi(x) \right|^2 d\xi \tag{7·9}$$

for some D in the range $1 \leqslant D \leqslant X^{\frac{1}{2}}$, by a simple splitting up argument.

The remaining sum in (7·9) is quite similar to one which occurs in the proof of the Barban–Davenport–Halberstam theorem, and our method of estimation is modelled on Hooley's approach to a generalized version of this theorem [**8**].

Let χ^* be the unique primitive character, modulo r^*, which is equivalent to χ, modulo r. Then $r^* | r$, and we write $r = r^* l$. We may therefore write

$$\sum_{\substack{\xi < x \leqslant \xi_1 \\ x \in d^{-1}\mathscr{A}}} \chi(x) = \sum_{\substack{x \in (\xi, \xi_1] \cap d^{-1}\mathscr{A} \\ (x, l) = 1}} \chi^*(x) = \sum_{h | l} \mu(h) \sum_{\substack{x \in (\xi, \xi_1] \cap d^{-1}\mathscr{A} \\ x \equiv 0 \,(\mathrm{mod}\, h)}} \chi^*(x).$$

By definition of $\mathscr{A}$ the inner sum here is zero unless $h \in \mathscr{D}$ and $(h, d) = 1$. Thus

$$\sum_{\substack{\xi \leqslant x \leqslant \xi_1 \\ x \in d^{-1}\mathscr{A}}} \chi(x) = \sum_{\substack{h | l \\ h \in \mathscr{D}}} \mu(h) \chi^*(h) \sum_{\substack{\xi h^{-1} < y \leqslant \xi_1 h^{-1} \\ y \in (dh)^{-1}\mathscr{A}}} \chi^*(y).$$

By a now familiar argument, the sum over h has only $O(1)$ terms (note that $l \leqslant X$). From Cauchy's inequality we now deduce that

$$\left| \sum_{\substack{\xi < x \leqslant \xi_1 \\ x \in d^{-1}\mathscr{A}}} \chi(x) \right|^2 \ll \sum_{\substack{hh_1 = l \\ h \in \mathscr{D},\, (h, d) = 1}} \left| \sum_{\substack{\xi h^{-1} < y \leqslant \xi_1 h^{-1} \\ y \in (dh)^{-1}\mathscr{A}}} \chi^*(y) \right|^2.$$

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Now write $r = r^*hh_1$ and use the fact that χ^* is uniquely determined by χ. This shows

$$\sum_{\substack{D<d\leqslant 2D\\ d\in\mathscr{D}}}\sum_{r\leqslant Xd^{-2}}\frac{rd^2}{\phi(r)}3^{\omega(r)}(r,d)\,\kappa^*(rd^2)^2\sum_{\chi\neq\chi_0}\Big|\sum_{\substack{\xi<x\leqslant\xi_1\\ x\in d^{-1}\mathscr{A}}}\chi(x)\Big|^2$$

$$\ll\sum_{\substack{D<d\leqslant 2D\\ d\in\mathscr{D}}}\sum_{r^*hh_1\leqslant Xd^{-2}}\frac{r^*hh_1d^2}{\phi(r^*hh_1)}3^{\omega(r^*hh_1)}(r^*h_1,d)\,\kappa^*(r^*hh_1d^2)^2\sum_{\chi^*}\Big|\sum_{\substack{\xi h^{-1}<y\leqslant\xi_1h^{-1}\\ y\in(dh)^{-1}\mathscr{A}}}\chi^*(y)\Big|^2$$

$$\ll\sum_{\substack{D<d\leqslant 2D\\ d\in\mathscr{D}}}\sum_{r^*h\leqslant Xd^{-2}}\frac{(r^*h)^{\frac12}}{\phi(r^*h)}d(r^*,d)\,3^{\omega(r^*h)}\sum_{\chi^*}\Big|\sum_{\substack{\xi h^{-1}<y\leqslant\xi_1h^{-1}\\ y\in(dh)^{-1}\mathscr{A}}}\chi^*(y)\Big|^2\sum_{h_1\leqslant X}\frac{3^{\omega(h_1)}h_1}{\phi(h_1)}\kappa^*(h_1)^2$$

$$\ll(\log X)^{30}\sum_{\substack{D<d\leqslant 2D\\ d\in\mathscr{D}}}\sum_{r^*h\leqslant Xd^{-2}}\frac{(r^*h)^{\frac12}}{\phi(r^*h)}d(r^*,d)\,3^{\omega(r^*h)}\sum_{\chi^*}\Big|\sum_{\substack{\xi h^{-1}<y\leqslant\xi_1h^{-1}\\ y\in(dh)^{-1}\mathscr{A}}}\chi^*(y)\Big|^2. \tag{7·10}$$

Here we used (3·15) and (7·7). Now carry out the sum over d first to remove the common divisor function. One has

$$\sum_{\substack{d\leqslant 2D\\ d\in\mathscr{D}}}d(r^*,d)\ll D^2\sum_{\substack{d\leqslant 2D\\ d\in\mathscr{D}}}\frac{(r^*,d)}{d}\ll D^2\sum_{\substack{\delta\leqslant 2D\\ \delta\mid r^*\\ \delta\in\mathscr{D}}}\sum_{d_1\leqslant 2D}d_1^{-1}\ll D^2\log X$$

since r^* has at most $O(1)$ divisors in $\mathscr{D}$. Splitting also the ranges for r^* and h in the sums in (7·10) we find that there are numbers $R\geqslant 1, H\geqslant 1, RH\leqslant XD^{-2}$ such that (7·10) is

$$\ll(\log X)^{33}D^2(RH)^{\frac12(1+\eta)}\max_{\substack{D<d\leqslant 2D\\ d\in\mathscr{D}}}\sum_{\substack{H<h\leqslant 2H\\ R<r^*\leqslant 2R}}\frac{1}{\phi(r^*)\,\phi(h)}\sum_{\chi^*}\Big|\sum_{\substack{\xi h^{-1}<y\leqslant\xi_1h^{-1}\\ y\in(dh)^{-1}\mathscr{A}}}\chi^*(y)\Big|^2. \tag{7·11}$$

The large sieve (see Gallagher [5]) gives

$$\sum_{R<r^*\leqslant 2R}\frac{1}{\phi(r^*)}\sum_{\chi^*}\Big|\sum_{\substack{\xi h^{-1}<y\leqslant\xi_1h^{-1}\\ y\in(dh)^{-1}\mathscr{A}}}\chi^*(y)\Big|^2\ll\frac{Q^2}{h^2d^2R}+\frac{QR}{hd}.$$

Therefore (7·11) is

$$\ll(\log X)^{33}D^2(RH)^{\frac12(1+\eta)}\left(\frac{Q^2}{H^2D^2R}+\frac{QR}{HD}\right)$$

$$\ll(\log X)^{33}Q^2(RH)^{-\frac13}+Q^{2-\eta},$$

since from $RHD^2\leqslant X=P^{\frac47-\eta}$ we have $DR^{\frac32+\frac12\eta}\leqslant X^{\frac32+\frac12\eta}\leqslant P^{\frac67-\eta}$. When $HR>(\log P)^{4A}$ we find that (7·11) is $\ll Q^2(\log P)^{-A}$.

Now suppose that $HR\leqslant(\log P)^{4A}$. For $d\leqslant X^{\frac12}$ and $H<h\leqslant 2H$ we then have $\log P\ll\log(Q/(hd))\ll\log P$, and it is readily seen from standard results in prime number theory that

$$\sum_{\substack{\xi h^{-1}<y\leqslant\xi_1h^{-1}\\ y\in(dh)^{-1}\mathscr{A}}}\chi^*(y)\ll\frac{Q}{d}(\log P)^{-4A}.$$

If this is used in (7·11) we again find that (7·11) is $\ll Q(\log P)^{-A}$, which is now verified in all cases. This bound for (7·11) being used in (7·10) we find from (7·9) that non-principal characters contribute at most (7·8) to (7·6), and therefore terms with $p\nmid q$ in (7·1) contribute at most (7·8), which is acceptable.

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It remains to consider the contribution to (7·1) arising from terms with $p \| q$. Unfortunately the analysis leading to (7·5) partially breaks down and has to be refined as follows. Again write $q = rd^2$. Then, since $Y < p \leqslant 2Y$ and $d \in \mathscr{D}$ implies $p \nmid d$, we have $p \| q$ if and only if $p \| r$. By (7·2) we have

$$\sum_{a=1}^{q} |\Delta_d(q, ap^3, p^3\beta)|^2$$

$$= \phi(r)^{-2} \sum_{\chi_1, \chi_2} J_d(\chi_1, p^3\beta) J_d(\overline{\chi_2}, -p^3\beta) \sum_{a=1}^{q} \sum_{b_1, b_2=1}^{r} \overline{\chi_1}(b_1) \chi_2(b_2) e\left(\frac{adp^3}{r}\left(b_1^3 - b_2^3\right)\right)$$

$$= \phi(r)^{-2} \sum_{\chi_1, \chi_2} J_d(\chi_1, p^3\beta) J_d(\overline{\chi_2}, -p^3\beta) \sum_{\substack{b_1, b_2=1 \\ d^3p^3(b_1^3 - b_2^3) \equiv 0 \,(\mathrm{mod}\, q)}}^{r} \overline{\chi_1}(b_1) \chi_2(b_2).$$

Now $d^3p^3(b_1^3 - b_2^3) \equiv 0 \,(\mathrm{mod}\, q)$ reduces to

$$d(b_1^3 - b_2^3) \equiv 0 \left(\mathrm{mod}\, \frac{r}{p}\right); \quad \text{or} \quad b_1^3 \equiv b_2^3 \left(\mathrm{mod}\, \frac{r}{p(r, d)}\right).$$

Hence the same argument as used to deduce (7·5) shows that the inner sum over b_1, b_2 vanishes unless $\chi_1 = \chi_2$ in which case it is $\ll \phi(r)\, p(r, d)\, 3^{\omega(r)}$. The analogue of (7·5) is thus the bound

$$\sum_{\substack{a=1 \\ (a, q)=1}}^{q} |\Delta_d(q, ap^3, p^3\beta)|^2 \ll \frac{q}{\phi(r)} p(r, d)\, 3^{\omega(r)} \sum_{\chi} |J_d(\chi, p^3\beta)|^2.$$

This is put into (7·1) to infer that terms with $p \| q$ contribute to (7·1) at most

$$\ll PQ \sum_{\substack{d \leqslant X^{\frac{1}{2}} \\ d \in \mathscr{D}}} \sum_{r \leqslant Xd^{-2}} \sum_{\substack{Y < p \leqslant 2Y \\ p \| r}} p \frac{rd^2}{\phi(r)} 3^{\omega(r)}(r, d)\, \kappa^*(rd^2)^2$$

$$\times \int_{-XP^{-3}}^{XP^{-3}} (1 + P^3|\beta|)^{-2} \sum_{\chi} |J_d(\chi, p^3\beta)|^2 \, d\beta. \quad (7·12)$$

The sum over p has only $O(1)$ terms so that (7·12) is bounded by (7·6). Therefore the contribution to (7·1) arising from terms with $p \| q$ is also bounded by (7·8). This establishes Lemma 4.

8. *A lower bound for* $\nu^{**}(n)$

In this section we shall give a lower bound for $\nu^{**}(n)$. In principal our arguments are standard endgame technique in the Hardy–Littlewood method but the occurrence of a divisor sum in the exponential sum (5·7) leads to a number of unfamiliar problems not experienced in the traditional theory of the singular series in Waring's problem.

By (5·8), (4·3), (3·11), (5·7), (5·4), (6·4), Lemma 5 and (6·11) we may rewrite $\nu^{**}(n)$ as

$$\nu^{**}(n) = \sum_{Y < p \leqslant 2Y} \sum_{\substack{q \leqslant X \\ p^2 \nmid q}} \sum_{\substack{d_1^2 \mid q \\ d_1 \in \mathscr{D}}} \sum_{\substack{d_2^2 \mid q \\ d_2 \in \mathscr{D}}} q^{-2} \left(d_1 d_2 \phi\left(\frac{q}{d_1}\right) \phi\left(\frac{q}{d_2}\right)\right)^{-1} \mathfrak{s}_p(q, n, d_1, d_2)\, \mathfrak{I}_p(q, n, d_1, d_2)$$

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where

$$\mathfrak{s}_p(q,n,d_1,d_2) = \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a,p)\,S(q,ap^3)\,S^*(q,ap^3d_1^3)\,S^*(q,ap^3d_2^3)\,e\Big(-\frac{an}{q}\Big).$$

$$\mathfrak{I}_p(q,n,d_1,d_2) = \int_{-XP^{-3}q^{-1}}^{XP^{-3}q^{-1}} v(\beta)\,w(p^3\beta)\,t_{d_1}(p^3\beta)\,t_{d_2}(p^3\beta)\,e(-\beta n)\,d\beta. \tag{8·1}$$

LEMMA 7. *Suppose* $p^2 \nmid q$; $(p, d_1 d_2) = 1$, $d_1 \in \mathscr{D}$, $d_2 \in \mathscr{D}$, $d_i^2 \mid q$ *and* $d_1 \neq d_2$. *Then* $\mathfrak{s}_p(q,n,d_1,d_2) = 0$.

Since $p \nmid d_1 d_2$ is always true for $Y < p \leqslant 2Y$ and $d_i \in \mathscr{D}$ we may further simplify our formula for $\nu^{**}(n)$ to

$$\nu^{**}(n) = \sum_{Y<p\leqslant 2Y} \sum_{\substack{q\leqslant X\\p^2\nmid q}} \sum_{\substack{d^2\mid q\\d\in\mathscr{D}}} \Big(q\,d\,\phi\Big(\frac{q}{d}\Big)\Big)^{-2} \mathfrak{s}_p(q,n,d)\,\mathfrak{I}_p(q,n,d). \tag{8·2}$$

Here we have removed an unnecessary parameter from $\mathfrak{s}_p(q,n,d_1,d_2)$ and $\mathfrak{I}_p(q,n,d_1,d_2)$ when $d_1 = d_2$. Our first goal is a proof of Lemma 7. It will then turn out that the main contribution to (8·2) comes from terms with $p \nmid q$ and $d = 1$.

Proof of Lemma 7. For $d_1, d_2 \in \mathscr{D}$ we write $(d_1, d_2) = d'$; $d_i = d' d_i'$. Then, since d_i is squarefree, $(d_1', d_2') = (d_i', d') = 1$. Also, from $d_i^2 \mid q$ we have $d_i'^2 \mid q, d'^2 \mid q$. It is now helpful to recall (6·2), and to observe that this identity is true with S replaced by S^* as well. Then, by standard methods (see Section 2·6 of Vaughan [**12**]; the proof of Lemma 2·10 there is relevant. Analogues for $S^*(r,b)$ can be found in Hua [**10**]), when $p \nmid q$ we find that

$$\mathfrak{s}_p(q,n,d_1,d_2) = \Big(1-\frac{1}{p}\Big) A(q_0,n) \prod_{\substack{p_1^{t_1}\parallel q\\p_1\mid d_1'd_2'\\t_1\geqslant 2}} A_1(p_1^{t_1},n) \prod_{\substack{p_2^{t_2}\parallel q\\p_2\mid d'\\t_2\geqslant 2}} A_2(p_2^{t_2},n) \tag{8·3}$$

where

$$q_0 = \prod_{\substack{\pi^t\parallel q\\\pi\nmid d_1d_2}} \pi^t$$

and

$$A(r,n) = \sum_{\substack{a=1\\(a,r)=1}}^{r} S(r,a)^2 S^*(r,a)^2\,e\Big(-\frac{an}{r}\Big), \tag{8·4}$$

$$A_1(p^t,n) = \sum_{\substack{a=1\\p\nmid a}}^{p^t} S(p^t,a)^2 S^*(p^t,a)\,S^*(p^t,ap^3)\,e\Big(-\frac{an}{p^t}\Big), \tag{8·5}$$

$$A_2(p^t,n) = \sum_{\substack{a=1\\p\nmid a}}^{p^t} S(p^t,a)^2 S^*(p^t,ap^3)^2\,e\Big(-\frac{an}{p^t}\Big).$$

By Lemma 4 of Hua [**10**], when $\pi \nmid b$,

$$S^*(\pi^t,b) = 0 \quad \text{when} \quad \pi \neq 3, t \geqslant 2 \quad \text{or} \quad \pi = 3, t \geqslant 3. \tag{8·6}$$

In particular, by (8·5), $A_1(p^t,n) = 0$ for $t \geqslant 2$. When $d_1 \neq d_2$ one has $d_1' d_2' > 1$ so that the first product in (8·3) is non-empty, and hence vanishes. This gives Lemma 7 when $p \nmid q$.

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When $p \| q$ the above argument needs only minor adjustments. In the same notation, and by the same reasoning we obtain this time that

$$\mathfrak{s}_p(q,n,d_1,d_2) = -\frac{1}{p}A_3(p,n)A(q_0/p,n)\prod_{\substack{p_1^{t_1}\| q/p\\ p_1 | d_1' d_2'\\ t_1\geqslant 2}} A_1(p_1^{t_1},n)\prod_{\substack{p_2^{t_2}\| q/p\\ p_2 | d'\\ t_2\geqslant 2}} A_2(p_2^{t_2},n) \tag{8·7}$$

where $$A_3(p,n) = \sum_{a=1}^{p-1} S(p,ap^3)^2 S^*(p,ap^3)^2 e\left(-\frac{an}{p}\right) = p^2(p-1)^2 c_p(n). \tag{8·8}$$

Here $c_r(n)$ denotes Ramanujan's sum

$$c_r(n) = \sum_{\substack{a=1\\(a,r)=1}}^{r} e\left(-\frac{an}{r}\right) = \phi(r)\frac{\mu(r/(r,n))}{\phi(r/(r,n))}, \tag{8·9}$$

the identity implicit in (8·9) being a consequence of theorems 271 and 67 of Hardy and Wright[7]. Again the first product in (8·7) vanishes for $d_1 \neq d_2$. This establishes Lemma 7.

We also require upper bounds for $\mathfrak{s}_p(q,n,d)$ which we summarize in the next Lemma.

LEMMA 8. *Let $d^2 | q$, $d \in \mathscr{D}$ and $Y < p \leqslant 2Y$. If $p \nmid q$, then*

$$q^{-2}d^{-4}\phi\left(\frac{q}{d}\right)^{-2}\mathfrak{s}_p(q,n,d) \ll q^{-\frac{5}{4}}d^{-\frac{1}{4}}(q,n).$$

If $q = pr$ with $p \nmid r$, then

$$q^{-2}d^{-4}\phi\left(\frac{q}{d}\right)^{-2}\mathfrak{s}_p(q,n,d) \ll p^{-1}|c_p(n)|\, r^{-\frac{5}{4}}d^{-\frac{1}{4}}(r,n).$$

Proof. We may recall the identities (8·3) and (8·7) which are simpler in the present context since the first products are now empty (that is, are equal to one) and $d = d'$. Therefore it suffices to estimate $A(q,n)$ and $A_2(p^t,n)$. By a standard argument (for which we again refer to Section 2·6 of Vaughan[**12**] for details), $A(q,n)$ is a multiplicative function of q. It now suffices to consider $q = \pi^t$. Let $\pi \neq 3$ for the moment. Then, by (8·6), $A(\pi^t,n) = 0$ for $t \geqslant 2$. But $S^*(\pi,a) = S(\pi,a) - 1$ so that

$$A(\pi,n) = B_4(\pi,n) - 2B_3(\pi,n) + B_2(\pi,n)$$

where $$B_k(\pi,n) = \sum_{a=1}^{\pi-1} S(\pi,a)^k e\left(-\frac{an}{q}\right).$$

From lemma 4·7 of Vaughan[**12**] we now deduce that

$$A(\pi,n) \ll \pi^{\frac{5}{2}}(\pi,n)^{\frac{1}{2}}. \tag{8·10}$$

For $\pi = 3$ it now suffices to refer to (8·6) again in order to extract from this the estimate

$$A(r,n) \ll r^{\frac{5}{2}+\epsilon}(r,n)^{\frac{1}{2}}. \tag{8·11}$$

We now turn our attention to $A_2(p^t,n)$. From (8·6) we immediately see that

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$S^*(p^t, ap^3) = 0$ for $t \geqslant 5$ and $p \nmid a$. Also $S(p^4, a) = 0$ when $p \nmid a$. This follows from lemmata 4·3 and 4·4 of Vaughan [12]. Hence

$$A_2(p^t, n) = 0 \quad \text{when} \quad t \geqslant 4. \tag{8·12}$$

Thus we may concentrate on the cases $t = 2$ and $t = 3$. Here we have

$$S^*(p^t, p^3 a) = (p-1)^t \quad \text{and} \quad S(p^t, a) = p^{t-1}$$

(the latter is shown in lemma 4·4 of Vaughan [12]). Therefore

$$A_2(p^t, n) = (p-1)^{2t} p^{2t-2} c_{p^t}(n) \quad \text{when} \quad 2 \leqslant t \leqslant 3. \tag{8·13}$$

Let $d^2 \mid q$ and $p \nmid q$. As in (8·3) let q_0 be the largest divisor of q coprime to d. In view of (8·12) we may suppose that for all $p' \mid (d, q)$ we have $p'^4 \nmid q$, and thus we may write $d = \delta_2 \delta_3$ where

$$\delta_k = \prod_{\substack{p \mid d \\ p^k \parallel q}} p.$$

From (8·3), (8·10) and (8·13),

$$\mathfrak{s}_p(q, n, d) \ll q_0^{\frac{5}{2}+\epsilon} (q_0, n)^{\frac{1}{2}} |c_{\delta_2^2 \delta_3^3}(n)| \, \delta_2^6 \, \delta_3^{10}$$

(note that $c_r(n)$ is a multiplicative function of r; compare (8·9)). By (8·9),

$$|c_{\delta_2^2}(n)| = \prod_{\substack{p \mid \delta_2 \\ p \parallel n}} p \prod_{\substack{p \mid \delta_2 \\ p^2 \mid n}} \phi(p^2) \leqslant (\delta_2^2, n).$$

Similarly, $|c_{\delta_3^3}(n)| \leqslant (\delta_3^3, n)$. Observe that the three numbers q_0, δ_2, δ_3 are mutually coprime, and therefore, since $q = q_0 \delta_2^2 \delta_3^3$,

$$q^{-2} d^{-4} \phi\left(\frac{q}{d}\right)^{-2} \mathfrak{s}_p(q, n, d) \ll q_0^{\epsilon - \frac{3}{2}} (q, n) \, \delta_2^{-4} \, \delta_3^{-4}$$

which is more than required.

Now let $q = pr$ with $p \nmid r$. In this case we can use (8·7) and (8·8). Since $(d, p) = 1$ we have $d^2 \mid r$. Thus the claim of Lemma 8 emerges from the same argument as before.

We proceed by removing unwanted terms from (8·2). To this end it suffices to have at hand the simplest estimate for $\mathfrak{I}_p(q, n, d)$ when $Y < p \leqslant 2Y$, namely

$$\mathfrak{I}_p(q, n, d) \ll P^{-2} Q^3 d^{-2} \tag{8·14}$$

which is available from (8·1), (5·6), (4·2) and (3·10). Thus we see from (8·14) and Lemma 8 that terms with $p \parallel q$ contribute to (8·2) an amount which does not exceed

$$P^{-2} Q^3 \sum_{Y < p \leqslant 2Y} p^{-1} |c_p(n)| \sum_{r \leqslant X/p} r^{-\frac{5}{4}} (r, n) \ll P^{-2} Q^3 n^{\epsilon}.$$

Now consider the contribution from terms with $p \nmid q$ to (8·2). Recall that $d \in \mathscr{D}$ and $d > 1$ implies $d \geqslant Q^{\eta}$. Any $q \leqslant X$ has only $O(1)$ prime divisors $d \in \mathscr{D}$. Thus we see from (8·14) and Lemma 8 that the contribution to (8·2) arising from terms with $p \nmid q$ and $d > 1$ is

$$YP^{-2} Q^{3 - \frac{1}{4}\eta} \sum_{q \leqslant X} q^{-\frac{5}{4}} (q, n) \ll YP^{-2} Q^{3 - \frac{1}{5}\eta}$$

for $N < n \leqslant 2N$. Similarly we see that the contribution to (8·2) from terms with $p \nmid q$ and $q > Y$ is

$$\ll Y^{\frac{3}{4}} P^{-2} Q^3 \sum_{q \leqslant X} q^{-1} (q, n) \ll Y^{\frac{3}{4}} P^{\epsilon - 2} Q^3$$

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when $N < n \leqslant 2N$. Now, for $q \leqslant Y$ we always have $p \nmid q$. Collecting together, we see from (8·2) and (8·3) that

$$\nu^{**}(n) = \sum_{Y<p\leqslant 2Y} \sum_{q\leqslant X} (q\phi(q))^{-2} \mathfrak{s}_p(q,n,1)\,\mathfrak{I}_p(q,n,1) + O(YP^{-2}Q^{3-\frac{1}{5}\eta})$$

$$= \sum_{q\leqslant Y} \frac{A(q,n)}{(q\phi(q))^2} \sum_{Y<p\leqslant 2Y} \left(1-\frac{1}{p}\right)\mathfrak{I}_p(q,n,1) + O(YP^{-2}Q^{3-\frac{1}{5}\eta}). \tag{8·15}$$

This formula still has two disadvantages. First of all $\mathfrak{I}_p(q,n,1)$ still depends on q. Also it involves a product of the exponential integrals (4·2) and (3·10), and the exponential sum (5·6). The latter problem could have been avoided by using either integrals or sums throughout but the difficulties which would then have arisen at earlier stages outweigh the simple argument which is now required. On writing

$$u(\beta,\Xi) = \tfrac{1}{3} \sum_{\Xi^3<x\leqslant 8\Xi^3} x^{-\frac{2}{3}} e(\beta x) \tag{8·16}$$

we see from (4·2), (3·10) and Euler's summation formula that

$$v(\beta) = u(\beta,P) + O(1+P|\beta|),$$

$$w(p^3\beta) = p^{-1}u(\beta,Qp) + O(1+P|\beta|)$$

when $Y < p \leqslant 2Y$ as we shall henceforth assume. The upper bound

$$u(\beta,\Xi) \ll \Xi(1+\Xi^3|\beta|)^{-1}$$

is provided by lemma 6·2 of Vaughan [**12**]. Hence for $q \leqslant Y$ we find from (8·1) that

$$\mathfrak{I}_p(q,n,1) = \mathfrak{I}_p^*(n) + O(P^{-2}Q^3YX^{-1}),$$

where

$$\mathfrak{I}_p^*(n) = \frac{1}{p}\int_{-\frac{1}{2}}^{\frac{1}{2}} u(\beta,P)\,u(\beta,Qp)\,t_1(p^3\beta)^2\,e(-\beta n)\,d\beta. \tag{8·17}$$

Now, by (8·11) and (8·15),

$$\nu^{**}(n) = \sum_{q\leqslant Y} (q\phi(q))^{-2}A(q,n) \sum_{Y<p\leqslant 2Y} \left(1-\frac{1}{p}\right)\mathfrak{I}_p^*(n) + O(YP^{-2}Q^{3-\frac{1}{5}\eta}).$$

It is obvious that $\mathfrak{I}_p^*(n) \ll P^{-2}Q^3$, and we may use (8·11) again to find that

$$\nu^{**}(n) = \mathfrak{S}_1(n) \sum_{Y<p\leqslant 2Y} \left(1-\frac{1}{p}\right)\mathfrak{I}_p^*(n) + O(YP^{-2}Q^{3-\frac{1}{5}\eta}) \tag{8·18}$$

where

$$\mathfrak{S}_1(n) = \sum_{q=1}^{\infty} (q\phi(q))^{-2}A(q,n). \tag{8·19}$$

By (8·17), (8·16) and (5·6),

$$\mathfrak{I}_p^*(n) = \frac{1}{81p} \sum_{x,y,z_1,z_2} (xyz_1z_2)^{-\frac{2}{3}} (\log z_1 \log z_2)^{-s-1}$$

where the sum is over all integer quadruples x, y, z_1, z_2 subject to

$$P^3 < x \leqslant 8P^3; \quad (Qp)^3 < y \leqslant 8(Qp)^3; \quad C_1 Q^3 < z_1, z_2 \leqslant Q^3;$$

$$x+y+z_1+z_2 = n.$$

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In particular, $\mathfrak{I}_p^*(n)$ is real and non-negative. Moreover, when $N < n \leqslant 2N$ and $Y < p \leqslant (1+\eta)Y$ the conditions for x, y, z_1, z_2 are satisfied by at least $\gg P^3Q^6$ quadruples, and for any solution,

$$(xyz_1z_2)^{-\frac{2}{3}}(\log z_1 \log z_2)^{-s-1} \gg P^{-4}Q^{-4}(\log Q)^{-2s-2}.$$

It follows that

$$\sum_{Y<p\leqslant 2Y}\left(1-\frac{1}{p}\right)\mathfrak{I}_p(n) \gg P^{-1}Q^2(\log Q)^{-2s-2}\sum_{Y<p\leqslant 2Y}\frac{1}{p}\left(1-\frac{1}{p}\right)$$
$$\gg P^{-2}Q^3Y(\log Q)^{-2s-3}. \tag{8·20}$$

It remains to discuss the singular series $\mathfrak{S}_1(n)$ which is the formal singular series in the problem of representing n as the sum of two cubes and two cubes of primes. By (8·19), (8·4) and (8·6), recalling that $A(q,n)$ is multiplicative, we have

$$\mathfrak{S}_1(n) = \left(1+\tfrac{1}{36}A(3,n)+\frac{1}{54^2}A(9,n)\right)\prod_{\pi\neq 3}(1+\pi^{-2}\phi(\pi)^{-2}A(\pi,n)). \tag{8·21}$$

By a standard argument (see lemma 2·12 of Vaughan [12]),

$$1+\pi^{-2}\phi(\pi)^{-2}A(\pi,n) = \pi^{-1}\phi(\pi)^{-2}M_n(\pi)$$

where $M_n(q)$ is the number of incongruent solutions of the congruence

$$x_1^3+x_2^3+y_1^3+y_2^3 \equiv n \pmod q$$

in residues x_1, x_2 and reduced residues y_1, y_2, modulo q. A similar relation exists between the first factor in (8·21) and $M_n(9)$, and we deduce that $\mathfrak{S}_1(n)$ is real and non-negative. It is not difficult to see that $M_n(\pi) \geqslant 1$ for all $\pi \neq 3$ and all n, and $M_n(9) \geqslant 1$ for all n (see Roth [11] for details in an even more ambitious situation). Now we deduce from (8·21) and (8·10) that for some constant $C > 0$ one has

$$\mathfrak{S}_1(n) \gg \prod_{\pi>2C}(1-C\pi^{-\frac{3}{2}})\prod_{\substack{\pi>2C\\ \pi\mid n}}(1-C\pi^{-1}) \gg (\log\log n)^{-C}.$$

Combining this with (8·20) and (8·18) we are led to the lower bound

$$\nu^{**}(n) \gg YP^{-2}Q^3(\log Q)^{-2s-3}(\log\log Q)^{-C}$$

which is valid for $N < n \leqslant 2N$. Proposition 2 is now available from (5·11) and Lemma 2.

9. *The approach to Theorem* 2

We shall now briefly indicate the necessary changes in the above argument to prove Theorem 2. We try to keep as much notation from earlier sections as possible. A major change is that we now put $Q = [(\frac{1}{3}N)^{\frac{1}{3}}]$. Using (2·4) let

$$\nu'(n,\mathfrak{B}) = \int_{\mathfrak{B}} g(\alpha)^2 h(\alpha)^2 e(-\alpha n)\,d\alpha$$

where $\mathfrak{B}$ is a measurable set of real numbers. Let $\mathfrak{U} = [Q^{-2}, 1+Q^{-2}]$. Then $\nu'(n,\mathfrak{U})$ equals the number of solutions to

$$x_1^3+x_2^3+z_1^3+z_2^3 = n$$

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subject to $Q < x_i \leqslant 2Q$ and $z_i \in \mathscr{A}$. The object is to show that

$$\nu'(n, \mathfrak{U}) \gg Q(\log Q)^{-2s-3}$$

for all but $O(N^{\frac{11}{12}+\epsilon})$ values of n in the range $N < n \leqslant 2N$. This would prove Theorem 2.

Let $\mathfrak{M}(q,a)$ denote the interval $|q\alpha - a| \leqslant Q^{-2}$, and write $\mathfrak{M}$ for the union of all $\mathfrak{M}(q,a)$ subject to $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$. Let $\mathfrak{m} = \mathfrak{U} \backslash \mathfrak{M}$. By Weyl's inequality (see Vaughan [**12**], lemma 2·4) we have $g(\alpha) \ll Q^{\frac{3}{4}+\epsilon}$ for $\alpha \in \mathfrak{m}$. Therefore, by Bessel's inequality and Lemma 1 (in the form of its corollary (3·4)),

$$\sum_{N < n \leqslant 2N} |\nu'(n, \mathfrak{m})|^2 \ll \int_{\mathfrak{m}} |g(\alpha)\, h(\alpha)|^4 \, d\alpha \ll Q^{\frac{19}{4}+\epsilon} \ll Q^8 N^{-1-\frac{1}{12}+\epsilon}$$

which is already the appropriate analogue of Proposition 1.

It remains to treat $\nu'(n, \mathfrak{M})$. Using (3·11) and (3·12) it is not difficult to show that

$$\int_{\mathfrak{M}} |(g(\alpha)^2 - g^*(\alpha)^2)\, h(\alpha)^2|^2 \, d\alpha \ll Q^8 N^{-1-\frac{1}{12}+\epsilon} \tag{9·1}$$

(for the corresponding work for Theorem 1 see Section 4, but (9·1) is considerably simpler). Now we wish to prune the set $\mathfrak{M}$. Let $\mathfrak{M}_1$ be the union of all $\mathfrak{M}(q,a)$ subject to $1 \leqslant a \leqslant q \leqslant X$ and $(a,q) = 1$ where we redefine X as $X = Q^{\frac{3}{8}}$. By (3·14),

$$g^*(\alpha) \ll q^{\epsilon - \frac{1}{2}} Q \left(1 + Q^3 \left|\alpha - \frac{a}{q}\right|\right)^{-1}$$

for $\alpha \in \mathfrak{M}(q,a)$. By (3·20), (3·21) and Schwarz's inequality we find that

$$\int_{\mathfrak{M} \backslash \mathfrak{M}_1} |g^*(\alpha)\, h(\alpha)|^4 \, d\alpha \ll Q^{2-\frac{1}{4}+\epsilon} (Q^{1+\epsilon})^{\frac{1}{2}} (Q^{5+\epsilon})^{\frac{1}{2}} \ll Q^8 N^{-\frac{13}{12}+\epsilon}.$$

Put

$$\nu''(n) = \int_{\mathfrak{M}_1} g^*(\alpha)^2 h(\alpha)^2 \, e(-\alpha n) \, d\alpha.$$

Then, as in the proof of Lemma 2 we see that for all but $O(N^{\frac{11}{12}+\epsilon})$ values of n in $N < n \leqslant 2N$ we have

$$\nu'(n, \mathfrak{M}) = \nu''(n) + O(Q^{1-\epsilon}).$$

Let $h^*(\alpha)$ be given by (5·7). Then, the method presented in Sections 5–7 is readily adapted to show that

$$\nu''(n) = \nu'''(n) + O(Q(\log Q)^{-A}) \tag{9·2}$$

for all $A > 0$, where

$$\nu'''(n) = \int_{\mathfrak{M}_1} g^*(\alpha)^2 h^*(\alpha)^2 \, e(-\alpha n) \, d\alpha.$$

It might be of interest that one can prove (9·2) with much larger values of X; in fact the method from Section 7 works as long as $X \leqslant Q^{\frac{2}{3}-\epsilon}$.

Imitating the work of Section 8 one finds that

$$\nu'''(n) \gg Q(\log Q)^{-2s-2} (\log\log Q)^{-C} \quad (N < n \leqslant 2N)$$

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for some $C > 0$. Thus the same lower bound holds for $\nu'(n, \mathfrak{M})$ for all but $O(N^{\frac{11}{12}+\epsilon})$ values of n in the same range. This establishes Theorem 2.

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