Math. Proc. Camb. Phil. Soc. (1990), **108**, 203
Printed in Great Britain

Scattering operator and Eisenstein integral for Kleinian groups

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(*Received* 18 *October* 1989)

1. *Introduction*

In this paper we study certain aspects of the problem of analytic continuation of the scattering operator and Eisenstein integral which we introduced in [**7**, **8**, **10**], for Kleinian groups Γ with exponent of convergence $\delta(\Gamma) \geqslant 1$. The corresponding problem for groups with $\delta(\Gamma) < 1$ was examined and solved in [**9**].

The present study is based on the techniques developed in [**10**]. In order to avoid unnecessary technical complications we have worked with Kleinian groups acting on $\mathbb{H}^3$, the 3-dimensional hyperbolic space. The same methods however apply exactly in the same way to Kleinian groups acting on any higher dimensional hyperbolic space $\mathbb{H}^{n+1}$, as described in ([**10**]; epilogue).

The structure of the paper is as follows. In Section 2 we recall all necessary analytical prerequisites from [**10**]. In Section 3 we show that the scattering operator S_s, $\mathrm{Re}\,(s) > \delta(\Gamma)$ introduced in [**7**, **10**] can be continued analytically in the region $\mathrm{Re}\,(s) > 1$. In Section 4 we show that the Eisenstein integral has an analytic continuation in the same region $\mathrm{Re}\,(s) > 1$. Finally in Section 5 we investigate the behaviour of the scattering operator as we approach the line $\mathrm{Re}\,(s) = 1$ which carries the continuous spectrum. One knows that this is not a trivial problem. We have shown that the L^2-norm of the compact part of the scattering operator in the L^2-'desingularization' of it ([**10**], §3) remains finite as we approach the line of the continuous spectrum. This has been obtained by using the second 'Maass-Selberg' relation developed in [**10**].

The present work, which is part of a series of papers [**8**, **9**, **10**] developing the theory of Eisenstein series and spectral theory for Kleinian groups, extends previous work of Elstrodt [**2**], Patterson [**12**] and Fay [**3**] on Eisenstein series and spectral theory for Fuchsian groups. Here we have not used the method of lifting the action of the discrete group on a higher-dimensional hyperbolic space to obtain the analytic continuation, as described in [**8**, **9**]. An outline of the whole theory of Eisenstein series for Kleinian groups and their applications, especially to spectral theory of hyperbolic manifolds, can be found in [**8**]. (See also [**13**], lecture 4.)

The process of getting the whole work published has been very slow on my part. It finally happened only after a strong recommendation from various people to whom I am particularly indebted. Among them I would like to mention S. J. Patterson, P. Sarnak and E. B. Davies in chronological order. Meanwhile other recent studies on the subject have now appeared, or are going to appear, in print: [**11**, **15**, **16**]. In these, the authors have dealt with certain aspects of the problem, using different methods. [**11**] deals with spaces with asymptotically constant negative curvature, but it cannot produce a functional equation. For that reason it has a

https://doi.org/10.1017/S0305004100069085 Published online by Cambridge University Press

rather complementary character. [15] on the other hand uses *a priori* estimates and stationary scattering theory, while [16] uses a time-dependent scattering theory.

2. *Analytical considerations and prerequisites*

In this section we shall recall all necessary prerequisites from [10].

Let $\mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{R}_+^\times$ be the upper half-space equipped with the Riemannian metric $ds^2 = t^{-2}(|dz|^2 + dt^2)$ at $w = (z, t) \in \mathbb{H}^3$, $z = (x, y) \in \mathbb{C}$, $t > 0$. This is a model for 3-dimensional hyperbolic space. The group of isometries of the structure is $\mathrm{PSL}(2, \mathbb{C})$. To this metric corresponds the invariant Laplacian

$$L = t^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}\right) - t\frac{\partial}{\partial t},$$

so that $-L \geqslant 0$, and the invariant measure $d\sigma(w) = t^{-3}\,dx\,dy\,dt$. Let

$$\sigma(w, w') = (|z - z'|^2 + (t + t')^2)/4tt'$$

be the standard point-pair invariant on $\mathbb{H}^3$. It satisfies $\sigma(gw, gw') = \sigma(w, w')$, for all $w, w' \in \mathbb{H}^3$ and $g \in \mathrm{PSL}(2, \mathbb{C})$.

Now let Γ be a Kleinian group, that is, a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ which satisfies $\Omega(\Gamma) \neq \varnothing$, where $\Omega(\Gamma) \subseteq \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the region of discontinuity of Γ on $\hat{\mathbb{C}}$. We denote by $L(\Gamma) = \hat{\mathbb{C}} \backslash \Omega(\Gamma)$ the limit set of Γ. We assume that Γ is geometrically finite without parabolic elements. The manifold $M(\Gamma) = \Gamma \backslash (\mathbb{H}^3 \cup \Omega(\Gamma))$ is a 3-dimensional manifold with boundary $\partial M(\Gamma) = \Gamma \backslash \Omega(\Gamma)$ a compact manifold; indeed a compact Riemann surface in this case. By a known theorem of Ahlfors [1], $m(L(\Gamma)) = 0$, where m is 2-dimensional Lebesgue measure. The exponent of convergence $\delta(\Gamma)$ of Γ is the invariant

$$\delta(\Gamma) = \inf\{s > 0 : \sum_{\gamma \in \Gamma} \sigma(w_1, \gamma w_2)^{-s} < +\infty\}.$$

Now let $P(w, \zeta) = t/(|z - \zeta|^2 + t^2)$, $w = (z, t) \in \mathbb{H}^3$, $\zeta \in \mathbb{C}$ be the Poisson kernel for $\mathbb{H}^3$. For $\mathrm{Re}\,(s) > \delta(\Gamma)$ and $\zeta \in \Omega(\Gamma)$ we form the Eisenstein series

$$E(w, \zeta; s) = \sum_{\gamma \in \Gamma} P(\gamma w, \zeta)^s,$$

and the 'S-matrix'

$$S(\zeta, \eta; s) = \sum_{\gamma \in \Gamma} |\gamma(\zeta) - \eta|^{-2s} |\gamma'(\zeta)|^s \quad (\eta, \zeta \in \Omega(\Gamma)),$$

which describes the boundary behaviour of $E(w, \zeta; s)$ in the sense that

$$S(\eta, \zeta; s) = \lim_{w \to \eta} t(w)^{-s} E(w, \zeta; s).$$

The Eisenstein series is an eigenfunction of the Laplacian

$$-LE(w, \zeta; s) = s(2 - s)\, E(w, \zeta; s).$$

Next we recall the following two spaces of functions. We use the symbol $X_s(\Gamma)$ to denote either of them. If $s \in \mathbb{C}$,

$$X_s(\Gamma) = \{f(s, .) : f(s, .)$$

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is a complex valued function on $\Omega(\Gamma)$ and satisfies:

(i) $f(s, \gamma\zeta)|\gamma'(\zeta)|^{2-s} = f(s, \zeta)$, for all $\gamma \in \Gamma$;

(ii) either $f(s, .) \in C^\infty(\Omega(\Gamma))$ or $f(s, .) \in L^2_{\text{loc}}\}$.

If $f(s, .) \in C^\infty(\Omega(\Gamma))$, respectively L^2_{loc}, we have denoted the corresponding space by $L_s(\Gamma)$, respectively $\mathscr{L}_s(\Gamma)$. Clearly $L_s(\Gamma) \subset \mathscr{L}_s(\Gamma)$ and the inclusion is dense. For each $s \in \mathbb{C}$ the space $\mathscr{L}_s(\Gamma)$ is isomorphic in $L^2(\Gamma\backslash\Omega(\Gamma))$.

The Eisenstein integral is the operator

$$E(w, ., s): L_s(\Gamma) \to C^\infty(\Gamma\backslash\mathbb{H}^3)$$

which in the region $\text{Re}\,(s) > \delta(\Gamma)$ is given by the expression

$$E(w, f, s) = \int_{\Gamma\backslash\Omega(\Gamma)} E(w, \zeta; s) f(s, \zeta)\, dm\,(\zeta),$$

where dm is Lebesgue measure on $\mathbb{R}^2$.

Similarly the scattering operator

$$S_s: L_s(\Gamma) \to L_{2-s}(\Gamma)$$

is defined in the region $\delta(\Gamma) < \text{Re}\,(s) < 2$ as follows:

$$S_s(f)(z) = \lim_{w \to z} t(w)^{-s}\left[E(w, f, s) - \frac{\pi}{s-1} f(s, z)\, t(w)^{2-s}\right].$$

This definition can be extended to the region $\text{Re}\,(s) > \delta(\Gamma)$ (see [**10**], §3).

If $A \in L_1(\Gamma)$ then $\tilde{S}_s(f)(z) = A(z)^{-s} S_s(f)(z)$ is an automorphic function with respect to Γ. We showed in [**10**], §3 that one can split $\tilde{S}_s$ as $\check{S}_s + \hat{S}_s$. The operator $\check{S}_s$ is defined for all s and depends meromorphically on s. It is a continuous $L^2(\Gamma\backslash\Omega(\Gamma))$ operator for $\text{Re}\,(s) < 1$ and a singular integral operator for $\text{Re}\,(s) = 1$ which can be extended to a bounded $L^2(\Gamma\backslash\Omega(\Gamma))$ operator. The operator $\hat{S}_s$ on the other hand, which reflects the character of the group Γ, is defined originally in the region $\text{Re}\,(s) > \delta(\Gamma)$ and it is a compact $L^2(\Gamma\backslash\Omega(\Gamma))$ operator. The above split is an L^2-'desingularization' of S_s in the terminology of [**10**].

Now we need to recall the Inner Product Formula and the 'Maass-Selberg' relations for Kleinian groups. If

$$E(w, f) = \int_{(\sigma)} E(w, f, s)\, |ds|,$$

where $(\sigma) = \{s : \text{Re}\,(s) = \sigma\}$, then under suitable conditions the following 'inner Product formula' (see [**7**, **10**]), holds:

$$\langle E(.\,, f_1), E(.\,, f_2)\rangle_{L^2(\Gamma\backslash H^3,\, d\sigma)} = \int_{\Gamma\backslash\Omega(\Gamma)} \int_{(\sigma)} \frac{2\pi^3}{(s-1)(1-s)} f_1(s, \zeta)\bar{f}_2(2-\bar{s}, \zeta)\, |ds|\, dm(\zeta)$$

$$+ \int_{\Gamma\backslash\Omega(\Gamma)} \int_{(\sigma)} \frac{2\pi^2}{s-1} S_s(f_1)(\zeta)\bar{f}_2(\bar{s}, \zeta)\, |ds|\, dm(\zeta). \quad (*)$$

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Now for $\lambda, \lambda_1 \in \mathbb{C}, A \in L_1(\Gamma)$ a fixed function and $f \in L_2(\Gamma)$, let us recall the following two modified complex powers of the Poisson kernel, F_A and $\tilde{F}_A$.

$$F_A(w,\zeta,f;\lambda,\lambda_1) := \begin{cases} \left(\frac{\lambda-1}{\pi}\left(\frac{P(w,\zeta)}{A(\zeta)}\right)^{\lambda} - \frac{\lambda_1-1}{\pi}\left(\frac{P(w,\zeta)}{A(\zeta)}\right)^{\lambda_1}\right) A(\zeta)^2 f(\zeta) & \text{if } P(w,\zeta) < A(\zeta) \\ 0 & \text{if } P(w,\zeta) > A(\zeta), \end{cases}$$

and for $\mathrm{Re}\,(\lambda) > \delta(\Gamma)$

$$\tilde{F}_A(w,\zeta,f;\lambda,\lambda_1) := \begin{cases} \left(\frac{\lambda-1}{\pi}\left(\frac{P(w,\zeta)}{A(\zeta)}\right)^{\lambda} - \frac{\lambda_1-1}{\pi}\left(\frac{P(w,\zeta)}{A(\zeta)}\right)^{\lambda_1}\right) A(\zeta)^2 f(\zeta) & \text{if } P(w;\zeta) < A(\zeta) \\ -\frac{\lambda-1}{\pi}\tilde{S}_{\lambda}(f)(\zeta) A(\zeta)^2 \left(\frac{1-\lambda}{\pi}\left(\frac{P(w,\zeta)}{A(\zeta)}\right)^{2-\lambda} - \frac{1-\lambda_1}{\pi}\left(\frac{P(w,\zeta)}{A(\zeta)}\right)^{2-\lambda_1}\right) & \text{if } P(w,\zeta) > A(\zeta), \end{cases}$$

where $\tilde{S}_{\lambda}(f)(\zeta) := A(\zeta)^{-\lambda} S_{\lambda}(F)(\zeta)$ and $F(\lambda,\zeta) := A(\zeta)^{2-\lambda} f(\zeta)$.

Let us also recall for $\mathrm{Re}\,(\lambda), \mathrm{Re}\,(\lambda_1) > \delta(\Gamma)$ the following two modified Eisenstein integrals [**10**]:

$$E_A(w,f,\lambda,\lambda_1) = \int_{\Gamma\backslash\Omega(\Gamma)} \sum_{\gamma\in\Gamma} F_A(\gamma w,\zeta,f;\lambda,\lambda_1)\, dm(\zeta)$$

and

$$\tilde{E}_A(w,f,\lambda,\lambda_1) = \int_{\Gamma\backslash\Omega(\Gamma)} \sum_{\gamma\in\Gamma} \tilde{F}_A(\gamma w,\zeta,f;\lambda,\lambda_1)\, dm(\zeta).$$

(The reader should notice the slight difference of the above notation with that of [**9**], §6: the above $F_A, \tilde{F}_A, E_A, \tilde{E}_A$ correspond respectively to $\tilde{F}_A, \tilde{\tilde{F}}_A, \tilde{E}_A, \tilde{\tilde{E}}_A$ in the notation of [**9**].

Then the 'Maass-Selberg' relations give the inner product of the above modified Eisenstein integrals in the space $L^2(\Gamma\backslash\mathbb{H}^3, d\sigma)$, in terms of combinations of the functions $f_i, S_s(f_i), \check{S}_s(f_i)$ and $\hat{S}_s(f_i)$, where $i = 1, 2$.

1. *First 'Maass-Selberg' relation* (*see* [**7**, **10**]).

THEOREM 2·1. *Let* $\max(1, \delta(\Gamma)) < \mathrm{Re}\,(\lambda), \mathrm{Re}\,(\mu), \mathrm{Re}\,(\lambda_1), \mathrm{Re}\,(\mu_1)$. *Also let* $A(\zeta) \in L_1(\Gamma)$ *be a fixed positive function and* $f_i(\zeta) \in L_2(\Gamma), i = 1, 2$. *Then the following identity holds*:

$$\begin{aligned} &\langle E_A(.,f_1,\lambda,\lambda_1), E_A(.,f_2,\mu,\mu_1)\rangle_{L^2(\Gamma\backslash H^3,\, d\sigma)} \\ &= \left[\frac{\bar{\mu}_1-\bar{\mu}}{(-2+\bar{\mu}+\lambda)(-2+\bar{\mu}_1+\lambda)} - \frac{\bar{\mu}_1-\bar{\mu}}{(-2+\bar{\mu}+\lambda_1)(-2+\bar{\mu}_1+\lambda_1)}\right]\langle f_1,f_2\rangle_0 \\ &\quad + \frac{\bar{\mu}_1-\bar{\mu}}{(\bar{\mu}-\lambda)(\bar{\mu}_1-\lambda)}\langle \dot{\hat{S}}_{\lambda_1}(f_1), f_2\rangle_0 + \frac{\lambda_1-\lambda}{(\lambda-\bar{\mu})(\lambda_1-\bar{\mu})}\langle f_1, \dot{\hat{S}}_{\mu}(f_2)\rangle_0 \\ &\quad - \frac{\bar{\mu}_1-\bar{\mu}}{(\bar{\mu}-\lambda_1)(\bar{\mu}_1-\lambda_1)}\langle \dot{\hat{S}}_{\lambda_1}(f_1), f_2\rangle_0 - \frac{\lambda_1-\lambda}{(\lambda-\bar{\mu}_1)(\lambda_1-\bar{\mu}_1)}\langle f_1, \dot{\hat{S}}_{\mu_1}(f_2)\rangle_0, \end{aligned}$$

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where $\dot{\hat{S}}_s := \dfrac{s-1}{\pi}\hat{S}_s$ *and* $\langle f, g\rangle_0 := \displaystyle\int_{\Gamma\backslash\Omega(\Gamma)} f(\zeta)\,\overline{g(\zeta)}\,A(\zeta)^2\,dm(\zeta)$,

for $f, g \in L_2(\Gamma)$.

2. *Second 'Maass-Selberg' relation (see* [**7**, **10**]).

THEOREM 2.2. *Let* $\delta(\Gamma) < \mathrm{Re}(\lambda)$, $\mathrm{Re}(\mu)$, $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\mu_1)$ *and* $F_i \in L_s(\Gamma)$, $i = 1, 2$, $F_i(s,\zeta) = A(\zeta)^{2-s} f_i(\zeta)$, *where* $A \in L_1(\Gamma)$ *is fixed and* $f_i \in L_2(\Gamma)$. *Then the following identity holds concerning the second modified Eisenstein integral*:

$$\langle \tilde{E}_A(.,f_1,\lambda,\lambda_1), \tilde{E}_A(.,f_2,\mu,\mu_1)\rangle_{L^2(\Gamma\backslash H^3,\,d\sigma)}$$

$$= \langle E_A(.,f_1,\lambda,\lambda_1), E_A(.,f_2,\mu,\mu_1)\rangle_{L^2(\Gamma\backslash H^3,\,d\sigma)}$$

$$+\left[\frac{\lambda_1-\lambda}{(-2+\lambda+\bar{\mu})(-2+\lambda_1+\bar{\mu})} - \frac{\lambda_1-\lambda}{(-2+\lambda+\bar{\mu}_1)(-2+\lambda_1+\bar{\mu}_1)}\right]\langle \dot{\tilde{S}}_\lambda(f_1), \dot{\tilde{S}}_\mu(f_2)\rangle_0$$

$$-\frac{\lambda_1-\lambda}{(-2+\bar{\mu}+\lambda)(-2+\bar{\mu}+\lambda_1)}\langle \dot{\tilde{S}}_{2-\bar{\mu}}(f_1), \dot{\tilde{S}}_\mu(f_2)\rangle_0 + \frac{\lambda_1-\lambda}{(-2+\lambda+\bar{\mu}_1)(-2+\lambda_1+\bar{\mu}_1)}$$
$$\langle \dot{\tilde{S}}_{2-\bar{\mu}_1}(f_1), \dot{\tilde{S}}_\mu(f_2)\rangle_0$$

$$-\frac{\bar{\mu}_1-\mu}{(-2+\bar{\mu}+\lambda)(-2+\bar{\mu}_1+\lambda)}\langle \dot{\tilde{S}}_\lambda(f_1), \dot{\tilde{S}}_\mu(f_2)\rangle_0 + \frac{\bar{\mu}_1-\mu}{(-2+\bar{\mu}+\lambda_1)(-2+\bar{\mu}_1+\lambda_1)}$$
$$\langle \dot{\tilde{S}}_{\lambda_1}(f_1), \dot{\tilde{S}}_\mu(f_2)\rangle_0$$

$$-\frac{\bar{\mu}_1-\bar{\mu}}{(-2+\bar{\mu}+\lambda)(-2+\bar{\mu}_1+\lambda)}\langle \dot{\tilde{S}}_{2-\lambda}(\dot{\tilde{S}}_\lambda(f_1)), f_2\rangle_0 + \frac{\bar{\mu}_1-\bar{\mu}}{(-2+\bar{\mu}+\lambda_1)(-2+\bar{\mu}_1+\lambda_1)}$$
$$\langle \dot{\tilde{S}}_{2-\lambda_1}(\dot{\tilde{S}}_\lambda(f_1)), f_2\rangle_0$$

$$-\frac{\lambda_1-\lambda}{(-2+\bar{\mu}+\lambda)(-2+\bar{\mu}+\lambda_1)}\langle \dot{\tilde{S}}_{\bar{\mu}}(\dot{\tilde{S}}_\lambda(f_1)), f_2\rangle_0 + \frac{\lambda_1-\lambda}{(-2+\bar{\mu}_1+\lambda)(-2+\bar{\mu}_1+\lambda_1)}$$
$$\langle \dot{\tilde{S}}_{\bar{\mu}_1}(\dot{\tilde{S}}_\lambda(f_1)), f_2\rangle_0$$

$$+\frac{\bar{\mu}_1-\bar{\mu}}{(\bar{\mu}-\lambda)(\bar{\mu}_1-\lambda)}\langle \dot{\tilde{S}}_{2-\lambda}(\dot{\tilde{S}}_\lambda(f_1)), \dot{\tilde{S}}_\mu(f_2)\rangle_0 + \frac{\lambda_1-\lambda}{(\lambda-\bar{\mu})(\lambda_1-\bar{\mu})}\langle \dot{\tilde{S}}_{2-\bar{\mu}}(\dot{\tilde{S}}_\lambda(f_1)), \dot{\tilde{S}}_\mu(f_2)\rangle_0$$

$$-\frac{\bar{\mu}_1-\bar{\mu}}{(\bar{\mu}-\lambda_1)(\bar{\mu}_1-\lambda_1)}\langle \dot{\tilde{S}}_{2-\lambda_1}(\dot{\tilde{S}}_\lambda(f_1)), \dot{\tilde{S}}_\mu(f_2)\rangle_0 - \frac{\lambda_1-\lambda}{(\lambda-\bar{\mu}_1)(\lambda_1-\bar{\mu}_1)}\langle \dot{\tilde{S}}_{2-\bar{\mu}_1}(\dot{\tilde{S}}_\lambda(f_1)), \dot{\tilde{S}}_\mu(f_2)\rangle_0,$$

where $\dot{\tilde{S}}_s := \dfrac{s-1}{\pi}\tilde{S}_s$ *and* $\dot{\check{S}}_s := \dfrac{s-1}{\pi}\check{S}_s$.

PROPOSITION 2·3. *Set in the second 'Maass-Selberg' relation* $\mu = \lambda$, $\mu_1 = \lambda_1$ *and* $f_1 = f_2 = f$. *Then if we make the substitution* $\dot{\tilde{S}}_s = \dot{\hat{S}}_s + \dot{\check{S}}_s$ *the right-hand side becomes quadratic in* $\|S\|_0$ *and for* $\mathrm{Re}(\lambda) < \mathrm{Re}(\lambda_1)$ *the coefficient* $\alpha(\lambda,\lambda_1)$ *of* $\langle S(f), S(f)\rangle_0$ *is negative and equal to*

$$\alpha(\lambda,\lambda_1) = \frac{1}{2}\,\frac{\mathrm{Re}(\lambda)-\mathrm{Re}(\lambda_1)}{(\mathrm{Re}(\lambda_1)-1)\,(\mathrm{Re}(\lambda)-1)}.$$

This is corollary of theorem 9·1 of [**10**].

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Finally, we need to recall two more relations concerning the first and second modified Eisenstein integral.

Let

$$P = \{(w,\zeta) \in \mathbb{H}^3 \times \Omega(\Gamma) : P(w,\zeta) \geqslant A(\zeta)\},$$

where $A \in L_1(\Gamma)$. The set P is a Γ-invariant subset of $\mathbb{H}^3 \times \Omega(\Gamma)$, in the sense that if $(w,\zeta) \in P$ then $(\gamma w, \gamma\zeta) \in P$ for all $\gamma \in \Gamma$.

Let

$$X_P = \begin{cases} 1 & \text{if} \quad (w,\zeta) \in P \\ 0 & \text{otherwise} \end{cases}$$

and

$$\check{E}_A(w,f,\lambda) = \int_{\Gamma\backslash\Omega(\Gamma)} \{\sum_{\gamma\in\Gamma} P(\gamma w,\zeta)^{\lambda} X_P(\gamma w,\zeta)\} F(\lambda,\zeta)\, dm(\zeta),$$

where $F(\lambda,\zeta) = A(\zeta)^{2-\lambda} f(\zeta)$, $f \in L_2(\Gamma)$. Then the following two identities concerning the first and second modified Eisenstein integral hold:

$$E_A(w,f,\lambda,\lambda_1) = \frac{\lambda-1}{\pi}(E(w,F,\lambda) - \check{E}_A(w,f,\lambda)) - \frac{\lambda_1-1}{\pi}(E(w,F,\lambda_1) - \check{E}_A(w,f,\lambda_1));$$

$$\tilde{E}_A(w,f,\lambda,\lambda_1) = E_A(w,f,\lambda,\lambda_1) - \frac{\lambda-1}{\pi}\left(\frac{1-\lambda}{\pi}\check{E}_A(w,\tilde{S}_\lambda(f),2-\lambda) - \frac{1-\lambda_1}{\pi}\check{E}_A(w,\tilde{S}_\lambda(f),2-\lambda)\right).$$

(See [**10**], pp. 57, 62, 63.)

3. *Analytic continuation of the scattering operator*

In this section we shall develop the analytic continuation of the scattering operator S_λ in the region $\{\lambda : \mathrm{Re}\,(\lambda) > 1, \lambda \notin (1, \delta(\Gamma))\}$. We have followed a method which is similar to that of [**4**] and [**5**] for arithmetic groups.

Let $\tilde{H}$ be the space of all functions $f(s,\zeta)$, $s \in \mathbb{C}$, $\zeta \in \Omega(\Gamma)$, which satisfy the conditions of validity of the Inner Product Formula (*). (See also [**10**].) The right-hand side of this formula defines an inner product which is positive semi-definite and which we denote by $\langle .,. \rangle_H$. If we form the quotient space $\tilde{H}/\{f : \langle f,f\rangle_H = 0\}$ and the completion of this space with respect to $\langle .,. \rangle_H$ we obtain a Hilbert space H.

Now let ϕ be a function of s which is bounded and analytic in some strip

$$\mathscr{E} = \{s : |\mathrm{Re}\,(s) - 1| < 1 + \epsilon, \epsilon > 0\},$$

and satisfies the relation

$$\phi(2-s) = \phi(s).$$

Then if $f_1, f_2 \in H$, we have

$$\langle \phi(.) f_1(.,.), f_2(.,.)\rangle_H = \langle f_1(.,.), \phi^*(.) f_2(.,.)\rangle_H$$

where

$$\phi^*(s) = \bar{\phi}(2-\bar{s}).$$

Let us also suppose that $\sup_{s\in\mathscr{E}} |\phi(s)| < k$. Then the function

$$g(s) = (k^2 - \phi^*(s)\,\phi(s))^{\frac{1}{2}}$$

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is analytic and bounded in $\mathscr{E}$. Moreover it satisfies the relations

$$g(2-s) = g(s) \quad \text{and} \quad g^*(s) = g(s).$$

Hence if $f \in H$ we get

$$\langle (k^2 - \phi^*(.)\phi(.))f(.,.), f(.,.)\rangle_H = \langle g(.)f(.,.), g(.)f(.,.)\rangle_H.$$

This implies $$\|\phi f\|_H^2 = k^2\|f\|_H^2 - \|gf\|_H^2$$

and subsequently $$\|\phi f\|_H^2 \leqslant k^2\|f\|_H^2.$$

Consequently the operator $$T: f \mapsto \phi f$$

(i.e. multiplication by the function $\phi(s)$), extends to a bounded linear operator on H with adjoint given by multiplication by ϕ^*; that is

$$T^*: f \mapsto \phi^* f.$$

Thus if $\phi = \phi^*$ this operator is self-adjoint. If $\operatorname{Re}(\lambda) > \delta(\Gamma)$, $\lambda \notin \mathscr{E}$, we may in particular take

$$\phi_\lambda(s) = (\lambda(2-\lambda) - s(2-s))^{-1}.$$

Then since the map $$f \mapsto \phi_\lambda f$$

is a one-to-one map of H onto itself, its range is dense and the associated operator is bounded, because ϕ_λ is a bounded analytic function on $\mathscr{E}$, and self-adjoint. From the above we also deduce that

$$T: f \mapsto s(2-s)f$$

defines an unbounded self-adjoint operator with resolvent given by multiplication by $\phi_\lambda(s)$. Indeed

$$\begin{aligned} R(\lambda(2-\lambda), T)f &= (\lambda(2-\lambda) - T)^{-1}f \\ &= (\lambda(2-\lambda) - s(2-s))^{-1}f. \end{aligned}$$

The following theorem is the main result of this section.

THEOREM 3·1. *Let* $K = \{\lambda : \operatorname{Re}(\lambda) > 1, \lambda \notin (1, \delta(\Gamma))\}$. *Let* $f \in L_2(\Gamma)$, (*that is, f is an automorphic function with respect to* Γ). *Then the operator* $\ddot{S}_\lambda$, (*the compact part of* S_λ), *which has been primarily defined for* $\operatorname{Re}(\lambda) > \delta(\Gamma)$, *can be extended to a continuous linear operator on* $L^2(\Gamma\backslash\Omega(\Gamma))$ *for* λ *in any compact subset of K. Moreover,* $\hat{S}_\lambda(f)$ *is, as an element of the Hilbert space* $L^2(\Gamma\backslash\Omega(\Gamma))$, *an analytic function of* λ *for* $\lambda \in K$.

For the proof we need the following three lemmas.

LEMMA 3·1. *Let* $A \in L_1(\Gamma)$, f, $\psi \in L_2(\Gamma)$ *and*

$$F(s,\zeta) = (s-1)\,e^{s(s-2)}A(\zeta)^{2-s}f(\zeta), \quad \Psi(s,\zeta) = (s-1)\,e^{s(s-2)}A(\zeta)^{2-s}\psi(\zeta).$$

Let T be the operator on H given by

$$T: F \mapsto s(2-s)\,F.$$

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Then if $R(\lambda(2-\lambda), T)$ is the resolvent operator of T, the following identity holds:

$$\langle R(\lambda(2-\lambda), T)F, \Psi\rangle_H = \int_{(\sigma)} \frac{2\pi^3}{(\lambda-s)(2-\lambda-s)} e^{2\pi i(s-2)} |ds| . \langle f, \psi\rangle_0$$

$$+ \int_{(\sigma_2)} \frac{2\pi^3}{(\lambda-s)(2-\lambda-s)} e^{2\pi i(s-2)} \langle \hat{S}_s(f), \psi\rangle_0 |ds| - \frac{2\pi^4}{\lambda-1} e^{2\pi i\lambda(\lambda-2)} \langle \hat{S}_\lambda(f), \psi\rangle_0$$

$$+ \int_{(\sigma_1)} \frac{2\pi^3}{(\lambda-s)(2-\lambda-s)} e^{2\pi i(s-2)} \langle \check{S}_s(f), \psi\rangle_0 |ds| + \frac{2\pi^4}{1-\lambda} e^{2\pi i\lambda(\lambda-2)} \langle \check{S}_{2-\lambda}(f), \psi\rangle_0,$$

where $(\sigma) = \{s : \mathrm{Re}\,(s) = \sigma\}, \quad 1 < \delta(\Gamma) < \sigma < 2, \mathrm{Re}\,(\lambda),$

$$\sigma_2 > \mathrm{Re}\,(\lambda), \quad \sigma_1 < 2 - \mathrm{Re}\,(\lambda),$$

and as before $$\langle f, \psi\rangle_0 = \int_{\Gamma\backslash\Omega(\Gamma)} f(\zeta)\overline{\psi(\zeta)} A(\zeta)^2 \, dm(\zeta).$$

Proof. One should observe that all operators $\check{S}$ and $\hat{S}$ which are involved in the above formula are $L^2(\Gamma\backslash\Omega(\Gamma))$ operators according to theorem 3·1 of [**10**].

We apply the Inner Product Formula to the pair $R(\lambda(2-\lambda), T)F$ and Ψ. We have

$$\langle R(\lambda(2-\lambda), T)F, \Psi\rangle_H = \int_{\Gamma\backslash\Omega(\Gamma)} \int_{(\sigma)} \frac{1}{\lambda(2-\lambda)-s(2-s)} \cdot \left\{ \frac{2\pi^3}{(s-1)(1-s)} F(s,\zeta)\,\overline{\Psi}(2-\bar{s},\zeta) \right.$$

$$\left. + \frac{2\pi^2}{s-1} S_s(F)(\zeta)\,\overline{\Psi}(\bar{s},\zeta) \right\} |ds|\, dm(\zeta) \equiv I + II.$$

On evaluating term I we get

$$I = \langle f, \psi\rangle_0 . \int_{(\sigma)} \frac{2\pi^3}{(\lambda-s)(2-\lambda-s)} e^{2\pi i(s-2)} |ds|.$$

Term II can be evaluated by using the same technique we used in the proof of theorem 8·1, of [**10**]. Let us recall briefly the main steps. Since $\max(1, \delta(\Gamma)) < \sigma < 2$, we can express $S_s(f)$ as an integral over $\mathbb{C}$. If

$$D_R = \{(\eta, \zeta) : |\eta - \zeta| \geqslant (A(\zeta)R)^{-1}\},$$

then

$$II = \int_{\Gamma\backslash\Omega(\Gamma)} \int_{(\sigma)} \frac{2\pi^2(s-1)}{(\lambda-s)(2-\lambda-s)} \lim_{R\to\infty} \left[\int [A(\eta)A(\zeta)|\eta-\zeta|^2]^{-s} . f(\eta) \right.$$

$$\left. \times \chi_{D_R}(\eta) A(\eta)^2 \, dm(\eta) - \frac{\pi}{s-1} f(\zeta) R^{2s-2} \right] e^{2\pi i(s-2)} \overline{\psi(\zeta)} |ds| . A(\zeta)^2 \, dm(\zeta).$$

Since the above integrals converge absolutely we may interchange the order of calculation by first integrating with respect to s. The integral which contains the factor R^{2s-2} gives at the end zero contribution. On the other hand let us recall from [**10**] that the evaluation of the integral of the other term with respect to s depends on whether $(\eta, \zeta) \in E$ (that is, on whether $A(\eta)A(\zeta)|\eta-\zeta|^2 > 1$ or not). On carrying out the calculation we finally obtain the result.

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LEMMA 3·2. *Let f be an automorphic function on $\Omega(\Gamma)$ such that $f \in L^2(\Gamma\backslash\Omega(\Gamma))$. Then the following hold*:

(i) $\hat{S}_s(f)$, $\mathrm{Re}\,(s) > \max(1, \delta(\Gamma))$, *is an analytic function of s, as an element of $L^2(\Gamma\backslash\Omega(\Gamma))$*;

(ii) $\hat{S}_s(f)$, $\mathrm{Re}\,(s) \leqslant 1$, *is also an analytic function of s, as an element of $L^2(\Gamma\backslash\Omega(\Gamma))$.*

Proof. (i) Since $\mathrm{Re}\,(s) > \max(1, \delta(\Gamma))$, we may use the expression which gives $\hat{S}_s(f)$ as an integral over $\mathbb{C}$. If

$$K(\eta, \zeta) = A(\zeta)^{-1} A(\eta)^{-1} |\eta - \zeta|^{-2},$$

then
$$\hat{S}_s(f)(\zeta) = \frac{s-1}{\pi} \int K(\eta, \zeta)^s \chi_E(\eta) f(\eta) A(\eta)^2 \, dm(\eta),$$

where the set E has the same meaning as before and the kernel of the above operator is absolutely integrable.

Let
$$\begin{aligned} I(\zeta) &= \int \frac{\partial}{\partial s} [K(\eta, \zeta)^s] \chi_E(\eta) f(\eta) A(\eta)^2 \, dm(\eta) \\ &= \int K(\eta, \zeta)^s (\log K(\eta, \zeta)) \chi_E(\eta) f(\eta) A(\eta)^2 \, dm(\eta). \end{aligned}$$

An elementary estimate shows that the last integral is absolutely convergent. Hence applying Leibnitz's rule we deduce that

$$I(\zeta) = \frac{\partial}{\partial s} \left(\frac{\pi}{s-1} \hat{S}_s(f)(\zeta) \right).$$

Since the estimate depends on $\mathrm{Re}\,(s)$ we can apply the above with s replaced by $\bar{s}$ and we get

$$\frac{\partial}{\partial \bar{s}} \left(\frac{\pi}{s-1} \hat{S}_s(f) \right) = 0,$$

since $\partial / \partial \bar{s} (K(\eta, \zeta)^s) = 0$. This proves (i).

(ii) This follows in the same way. ∎

LEMMA 3·3. *Let $\Psi(s, \zeta) = (s-1) e^{s(s-2)} A(\zeta)^{2-s} \psi(\zeta)$, where $A \in L_1(\Gamma)$, $\psi \in L_2(\Gamma)$ and $\max(1, \delta(\Gamma)) < \mathrm{Re}\,(s) < 2$. Then $\|\Psi\|_H \leqslant c \|\psi\|_0$, where $c > 0$ is a constant and $\|\cdot\|_0$ is the $L_2(\Gamma\backslash\Omega(\Gamma), A^2\, dm)$ norm.*

Proof. The following holds:

$$\begin{aligned} \|\Psi\|_H^2 &= \int_{\Gamma\backslash\Omega(\Gamma)} \int_{(\sigma)} \frac{2\pi^3}{(s-1)(1-s)} \Psi(s, \zeta) \bar{\Psi}(2-\bar{s}, \zeta) |ds| \, dm(\zeta) \\ &\quad + \int_{\Gamma\backslash\Omega(\Gamma)} \int_{(\sigma)} \frac{2\pi^2}{s-1} S_s(\Psi)(\zeta) \bar{\Psi}(\bar{s}, \zeta) |ds| \, dm(\zeta) \\ &= 2\pi^3 \left(\int_{(\sigma)} e^{2s(s-2)} |ds| \right) \|\psi\|_0^2 \\ &\quad + 2\pi^2 \int_{\Gamma\backslash\Omega(\Gamma)} \int_{(\sigma)} e^{2s(s-2)} (s-1) \tilde{S}_s(\psi)(\zeta) |ds| \, \overline{\psi(\zeta)} A(\zeta)^2 . \, dm(\zeta). \end{aligned}$$

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On evaluating the second term we substitute

$$\tilde{S}_s(\psi)(\zeta) = \lim_{R\to\infty}\left\{\int [A(\eta)A(\zeta)|\eta-\zeta|^2]^{-s}\psi(\eta)\chi_{D_R}(\eta)A(\eta)^2\,dm(\eta) - \frac{\pi}{s-1}\psi(\zeta)R^{2s-2}\right\},$$

and we interchange the order of calculation by first calculating the integrals with respect to s. An easy calculation using the residue theorem shows that the corresponding integral which contains the factor R^{2s-2} is zero. It remains to evaluate the integral which contains the factor $[A(\eta)A(\zeta)|\eta-\zeta|^2]^{-s}$. We have already indicated above how to perform a similar calculation. Repeating the same method once more we obtain

$$II = 2\pi^3\int_{(\sigma_2)} e^{2s(s-2)}\langle\hat{S}_s(\psi),\psi\rangle_0\,|ds| + 2\pi^3\int_{(\sigma_1)} e^{2s(s-2)}\langle\check{S}_s(\psi),\psi\rangle_0\,|ds|,$$

where $\sigma_2 > \delta(\Gamma), \sigma_1 < 1$. But according to [**10**], theorem 3·1 we have the relations

$$|\langle\hat{S}_s(\psi),\psi\rangle_0| \leqslant c_1|s-1|\,\|\psi\|_0^2 \quad (\mathrm{Re}\,(s) > \delta(\Gamma)),$$

and

$$|\langle\check{S}_s(\psi),\psi\rangle_0| \leqslant c_2|s-1|\,\|\psi\|_0^2 \quad (\mathrm{Re}\,(s) \leqslant 1).$$

Consequently we have obtained

$$|II| \leqslant c_3\|\psi\|_0^2,$$

and the result now follows. ∎

Let us now go back to the proof of Theorem 3·1. We recall the formula for the resolvent from Lemma 3·1. The resolvent $R(\lambda(2-\lambda),T)$ is analytic outside the spectrum of T, namely in the domain $\mathrm{Re}\,(\lambda) > 1, \lambda \notin (1,\delta(T)]$, since T is a self-adjoint operator. On the other hand according to Lemma 3·2, the functions $\hat{S}_s(f)$, $\mathrm{Re}\,(s) > \delta(\Gamma)$ and $\check{S}_s(f), \mathrm{Re}\,(s) \leqslant 1$ are analytic functions of s as elements of the space $L^2(\Gamma\backslash\Omega(\Gamma))$, hence they are also weakly analytic. This implies that all terms on the right-hand side of the resolvent formula, apart from the term

$$\frac{2\pi^4}{\lambda-1}e^{2\lambda(\lambda-2)}\langle\hat{S}_\lambda(f),\psi\rangle_0$$

are analytic functions of λ for $\lambda\in K$. As a consequence it follows that $\langle\hat{S}_\lambda(f),\psi\rangle_0$ can be continued as an analytic function of λ for $\lambda\in K$.

Next, we show that $\hat{S}_\lambda$ is a bounded operator on $L^2(\Gamma\backslash\Omega(\Gamma))$, for λ in a compact subset of K. For this we estimate the terms which are involved in the formula for the resolvent (see Lemma 3·1).

The following holds:

$$|\langle R(\lambda(2-\lambda),T)F,\Psi\rangle_H| \leqslant \|R(\lambda(2-\lambda),T\|\,\|F\|_H\,\|\Psi\|_H.$$

But

$$\|R(\lambda(2-\lambda),T)\| \leqslant (\mathrm{dist}\,(\lambda(2-\lambda),\text{ spectrum of } T))^{-1},$$

and this is bounded for λ in any compact subset of K. Moreover according to Lemma 2·3 we have

$$\|F\|_H\,\|\Psi\|_H \leqslant c_1\|f\|_0\,\|\psi\|_0.$$

We estimate the other terms in a straightforward manner and obtain

$$\frac{2\pi^4}{|\lambda-1|}|e^{2\lambda(\lambda-2)}|\,|\langle\hat{S}_\lambda(f),\psi\rangle_0| \leqslant c\|f\|_0\,\|\psi\|_0.$$

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This implies that for λ in a compact subset of K we have

$$\sup_{\|\psi\|_0 \leqslant 1} |\langle \hat{S}_\lambda(f), \psi \rangle_0| \leqslant c(\lambda) \|f\|_0,$$

which implies $$\|\hat{S}_\lambda(f)\|_0 \leqslant c(\lambda) \|f\|_0.$$

Finally, since $\hat{S}_\lambda(f) \in L^2(\Gamma\backslash\Omega(\Gamma))$ for $\lambda \in K$, and in a Hilbert space weak and strong analyticity are equivalent, we deduce from the fact that $\langle \hat{S}_\lambda(f), \psi \rangle_0$ is an analytic function of λ for $\lambda \in K$ that $\hat{S}_\lambda(f)$ itself is analytic for $\lambda \in K$. This completes the proof of the theorem. ∎

4. *Analytic continuation of the Eisenstein integral*

In this section we shall show how to obtain the analytic continuation of the Eisenstein integral in the region $\{\lambda : \mathrm{Re}\,(\lambda) > 1, \lambda \notin (1, \delta(\Gamma)]\}$.

Let us recall the first 'Maass-Selberg' relation. This formula is valid for $\mathrm{Re}\,(\lambda)$, $\mathrm{Re}\,(\mu)$, $\mathrm{Re}\,(\lambda_1)$, $\mathrm{Re}\,(\mu_1) > \delta(\Gamma)$. In particular for $\mu = \lambda$, $\mu_1 = \lambda_1$, and $f_1 = f_2 = f \in L^2(\Gamma\backslash\Omega(\Gamma))$ we see that $$\|E_A(.\,, f, \lambda, \lambda_1)\|_{L^2(\Gamma\backslash\mathbb{H}^3)} < +\infty.$$

Let us regard the right-hand side of this formula as a function $g(\lambda, \bar{\mu})$ of two complex variables. (We keep λ_1, μ_1 fixed with $\mathrm{Re}\,(\lambda_1), \mathrm{Re}\,(\mu_1) > \delta(\Gamma)$.) As we proved in Section 3, $\hat{S}_\lambda(f)$ can be continued analytically as a function of λ with values in $L^2(\Gamma\backslash\Omega(\Gamma))$, in the region $\{\lambda : \mathrm{Re}\,(\lambda) > 1, \lambda \notin (1, \delta(\Gamma)]\}$. Let λ_0 be any point with $\mathrm{Re}\,(\lambda_0) > \delta(\Gamma)$. Then we deduce that $g(\lambda, \bar{\mu})$ is a holomorphic function of $\lambda, \bar{\mu}$ whenever

$$|\lambda - \lambda_0| < \min\,(\mathrm{Re}\,(\lambda_0) - 1, |\mathrm{Im}\,(\lambda_0)|),$$

$$|\bar{\mu} - \bar{\lambda}_0| < \min\,(\mathrm{Re}\,(\lambda_0) - 1, |\mathrm{Im}\,(\bar{\lambda}_0)|),$$

because the circle $$|\lambda - \lambda_0| = \min\,(\mathrm{Re}\,(\lambda_0) - 1, |\mathrm{Im}\,(\lambda_0)|)$$

does not touch the lines of the spectrum $\mathrm{Re}\,(\lambda) = 1, \mathrm{Im}\,(\lambda) = 0$. Then according to the theory of holomorphic functions of many complex variables we have the relation

$$\overline{\lim_{m \to \infty}} \left| \frac{1}{(m!)^2} \left(\frac{\partial}{\partial \lambda}\right)^m \left(\frac{\partial}{\partial \bar{\mu}}\right)^m g(\lambda, \bar{\mu}) \right|^{1/2m}_{\mu - \lambda = \lambda_0} \leqslant (\min\,(\mathrm{Re}\,(\lambda_0) - 1, |\mathrm{Im}\,(\lambda_0)|))^{-1}.$$

(This is a simple consequence of the fact that the Taylor series of $g(\lambda, \bar{\mu})$ is convergent at $\lambda = \mu = \lambda_0$.) But

$$\left(\frac{\partial}{\partial \lambda}\right)^m \left(\frac{\partial}{\partial \bar{\mu}}\right)^m g(\lambda, \bar{\mu}) = \left\langle \frac{\partial^m}{\partial \lambda^m} E_A(.\,, f, \lambda, \lambda_1), \frac{\partial^m}{\partial \mu^m} E_A(.\,, f, \mu, \mu_1) \right\rangle_{L^2(\Gamma\backslash\mathbb{H}^3)}. \tag{4·1}$$

Hence

$$\overline{\lim_{n \to \infty}} \left\| \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} E_A(.\,, f, \lambda, \lambda_1) \right\|^{1/n}_{L^2(\Gamma\backslash\mathbb{H}^3)} \leqslant (\min\,(\mathrm{Re}\,(\lambda_0) - 1, |\mathrm{Im}\,(\lambda_0)|))^{-1}.$$

This implies that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left\| \frac{\partial^n}{\partial \lambda^n} E_A(.\,, f, \lambda_0, \lambda_1) \right\|_{L^2(\Gamma\backslash\mathbb{H}^3)} |\lambda - \lambda_0|^n$$

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converges whenever

$$|\lambda-\lambda_0| < \min(\mathrm{Re}\,(\lambda_0)-1, |\mathrm{Im}\,(\lambda_0)|),$$

and therefore that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n} E_A(\omega,f,\lambda,\lambda_1)|_{\lambda=\lambda_0}(\lambda-\lambda_0)^n$$

is strongly convergent for each of the same values of λ in the Hilbert space $L^2(\Gamma\backslash\mathbb{H}^3, d\sigma)$.

After a finite number of steps, since (4·1) persists in any region in $\mathrm{Re}\,(\lambda)$, $\mathrm{Re}\,(\mu) > 1, \lambda, \mu \notin (1, \delta(\Gamma)]$ to which the functions in it can be analytically continued we deduce that

$$\lambda \mapsto E_A(\omega,f,\lambda,\lambda_1), \quad f \in L^2(\Gamma\backslash\Omega(\Gamma)),$$

may be analytically continued as a function with values in $L^2(\Gamma\backslash\mathbb{H}^3)$, to the region $\mathrm{Re}\,(\lambda) > 1, \lambda \notin (1, \delta(\Gamma)]$.

Since

$$\frac{\lambda-1}{\pi}E(\omega,F,\lambda) - E_A(\omega,f,\lambda,\lambda_1) = \frac{\lambda-1}{\pi}\check{E}_A(\omega,f,\lambda) + \frac{\lambda_1-1}{\pi}(E(\omega,F,\lambda_1) - \check{E}_A(\omega,f,\lambda_1))$$

(see the identities at the end of Section 2), and the right-hand side is an analytic function of $\lambda, E(\omega,F,\lambda)$ can itself be continued analytically in the same region. Thus we have obtained the analytic continuation of the Eisenstein integral.

We can go one step further and prove that the function

$$\lambda \mapsto \tilde{E}_A(\omega,f,\lambda,\lambda_1),$$

where $f \in C^\infty(\Gamma\backslash\Omega(\Gamma))$, can also be continued analytically in the above region with values in the space $L^2(\Gamma\backslash\mathbb{H}^3)$. Indeed let us recall from Section 2 the identity

$$\tilde{E}_A(\omega,f,\lambda,\lambda_1) = E_A(\omega,f,\lambda,\lambda_1) - \frac{\lambda-1}{\pi}\left(\frac{1-\lambda}{\pi}\check{E}_A(\omega,\tilde{S}_\lambda(f),2-\lambda)\right.$$

$$\left. - \frac{1-\lambda_1}{\pi}\check{E}_A(\omega,\tilde{S}_\lambda(f),2-\lambda_1)\right), \quad (4\cdot2)$$

which also appeared in examining term $I_{5,3}$ in the proof of theorem 7·1 of [**10**]. As we proved there, the second term of the right-hand side is, for ω near the boundary, equal to

$$\frac{\lambda-1}{\pi}S_\lambda(F)(z)\,t^\lambda - \frac{\lambda-1}{\pi}A(z)^{\lambda_1}\tilde{S}_\lambda(f)\,t^{\lambda_1} + R,$$

(see p. 63 of [**10**]), where R is square-integrable in a neighbourhood of the boundary with respect to the hyperbolic measure $d\sigma$ and λ in the region $1 \leqslant \mathrm{Re}\,(\lambda) < 2$. Since the other two terms of the last expression are square-integrable for $\mathrm{Re}\,(\lambda) > 1$, we deduce that the second term on the right-hand side of (4·2) and hence $\tilde{E}_A$ itself is square-integrable near the boundary for $1 < \mathrm{Re}\,(\lambda) < 2$. This implies at once our assertion. The above also imply that both 'Maass-Selberg' relations have an analytic continuation to, and hence are valid in, the region

$$\mathrm{Re}\,(\lambda), \mathrm{Re}\,(\mu) > 1 \quad \text{and} \quad \lambda, \mu \notin (1, \delta(\Gamma)].$$

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We summarize the results of this section in the following theorem.

THEOREM 4·1. (i) *The first modified Eisenstein integral*

$$E_A(\omega, f, \lambda, \lambda_1) \quad (f \in L^2(\Gamma\backslash\Omega(\Gamma)), \operatorname{Re}(\lambda), \operatorname{Re}(\lambda_1) > \delta(\Gamma))$$

may be analytically continued to the region $\operatorname{Re}(\lambda) > 1, \lambda \notin (1, \delta(\Gamma)]$, *as a function with values in* $L^2(\Gamma\backslash\mathbb{H}^3)$.

(ii) *The Eisenstein integral* $E(\omega, ., \lambda), \operatorname{Re}(\lambda) > \delta(\Gamma)$ *has an analytic continuation to the region* $\operatorname{Re}(\lambda) > 1, \lambda \notin (1, \delta(\Gamma)]$.

(iii) *The second modified Eisenstein integral*

$$\tilde{E}_A(\omega, f, \lambda, \lambda_1) \quad (f \in C^\infty(\Gamma\backslash\Omega(\Gamma)), \operatorname{Re}(\lambda), \operatorname{Re}(\lambda_1) > \delta(\Gamma))$$

has an analytic continuation to the region $\operatorname{Re}(\lambda) > 1, \lambda \notin (1, \delta(\Gamma)]$ *with values in* $L^2(\Gamma\backslash\mathbb{H}^3)$.

(iv) *Both the first and second 'Maass-Selberg' relations have an analytic continuation to, and hence they are valid in, the region* $\operatorname{Re}(\lambda), \operatorname{Re}(\mu) > 1, \lambda, \mu \notin (1, \delta(\Gamma)]$.

5. *Approaching the continuous spectrum*

The purpose of this section is to show that as we approach the line of the continuous spectrum $\{\lambda : \operatorname{Re}(\lambda) = 1\}$, the $L^2(\Gamma\backslash\Omega(\Gamma))$ norm of $\dot{S}_\lambda(f), f \in L^2(\Gamma\backslash\Omega(\Gamma))$, remains finite. Indeed the following theorem holds.

THEOREM 5·1. *Let* (λ_n) *be a sequence such that* $\lambda_n \to \lambda_0, \operatorname{Re}(\lambda_n) > 1, \operatorname{Re}(\lambda_0) = 1$, $\lambda_0 \neq 1$. *Then there exists a positive constant* $c(\lambda_0)$ *such that*

$$\limsup_{n\to\infty} \|\dot{S}_{\lambda_n}(f)\|_0 \leqslant c(\lambda_0) \quad (f \in L^2(\Gamma\backslash\Omega(\Gamma))).$$

Proof. Let us recall the second 'Maass-Selberg' relation (Theorem 2·2). We write the right-hand side of this relation in terms of the operators $\check{S}$ and $\hat{S}$ by using the relation

$$\dot{\check{S}}_s = \dot{\check{S}}_s + \dot{\hat{S}}_s.$$

According to Section 4 this relation is now valid for $\operatorname{Re}(\lambda)$, $\operatorname{Re}(\mu) > 1$ and λ, $\mu \notin (1, \delta(\Gamma)]$. (We have already assumed that $\operatorname{Re}(\lambda_1), \operatorname{Re}(\mu_1) > \delta(\Gamma)$.) Putting $\mu = \lambda$, $\mu_1 = \lambda_1, f_1 = f_2 = f \in H^\infty(\Gamma\backslash\Omega(\Gamma))$ into this relation we see that the left-hand side is non-negative. The crux of the matter now lies in Proposition 2·3. Indeed according to this proposition we know that the coefficient $\alpha(\lambda, \lambda_1)$ of $\langle \dot{\hat{S}}_\lambda(f), \dot{\hat{S}}_\lambda(f) \rangle_0$ is negative. This implies that we can estimate $\|\dot{\hat{S}}_\lambda\|_0$. However, there is a point which needs attention.

We proceed as follows. We move the term $\alpha(\lambda, \lambda_1) \|\dot{\hat{S}}_\lambda(f)\|_0^2$ to the left-hand side and we estimate all other terms of the right-hand side by using the inequality

$$|\langle x, y \rangle_0| \leqslant \|x\|_0 \|y\|_0,$$

where $x, y \in H^\infty$. (The left-hand side of this inequality is a continuous conjugate bilinear pairing of $H^{-\infty}$ with H^∞.) But now four more terms appear containing the factor $\|\dot{\hat{S}}_\lambda(f)\|_0^2$ with positive coefficients. Indeed this is the case with the last four terms of the second 'Maass-Selberg' relation. For example the term

$$\frac{\bar{\lambda}_1 - \bar{\lambda}}{(\bar{\lambda} - \lambda)(\bar{\lambda}_1 - \lambda)} \langle \dot{\check{S}}_{2-\lambda}(\dot{\hat{S}}_\lambda(f)), \dot{\hat{S}}_\lambda(f) \rangle_0$$

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is dominated in absolute value by

$$\left|\frac{\bar{\lambda}_1-\bar{\lambda}}{(\bar{\lambda}-\lambda)(\bar{\lambda}-\lambda)}\right| \|\dot{\bar{S}}_{2-\lambda}\|_0 \|\dot{\hat{S}}_\lambda(f)\|_0^2.$$

What saves the situation here is the fact that according to the above-mentioned corollary

$$-\alpha(\lambda,\lambda_1)=\frac{1}{2}\frac{\mathrm{Re}\,(\lambda_1)-\mathrm{Re}\,(\lambda)}{(\mathrm{Re}\,(\lambda_1)-1)(\mathrm{Re}\,(\lambda)-1)},$$

and this tends to ∞ for $\mathrm{Re}\,(\lambda)\to 1$, something which does not happen with the above-mentioned four terms that produce the factor $\|\dot{\hat{S}}_\lambda(f)\|_0^2$.

Finally we obtain an inequality of the form

$$A_0(\lambda,\lambda_1)\|\dot{\hat{S}}_\lambda(f)\|_0^2-A_1(\lambda,\lambda_1)\|\hat{S}_\lambda(f)\|_0-A_2(\lambda,\lambda_1)\|f\|_0^2-A_3(\lambda,\lambda_1)\|f\|_0\|\check{S}_\lambda(f)\|_0\leqslant 0,$$

where all the coefficients $A_i(\lambda,\lambda_1), i=0,1,2,3$ are positive numbers for λ in a sufficiently small neighbourhood of $\lambda_0, \mathrm{Re}\,(\lambda)>1$.

The above inequality implies

$$\|\dot{\hat{S}}_\lambda(f)\|_0\leqslant c(\lambda,\lambda_1),$$

for each $f\in H^\infty(\Gamma\backslash\Omega(\Gamma))$, and $c(\lambda,\lambda_1)$ remains finite for $\lambda\to 0$, since $\|\dot{\bar{S}}_\lambda\|_0$ is well defined and finite on $\mathrm{Re}\,(\lambda)=1, \lambda\neq 1$, according to theorem 3·1(vi) of [**10**]. This completes the proof since H^∞ is dense in L^2. ∎

The author would like to express his sincere gratitude to Professor S. J. Patterson for many valuable comments during the preparation of the work, and he thanks the SERC for financial support. Part of this work was carried out while the author was a Research Fellow at Girton College, Cambridge.

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