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INTERVAL VALUES AND RATIONAL CHOICE

MARTIN PETERSON*

Abstract: Decision theorists have suggested that it is sometimes appropriate to represent the value of certain options as intervals of real numbers. In this note I show that doing so is incompatible with some deeply held intuitions about rational choice.

Keywords: Pasadena problem, Oscillating gamble, Interval value.

1. INTRODUCTION

Decision theorists have suggested that it is sometimes appropriate to represent the value of certain options as intervals of real numbers.[1] In this note I show that doing so is incompatible with some deeply held intuitions about rational choice.

Claims about interval values can be motivated in different ways.[2] Here I focus on Lauwers and Vallentyne's (2016) recent suggestion to assign interval values to some Pasadena-style gambles.[3] The original version of the Pasadena gamble can be formulated as follows:[4] You toss a fair coin n times until it lands heads up for the first time; if

* Department of Philosophy, Texas A&M University, College Station, TX 77840, USA. Email: martinpeterson@tamu.edu. URL: www.martinpeterson.org

[1] Fishburn (1985: Ch. 7) states necessary and sufficient conditions for the existence of real-valued interval representations. However, he does not discuss decision-theoretic aspects of interval representations.

[2] Gert (2004) argues that Chang's notion of parity can be modelled by interval values. Chang (2005) denies this; her argument draws on work reported in Fishburn (1985: 78).

[3] It is worth keeping in mind that the point about interval values articulated in this paper applies no matter how such values are motivated. The scope of the discussion is quite general, as explained in Section 3.

[4] The Pasadena gamble was introduced by Nover and Hajek (2004). See Hajek and Nover (2008) for a helpful discussion of some parts of the literature.

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n is odd you win $(2^n)/n$ units of value, but if n is even you have to pay $(2^n)/n$. It is tempting to claim that the expected value of the Pasadena gamble is $\frac{1}{2} \cdot \frac{2}{1} - \frac{1}{4} \cdot \frac{4}{2} + \frac{1}{8} \cdot \frac{8}{3} - \frac{1}{16} \cdot \frac{16}{4} + \frac{1}{32} \cdot \frac{32}{5} \ldots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots = \sum_n \frac{(-1)^{n-1}}{n}$. However, according to the Riemann Rearrangement Theorem the sum of these terms, known as the alternating harmonic series, depends on the *order* in which they are summed up. If the terms are added together in the manner indicated above the sum is ln 2, but by rearranging the order we can make the sum converge to any real number, including $-\infty$ and $+\infty$.[5] Decision theorists have been puzzled by this mathematical fact. What is the value of something if the sum of its parts depends on the order in which they are added up? Easwaran (2008) famously introduces a distinction between weak and strong expectations.[6] The weak expected value of the Pasadena gamble is ln 2, which Easwaran takes to be the value that should guide our choices. However, as Easwaran himself is aware, some slightly modified versions of the Pasadena gamble have neither strong nor weak expectations.[7] Consider, for instance, the *oscillating gamble* (see Table 1) introduced by Lauwers and Vallentyne (2016: 392).

State	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	...
Probability	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	...
Outcome	2	−4/2	−8/3	16/4	32/5	64/6	128/7	−256/8	...

TABLE 1. The Oscillating Gamble.

In the oscillating gamble the *first* term in the series of possible outcomes is positive, the next *two* are negative, the next *four* are positive, and the next *eight* are negative, and so on. By multiplying the probability of each state with the value of the corresponding outcome we get $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots$. As we keep adding up these terms, the sum is sometimes close to 0 and sometimes close to 1, but it oscillates up and down no matter how long we keep adding the terms together. As explained by Lauwers and Vallentyne, this ensures that the oscillating gamble has no weak expected value.[8] Their proposal is that we should think of the oscillating gamble as having an "interval value". On their view, the value of the oscillating gamble *is* the interval [0,1]. This interval is no mere approximation of some yet unknown point value. The entire

[5] For details, see Nover and Hajek (2004).
[6] I will not restate the distinction here. See Easwaran (2008: 635) for a very clear formulation.
[7] Easwaran (2008: 639).
[8] See Lauwers and Vallentyne (2016: 392) for details.

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interval between 0 and 1, including its endpoints, represents the value of the gamble.

Although Lauwers and Vallentyne do not make any claim about rational choice themselves (but rather about value), nearly all decision theorists would agree that evaluations of options that have nothing to say about how decision makers should *choose* are at best incomplete. In the next section we state a simple yet powerful impossibility result, which is applicable not just to interval values assigned to Pasadena-style gambles but to every interval value no matter how it is derived.

2. AN IMPOSSIBILITY RESULT

If we believe that the value of the oscillating gamble is [0,1], then it is appropriate to ask how a rational agent should choose if offered a choice between something worth [0,1] and some other option worth, say, 1/3 or 1/2 units of value. Lauwers and Vallentyne correctly point out that if the minimum value of some interval [*a*,*b*] exceeds the maximum of some other interval [*c*,*d*], then [*a*,*b*] is better than [*c*,*d*], meaning that it is not rationally permissible to choose the latter option if the former is available.[9] However, this observation does not cover choices between overlapping intervals (e.g. a choice between [0,5] and [2,3]) or between intervals and real-valued points within an interval (e.g. a choice between [0,1] and 0.5). The claim that the oscillating gamble is worth [0,1] is, arguably, hard to evaluate until we know how to choose among *all* possible combinations of options.

In what follows we will consider some general conditions a decision theory for interval values ought to satisfy. To start with, it is reasonable to assume that interval values cannot be reduced to point values: For every interval value [*x*,*y*], with $x \neq y$, there is no real-valued point *a* such that $[x,y] \sim a$, where $\sim$ means that the two options are equally valuable. If the interval value [*x*,*y*] turned out to be worth exactly as much as some real-valued point *a*, then there would be no need for interval values in decision theory. Suppose, for instance, that the interval value of the oscillating gamble, [0,1], is worth exactly 0.5. The value of the oscillating gamble would then simply be that particular real number. This would make the interval value redundant. In the formal impossibility result stated at the end of this section we invoke the following, somewhat stronger version of this idea, in which $a > b$ means that *a* is more valuable than *b*.

Local Incomparability. For every interval value [x,y], where $x \neq y$, there exist at least two real-valued points *a*, *b* such that $a > b$ and:

[9] This holds true irrespective of whether the intervals convey ordinal or cardinal information. If, for instance, [0,1] and [2,3] represent ordinal values, then [2,3] is better than [0,1], and if the numbers convey cardinal information the first interval is exactly 2 units better than the second.

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(i) $\neg([x,y] > a)$ and $\neg([x,y] \sim a)$ and $\neg([x,y] < a)$
(ii) $\neg([x,y] > b)$ and $\neg([x,y] \sim b)$ and $\neg([x,y] < b)$

The motivation for this condition can be broken down into two parts. First, although some point-values are better or worse than some interval values (e.g. a point value that exceeds the maximum of the interval), interval values would be of little interest unless there exists at least *one* point that is neither better, worse, nor equally good as the interval. As noted above, interval values must not be reducible to point values.[10] Second, if there is one such point, there has to be *two*. Why? Because otherwise, for every other point b located arbitrary close to a, the point b would have to be better, worse or equally good as the interval. We can exclude the possibility that b is equally good as the interval, because then the value of the interval could obviously be reduced to a point-value, which we are not willing to accept, as noted above. We are thus left with the possibility that every point $b < a$ is worse than the interval while every $b > a$ is better, which leads to further trouble. To see this, we assume that a rational and fully informed agent is indifferent between options just in case they are of equal value. We then apply Savage's well-known definition of indifference:

> If the person really does regard [two options] as equivalent, that is, if he is indifferent between them, then, if [one of the options] were modified by attaching an arbitrarily small bonus to its consequences in every state, the person's decision would presumably be for whichever act was thus modified. (Savage 1954: 17)

Savage's definition entails that if there is only one point a of the type described in *Local Incomparability*, then the agent would, according to this widely accepted definition of indifference, be indifferent between a and $[x,y]$, because every slightly improved version of a (including b) would be preferred to $[x,y]$. So $[x,y] \sim a$, which contradicts our initial assumption. The upshot is that the distinction between indifference and incomparability requires the existence of *at least two* points that are neither better, worse, or equally good as something else. Incomparability cannot be *fully* local.

The remaining conditions should be familiar to the reader.[11] Let S denote the set of feasible options and let C(S) be a subset of rationally

[10] In addition to stipulating that there is no real-valued point a such that $[x,y] \sim a$, *Local Incomparability* excludes the possibility that every a is better, or worse, than $[x,y]$. The real number line is continuous, so it would be odd to maintain that, for every interval $[x,y]$, the real numbers switch from being better than $[x,y]$ to being worse than $[x,y]$, without ever being equally good as $[x,y]$. See the discussion of Savage's definition of indifference below.

[11] These conditions are similar to those discussed in e.g. Peterson (2007).

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permissible options in S. Our second condition states that it is not permissible to choose an option if some better option is available:

Better. For all $a, b \in S$, if $a > b$ then $b \notin C(S)$

The next condition holds that at least one option in each choice problem is rationally permissible.

No Dilemmas. For every S, there is some a such that $a \in C(S)$.

The fourth and final condition is designed to exclude the possibility that one of two incomparable options is rationally permissible while the other is not. This is probably the most controversial condition. I will discuss it at length in the next section:

Choice Parity. For all $a, b \in S$, if $\neg(a > b)$ and $\neg(a \sim b)$ and $\neg(b < a)$, then $a \in C(a)$ iff $C(b)$.

It can be easily verified that at least one of the four conditions stated here has to be rejected:

Observation. *Local Incomparability*, *Better, No Dilemmas* and *Choice Parity* are logically inconsistent.

Proof. We assume that any set of values is a feasible set and that options can be identified with their values.[12] Consider the interval $[x,y]$. *Local Incomparability* guarantees the existence of points a and b, with $a > b$ such that [x,y] is incomparable with both a and b. (Formally: (i) $\neg([x,y] > a)$ and $\neg([x,y] \sim a)$ and $\neg([x,y] < a)$ and (ii) $\neg([x,y] > b)$ and $\neg([x,y] \sim b)$ and $\neg([x,y] < b)$.) Let $S = \{a, b, [x, y]\}$ be a set of feasible options. Because $a > b$, *Better* entails that $b \notin C(S)$. Hence, *No Dilemmas* entails that $a \in C(S)$ or $[x, y] \in C(S)$. If $a \in C(S)$ then *Choice Parity* entails that $[x, y] \in C(S)$, and by applying *Choice Parity* again we find that $b \in C(S)$, which contradicts the conclusion (above) that $b \notin C(S)$. The second case, $[x, y] \in C(S)$, is analgous to the first.

3. DISCUSSION

Should we feel worried by this impossibility result? Although it is not very exciting from a technical point of view it makes the decision theorist's worries about interval values explicit. Lauwers and Vallentyne, as well as other advocates of interval values, have to reject at least one of the four conditions.

Vallentyne believes that *Choice Parity* is too strong.[13] In his view, we should conclude that sometimes only one of two incomparable options is permissible. Imagine, for instance, that we are offered three options worth [0,1], 0.49 and 0.51 units of value, and conclude that [0,1] and 0.49, as well as [0,1] and 0.51, are incomparable. Would it then be reasonable to maintain that [0,1] is permissible if 0.49 is permissible? The fact that we can choose 0.51, which is better than 0.49, might be taken to indicate that

[12] Example: 0.5 is an option worth 0.5 units of value, and so on.
[13] In conversation, October 2016.

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although [0,1] and 0.49 are incomparable, [0,1] but not 0.49 is permissible. If so, *Choice Parity* is false.

I agree that this manoeuvre would avoid the inconsistency outlined above. But what is the reason for accepting all this, apart from the fact that it solves the problem? Perhaps it could be argued that one of the options in the example has a property the other lacks.[14] The option worth 0.49 has the property of being *less valuable than* some feasible option (viz. 0.51); the interval [0.1] does not have that property. Of course, we all agree that this makes it rationally impermissible to choose 0.49 whenever 0.51 is available, as stipulated by *Better*. But why would this be a reason for rejecting *Choice Parity*? The latter condition merely asserts that *if* [0,1] is rationally permissible then so is 0.49, and vice versa. What is missing here is a reason for thinking that [0,1] would be rationally permissible when 0.51 and 0.49 are available.[15]

It is also worth pointing out that the refutation of *Choice Parity* leads to another problem. If the mere presence of a third option affects the permissibility of the two other options, then this violates Sen's principle beta. (If x and y are rationally permissible options in A, and $A \subseteq B$, then x is a rationally permissible option in B iff y is a rationally permissible option in B.) Principle beta is not accepted by everyone, but my view is that it is not reasonable to give up this principle just to make room for interval values.[16] If we let pairwise comparisons between two options depend on the presence of a third option, it seems to me that the characterization of the normatively relevant properties of those options may have been incomplete.[17]

A more promising response to the impossibility result could be to make room for genuine dilemmas in decision theory, that is, reject *No Dilemmas*. Although it seems implausible to claim that all options may be rationally impermissible, one can make roughly the same point by questioning the assumption that rational permissibility is a binary property. Perhaps it is neither fully permissible nor fully impermissible to choose [0,1] in a choice between that option and 0.51. All things considered, each option might be somewhat impermissible and somewhat permissible. Claims about rationality need not be black or white. Perhaps

[14] I would like to thank one of the reviewers for suggesting this argument.

[15] It would make little sense to argue that [0,1] is rationally permissible *because* this option is incomparable with 0.51. First, we do not know whether 0.51 is permissible. Second, such an argument would rest on the assumption that if one of two incomparable options is permissible then so is the other, which is the very idea we are discussing.

[16] Principle beta was introduced by Sen (1969, 1970). It is rejected by e.g. Weirich (1998: 117). See also Sen (1993).

[17] Sen (1993: 501) discusses some counterexamples to another of his principles, the famous principle alpha. It seems to me that the characterization of the normatively relevant properties in those examples are also underspecified.

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rational permissibility comes in different shades of grey.[18] On such a nonbinary view, it is not implausible to think that it is somewhat less impermissible, but not fully permissible, to choose 0.51 instead of [0,1].

However, even if we accept this somewhat exotic idea, the impossibility result would still go through as long as we can construct a case in which one option is strictly better (in the binary sense) than another and both these options are permissible *to the same degree* when compared to some third option. Note that this is true irrespective of which (minimally plausible) ordinal or cardinal representation of degrees of permissibility we prefer.[19]

To reject *Better* is, in my opinion, out of the question. If one option is better than another in the traditional, binary sense, then it is not rationally permissible to choose the second option, period.

Although the remarks in this paper are primarily motivated by Lauwers and Vallentyne's analysis of the oscillating gamble, it is worth noting that the scope of the impossibility result is broader. For instance, authors who believe that options can be of 'roughly equal' value (Parfit 1984) or be 'on a par' (Chang 2002) also have reason to worry about this result. The *Local Incomparability* condition seems to be equally applicable to options that are roughly equal or on a par, as are the three other ones. If some options are roughly equal or on a par, then there are at least two local point-values that are neither better, worse or equally good as options to which they stand in that relation to. Neither rough equality nor parity can be fully local phenomena.

It is also worth keeping in mind that the probabilistic nature of the Pasadena-style choices discussed by Lauwers and Vallentyne is not essential for their appeal to interval values. Pasadena-style problems can arise in nonprobabilistic contexts; see Peterson (2013a).[20]

[18] In Peterson (2013b: chapter 2) I defend the analogous view about moral rightness: some acts are neither entirely right nor entirely wrong; rightness and wrongness come in degrees.

[19] Kenny Easwaran has pointed out to me that this non-binary version of the impossibility result relies on the assumption that the entity that comes in degrees (rational impermissibility) is linearly ordered. Perhaps rational impermissibility is indeterminate in some more fundamental sense.

[20] Here is an example: Imagine that the aesthetic value of your favorite painting in the Museum of Modern Arts is the sum of some countably infinite set of valuable points on the painting to which we enumerate in an arbitrary order. If the value of each point is $1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \ldots$ then the aesthetic value of the painting would, according to Lauwers and Vallentyne, be $[-\infty, +\infty]$ (i.e. completely indeterminate value). Here is another example: Imagine that we wish to apply utilitarianism (or any other additive ethical theory) to infinite populations. If the members of some countably infinite populations are numbered arbitrarily, and the series representing the individual's wellbeing is $1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}$, then the value of the entire population is $[-\infty, +\infty]$. See Peterson (2013a) for detailed discussions of these examples.

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All this suggests that the scope of the impossibility theorem stated in this paper is fairly large. It applies to a wide range of gambles and nonprobabilistic choice situations and it is not the principle of maximizing expected value that is at stake in these problems, but rather the idea that the value of the whole is equal to some interval of real numbers. We may, therefore, wish to reject the proposal that sometimes the value of an option is an interval of real numbers.

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BIOGRAPHICAL INFORMATION

Martin Peterson is Sue G. and Harry E. Bovay Jr. Professor in the Department of Philosophy at Texas A&M University. His most recent books are *The Ethics of Technology* (Oxford University Press, 2017) and *The Dimensions of Consequentialism* (Cambridge University Press, 2013). He is also the auhor of a textbook on decision theory, *An Introduction to Decision Theory* (Cambridge University Press, 2nd edn, 2017).

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