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Extended generalized hexagons

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1. *Introduction*

As in [2], we define a geometry $\Gamma = (\mathscr{B}_1, \ldots, \mathscr{B}_r; *)$ to be an ordered sequence of r pairwise disjoint non-empty sets $\mathscr{B}_i$ together with a symmetric incidence relation $*$ on their union $\mathscr{B} = \mathscr{B}_1 \cup \ldots \cup \mathscr{B}_r$ such that if F is any maximal set of pairwise incident elements (i.e. a maximal flag), then $|F \cap \mathscr{B}_i| = 1$ for $i = 1, \ldots, r$. The number r is called the rank of Γ. The geometry Γ is called connected if the r-partite graph $(\mathscr{B}, *)$ is connected.

We recall that a generalized n-gon (for $n \geqslant 2$) is a geometry $\Gamma = (\mathscr{P}, \mathscr{L}; *)$ of rank 2 such that the bipartite graph $(\mathscr{P} \cup \mathscr{L}, *)$ has diameter n and girth $2n$ and every vertex of this graph has at least three neighbours. The elements of $\mathscr{P}$ are called points and the elements of $\mathscr{L}$ lines. The building attached to the group $G_2(p^n)$ is a generalized hexagon; see [3]. We distinguish the point set from the line set by declaring that $|Z(O_p(G_P))| = p^n$ for points P, and denote this hexagon by $\mathscr{H}_1(p^n)$ (which is self-dual if $p = 3$). The building attached to the group ${}^3D_4(p^n)$ is also a generalized hexagon; we define points and lines so that $p^{3n}+1$ lines pass through a point and p^n+1 points lie on a line and call this hexagon $\mathscr{H}_2(p^n)$. We denote the dual of $\mathscr{H}_i(p^n)$ by $\mathscr{H}_i(p^n)^\circ$ for $i = 1$ and 2.

Let $\mathscr{B}_1 = \{1, \ldots, m\}$ for some integer $m \geqslant 2$ and let $\mathscr{B}_2$ denote the set of all 2-element subsets of $\mathscr{B}_1$. Let $\Omega_m = (\mathscr{B}_1, \mathscr{B}_2; *)$, where $*$ is given by containment. We will call any rank 2 geometry isomorphic to Ω_m (for some $m \geqslant 2$) a *subset geometry*.

Now let $\Gamma = (\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3; *)$ be an arbitrary geometry of rank 3. Choose $i \in \{1, 2, 3\}$ and let $j < k$ be the two other elements of $\{1, 2, 3\}$. For each $x \in \mathscr{B}_i$, let $\Gamma_{\mathscr{B}_l}(x) = \{y \in \mathscr{B}_l \mid x * y\}$ for $l = j$ and k. The residue Γ_x of Γ at x is defined to be the rank 2 subgeometry $(\Gamma_{\mathscr{B}_j}(x), \Gamma_{\mathscr{B}_k}(x); *)$.

Definition 1·1. An extended generalized n-gon (for $n \geqslant 3$) is a connected rank 3 geometry $\Gamma = (\mathscr{P}, \mathscr{L}, \mathscr{C}; *)$ with the following residue types: (i) Γ_P is a generalized n-gon if $P \in \mathscr{P}$, (ii) Γ_l is a generalized 2-gon if $l \in \mathscr{L}$ and (iii) Γ_γ is a subset geometry if $\gamma \in \mathscr{C}$. The elements of $\mathscr{P}$, $\mathscr{L}$ and $\mathscr{C}$ are called points, lines and circles, respectively.

In this paper, we will examine the case $n = 6$ under the assumption that there is a group G acting flag-transitively on Γ. Note that extended generalized quadrangles are just the $c.C_2$-geometries first studied in [1]; see also [8].

There exist well known flag-transitive extended generalized hexagons associated with the sporadic groups J_2 and Suz [6]. The point-residues are $\mathscr{H}_1(2)$ for J_2 and $\mathscr{H}_1(4)$ for Suz. The universal covers of both of these geometries are infinite flag-transitive extended generalized hexagons with the same residues [4]. Thus, to characterize the

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quotient geometries associated with J_2 and Suz, some additional condition is needed. We prove here the following result:

THEOREM 1·2. *Let $\Gamma = (\mathscr{P}, \mathscr{L}, \mathscr{C}; *)$ be an extended hexagon and let $G \leqslant \mathrm{Aut}(\Gamma)$ be a group acting flag-transitively on Γ. Let $P \in \mathscr{P}$ and suppose that $\Gamma_P \cong \mathscr{H}_1(q)$, $\mathscr{H}_1(q)^\circ$, $\mathscr{H}_2(q)$ or $\mathscr{H}_2(q)^\circ$ for some q. Suppose, too, that*

(*) *there exist triples of pairwise collinear points which do not lie on any circle.*

Then one of the following holds:

(*a*) $\Gamma_P \cong \mathscr{H}_1(2)$, $G \cong J_2$ *or* $\mathrm{Aut}(J_2)$ *and* $|\mathscr{P}| = 100$;

(*b*) $\Gamma_P \cong \mathscr{H}_1(4)$, $G \cong \mathrm{Suz}$ *or* $\mathrm{Aut}(\mathrm{Suz})$ *and* $|\mathscr{P}| = 1782$;

(*c*) $\Gamma_P \cong \mathscr{H}_1(2)^\circ$, $G \cong 2^7 : G_2(2)'$ *or* $2^7 : G_2(2)$ *and* $|\mathscr{P}| = 128$; *or*

(*d*) $\Gamma_P \cong \mathscr{H}_1(2)^\circ$, $G \cong 2^6 : G_2(2)'$ *or* $2^6 : G_2(2)$ *and* $|\mathscr{P}| = 64$.

In each case (*a*)–(*d*), Γ *is uniquely determined.*

Note that condition (*) is just the opposite of condition (3) on p. 394 of [1]. Since G_P acts as a rank 4 permutation group on the points of the hexagon Γ_P (except in the case $\Gamma_P \cong \mathscr{H}_1(2)^\circ$ when the rank is 5 if G_P induces only $G_2(2)'$ on Γ_P), condition (*) implies that G_P acts transitively on the set $\Delta_2(P)$ of points at distance two from P in the collinearity graph Δ on the points of Γ. For flag-transitive generalized quadrangles, G_P acts, in general, as a rank 3 permutation group on the points of Γ_P; in this case the transitivity of G_P on $\Delta_2(P)$ holds without any additional conditions (and condition (3) on p. 394 of [1] is superfluous).

Our proof of Theorem 1·2 requires coset enumerations which have been carried out using CAYLEY.

2. *Preliminary observations*

Suppose that Γ and G fulfil the hypotheses of (1·2). Let G_P° denote the group induced by G_P on the residue Γ_P.

LEMMA 2·1. *If $\Gamma_P \cong \mathscr{H}_1(2)$ or $\mathscr{H}_1(2)^\circ$, then $G_P^\circ \cong G_2(2)'$ or $G_2(2)$. If $\Gamma_P \cong \mathscr{H}_1(q)$ or $\mathscr{H}_1(q)^\circ$ for $q > 2$, then, up to isomorphism, $G_2(q) \leqslant G_P^\circ \leqslant \mathrm{Aut}(G_2(q))$ and if $\Gamma_P \cong \mathscr{H}_2(q)$ or $\mathscr{H}_2(q)^\circ$ for arbitrary q, then, up to isomorphism, ${}^3D_4(q) \leqslant G_P^\circ \leqslant \mathrm{Aut}({}^3D_4(q))$.*

Proof. This follows from [5] since the group G_P° acts flag-transitively on Γ_P. ∎

Let $l \in \mathscr{L}$ and $\gamma \in \Gamma_{\mathscr{C}}(l)$. Then $\Gamma_{\mathscr{P}}(l) = \Gamma_{\mathscr{P}}(l) \cap \Gamma_{\mathscr{P}}(\gamma)$ since Γ_l is a generalized 2-gon, so $|\Gamma_{\mathscr{P}}(l)| = 2$ since Γ_γ is a subset geometry. We let Δ be the collinearity graph on $\mathscr{P}$. Since Γ is connected, so is Δ. We define a map ϕ from $\Gamma_{\mathscr{L}}(P)$ to $\Delta(P)$, the set of neighbours of P in Δ, by setting $\phi(l) = Q$ where $\Gamma_{\mathscr{P}}(l) = \{P, Q\}$. The map ϕ is of course surjective. By (2·1), G_P acts primitively on $\Gamma_{\mathscr{L}}(P)$. This implies that either ϕ is a bijection or $|\Delta(P)| = 1$. In the latter case, $|\mathscr{P}| = 2$ since Δ is connected. But if $\gamma \in \Gamma_{\mathscr{C}}(P)$, then $|\Gamma_{\mathscr{L}}(P) \cap \Gamma_{\mathscr{L}}(\gamma)| \geqslant 3$ since Γ_P is a generalized polygon, and thus $|\mathscr{P}| \geqslant |\Gamma_{\mathscr{P}}(\gamma)| \geqslant 3$ since Γ_γ is a subset geometry. We conclude that ϕ is a bijection. If γ and γ' are distinct circles in $\Gamma_{\mathscr{C}}(P)$, then

$$\Gamma_{\mathscr{L}}(P) \cap \Gamma_{\mathscr{L}}(\gamma) \neq \Gamma_{\mathscr{L}}(P) \cap \Gamma_{\mathscr{L}}(\gamma') \quad \text{and so} \quad \Gamma_{\mathscr{P}}(\gamma) \neq \Gamma_{\mathscr{P}}(\gamma').$$

Thus, we can identify an element $x \in \mathscr{L} \cup \mathscr{C}$ with the set $\Gamma_{\mathscr{P}}(x)$ and we will do so from now on. Let $\gamma \in \mathscr{C}$. We denote by G_γ° the group induced by the stabilizer G_γ on γ, i.e. on $\Gamma_{\mathscr{P}}(\gamma)$.

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LEMMA 2·2. $|\gamma| = q+2 \leqslant 6$, $G_\gamma^\circ \cong A_{q+2}$ *or* S_{q+2} *for* $\gamma \in \mathscr{C}$ *and* $\Gamma_P \cong \mathscr{H}_1(q)$, $\mathscr{H}_1(q)^\circ$ *or* $\mathscr{H}_2(q)$.

Proof. Let $\gamma \in \Gamma_{\mathscr{C}}(P)$. Then $|\Gamma_{\mathscr{L}}(P) \cap \Gamma_{\mathscr{L}}(\gamma)| = s+1$, where $s+1$ is the number of points on a line in the generalized hexagon Γ_P. Thus $|\Gamma_{\mathscr{P}}(\gamma)| = s+2$. Up to isomorphism

$$\mathrm{PGL}_2(s) \leqslant (G_\gamma^\circ)_P \leqslant \mathrm{P\Gamma L}_2(s).$$

Since G acts flag-transitively, G_γ° is a transitive extension of $(G_\gamma^\circ)_P$. By [7], $s \leqslant 4$. We have $s = q^3$ if $\Gamma_P \cong \mathscr{H}_2(q)^\circ$ and $s = q$ in the other cases, so the conclusion follows. ∎

LEMMA 2·3. G_P *acts faithfully on* $\Delta(P)$.

Proof. For each $U \in \mathscr{P}$, let N_U denote the kernel of the action of G_U on $\Delta(U)$. Choose $Q \in \Delta(P)$ and let $l = \{P, Q\}$. Let X_i denote the set of lines in $\Gamma_{\mathscr{L}}(P)$ at distance $2i$ from l in the graph $(\Gamma_{\mathscr{L}}(P) \cup \Gamma_{\mathscr{C}}(P), *)$, and let $Y_i = \{T \in \mathscr{P} \mid \{P, T\} \in X_i\}$ for $1 \leqslant i \leqslant 3$. We claim that N_Q acts trivially on $\Delta(P)$. Let H be the pointwise stabilizer of Y_1 in $G_{P,Q}$, let K be the pointwise stabilizer of Y_2 in H and let $S \in Y_3$ be arbitrary. By the definition of a generalized hexagon, both $H \cap G_S$ and K lie in N_P. If $Y_2 \cap \Delta(Q) \neq \varnothing$, then $Y_2 \subseteq \Delta(Q)$ since $G_{P,Q}$ acts transitively on Y_2. Condition (*) implies, therefore, that $N_Q \leqslant N_P$. Since $|N_P| = |N_Q|$, we have $N_P = N_Q$, and so N_P is normal in $\langle G_P, G_l \rangle$. Since Δ is connected, $\langle G_P, G_l \rangle$ acts transitively on $\mathscr{P}$. Thus $N_P = 1$. ∎

3. *The case* $\mathscr{H}_1(2)$

In this section we suppose that $\Gamma_P \cong \mathscr{H}_1(2)$. By (2·1) and (2·3) we have $G_P \cong G_2(2)'$ or $G_2(2)$. Thus G_P has generators a, A, b, B, c, C, d and, if $G_P \cong G_2(2)$, also w such that

$$\left.\begin{array}{l} a = A^2, \quad b = B^2, \quad c = C^2, \quad a^2 = b^2 = c^2 = d^2 = [A,B] = [B,C] = [C,d] = 1, \\ [A,c] = aB^{-1}, \quad [B,d] = bC, \quad [a,C] = B^{-1}c, \quad (AC)^3 = (ad)^3 = [C^A, a^d]^3 = 1, \\ w^2 = (Aw)^2 = (Bw)^2 = (Cw)^2 = [d,w] = 1. \end{array}\right\} \quad (3\cdot1)$$

If we set $x_0 = Aw$, $x_1 = ab$, $x_2 = Bw$, $x_3 = b$, $x_4 = w$, $x_5 = c$, $x_6 = wC$ and $x_7 = d$, then

$$\left.\begin{array}{l} x_i^2 = 1 \quad \text{for} \quad 0 \leqslant i \leqslant 7, \quad x_i^r = x_{6-i} \quad \text{for} \quad r = x_0^{x_6} \quad \text{and} \quad 0 \leqslant i \leqslant 6, \quad x_i^s = x_{8-i} \\ \text{for} \quad s = x_1^{x_7} \quad \text{and} \quad 1 \leqslant i \leqslant 7, \quad [x_0, x_1] = [x_1, x_3] = [x_0, x_3] = 1, \quad [x_0, x_2] = x_1, \\ [x_0, x_4] = x_1 x_3, \quad [x_1, x_5] = x_3, \quad [x_0, x_5] = x_1 x_2 x_3 x_4 \quad \text{and} \quad (rs)^6 = 1. \end{array}\right\} \quad (3\cdot2)$$

These are the Steinberg relations [3] defining $G_2(2)$, from which the relations 3·1 have been deduced. The points and lines of $\mathscr{H}_1(2)$ are, by definition, the left cosets of $\langle x_0, \ldots, x_6 \rangle$ and $\langle x_1, \ldots, x_7 \rangle$, respectively, $G_2(2)$. (Two of these cosets are incident in $\mathscr{H}_1(2)$ whenever their intersection is non-empty.) Thus there exists a point $Q \in \Delta(P)$ and a circle $\gamma \in \Gamma_{\mathscr{C}}(P) \cap \Gamma_{\mathscr{C}}(Q)$ such that $G_{P,Q} = \langle A, B, C, w \rangle$ and $G_{P,\gamma} = \langle a, B, C, d, w \rangle$, where the element w is to be ignored if $G_P \cong G_2(2)'$.

We have $N_\gamma = \langle B, C, w \rangle$ and $\langle a, d \rangle N_\gamma / N_\gamma \cong S_3$. Since $G_\gamma^\circ \cong S_4$, there exists an element $v \in G_\gamma$ inducing the transposition (P, Q) on γ. Then

$$v^2 \equiv [v, a] \equiv (vd)^3 \equiv 1 \,(\mathrm{mod}\, N_\gamma). \quad (3\cdot3)$$

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In fact we have

LEMMA 3·4. *The element v can be chosen so that $v^2 = (vd)^3 = 1$ and one of the following holds:*

(1) $B^v = B$, $C^v = BC^{-1}$ *and* $A^v = A$;
(2) $B^v = B$, $C^v = BC^{-1}$ *and* $A^v = A^C$;
(3) $B^v = B^{-1}$, $C^v = BC$ *and* $A^v = AB^{-1}$;
(4) $B^v = B^{-1}$, $C^v = BC$ *and* $A^v = A^{-1}b$.

Proof. We observe first that v can be chosen conjugate to d in G_γ. Then $(vd)^3 \in \langle B, C, w\rangle$ has order at most four; replacing v by d^{vdv}, we obtain the relations $v^2 = (vd)^3 = 1$.

If $G_P \cong G_2(2)$, then $N'_\gamma = \langle b, c\rangle$ and BN'_γ, CN'_γ and BCN'_γ are the only cosets of N'_γ in N_γ containing elements of order four, from which it follows that $\langle B, C\rangle$ is characteristic in N_γ. Hence v acts on $\langle B, C\rangle/\langle b, c\rangle$ both when $G_P \cong G_2(2)'$ and when $G_P \cong G_2(2)$. Since $\langle a, d, v\rangle N_\gamma/N_\gamma \cong S_4$ and $|\langle B, C\rangle/\langle b, c\rangle| = 4$, it follows that av acts trivially on $\langle B, C\rangle/\langle b, c\rangle$. Thus

$$B^v = Bb^x c^y \quad \text{and} \quad C^v = BC^{-1}b^z c^w$$

with $x, y, z, w = 0$ or 1. Then $B = (B^v)^v = Bb^y$ so $y = 0$, and $C = (C^v)^v = Cb^{x+w}$ so $w = x$. We then note that $B = B^{vdvdvd} = Bb^z$, so $z = 0$.

Since v exchanges P and Q, it normalizes $G_{P,Q} = \langle A, B, C, w\rangle$. Let $M = O_2(G_{P,Q})$. Then $M = \langle a, B, c, w\rangle$ and $G_{P,Q} \cong M.S_3$. In particular, $A^v \equiv A$ or $A^C (\mathrm{mod}\, M)$. Since A^v, A and A^C commute with B, it follows that in fact $A^v \equiv A$ or $A^C (\mathrm{mod}\, \langle a, B, c\rangle)$. We begin with the case $A^v \equiv A (\mathrm{mod}\, \langle a, B, c\rangle)$. Then

$$A^v = Aa^r B^s b^t c^u$$

for some $r, s, t, u = 0$ or 1 and $a^v = (A^v)^2 = a^{u+1}B^u b^{s+u+ru}$. Since $a^2 = 1$, $u = 0$. Then

$$A^{vc} = Aa^{r+1}B^{s+1}b^{r+t+1} \quad \text{and} \quad A^{bcv} = (AaBb)^v = Aa^{r+1}B^{s+1}b^{s+t+x+1}.$$

Since $[v, c] = b$, it follows that $r \equiv s + x (\mathrm{mod}\, 2)$. Next we observe that $(AgC)^3 = 1$ for $g \in \langle a, B, c\rangle$ if and only if $g = 1$, Bc, ac or aB^{-1}. Also $(AC)^v = Aa^{s+x}B^{s+1}b^t c^{x+1}C$. It follows that either A^v, B^v and C^v are as required or $x = 0$ and $s = t = 1$. In this latter case, however, we can obtain $A^v = A$ without disturbing the other relations by replacing v with vBc.

It remains to consider the case $A^v \equiv A^C (\mathrm{mod}\, \langle a, B, c\rangle)$. Let

$$A^v = A^C a^r B^s b^t c^u$$

with $r, s, t, u = 0$ or 1. We have $[A^C, c] = abc$ and $[A^C, a] = [A, a^{C^{-1}}]^C = abc$. Thus

$$a^v = (A^v)^2 = a^{r+u+1}Bb^{s+ru+1}c^{r+u+1}.$$

Since $a^2 = 1$, it follows that $r = u$ and $a^v = aBb^{r+s+1}c$. Then

$$A = (A^v)^v = Aa^x B^x b^{x+rx+s+rs+sx},$$

so $x = 0$. We have $(A^C gC)^3 = 1$ for $g \in \langle a, B, c\rangle$ if and only if $g = 1$, Bc, aB^{-1} or abc. Since

$$(AC)^v = A^C a^r B^{s+1} b^t c^{r+1} C,$$

it follows that $A^v = A^C$ or $A^C abc$. In the latter case, we replace v by vBc. ∎

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LEMMA 3·5. *If* $G_P \cong G_2(2)$, *then* $w^v = wB^m b^n c^{m+mx}$ *for some* $m, n = 0$ *or* 1.

Proof. We know that $w^v \equiv w \pmod{\langle B, C\rangle}$. Since $w = (w^v)^v$, the claim follows. ∎

Let X_i and Y_i be as in the proof of (2·3). Let $\epsilon = 0$ if $G_P \cong G_2(2)'$ and $\epsilon = 1$ if $G_P \cong G_2(2)$. Then for each $T \in Y_i$, $|G_{P,Q,T}| = 2^{4+\epsilon}$ if $i = 1$, $2^{2+\epsilon}$ if $i = 2$ and 3.2^ϵ if $i = 3$.

LEMMA 3·6. $Y_2 \notin \Delta(Q)$.

Proof. Let $R = Q^{dAd}$. Then $R \in Y_2 \cap \Delta(Q^d)$ and $G_{P,Q,R} = \langle BC^{-1}, bw\rangle \leqslant G_{P,Q,Q^d}$. If $R \in \Delta(Q)$, then $R^v \in \Delta(P)$. Since $|G_{P,Q,R}| = |G_{P,Q,R^v}|$, we have $R^v \in Y_2$. Moreover $G_{P,Q,R^v} \leqslant G_{P,Q,Q^d}$ since $v \in G_{Q^d}$. Since $\langle a, B\rangle$ acts transitively on

$$\{T \in Y_2 \mid G_{P,Q,T} \leqslant G_{P,Q,Q^d}\},$$

it follows that $vg \in G_R$ for some $g \in \langle a, B\rangle$. This implies that vg normalizes $G_{P,Q,R} = \langle BC^{-1}, bw\rangle$ so, in particular, $(BC^{-1})^{vg} \in \langle BC^{-1}\rangle$. This implies that we are in Case 1 or 2 of (3·4) and that $g = aB^\alpha b^\beta$ for $\alpha, \beta = 0$ or 1. We calculate that $(vg)^2 = b^\alpha$ in Case 1 and $(vg)^2 = Bb^\alpha c$ in Case 2. Since $(vg)^2 \in G_{P,Q,R} = \langle BC^{-1}, bw\rangle$, we have Case 1 and $\alpha = 0$. Since $G_{R,\{P,Q\}}$ and $G_{P,\{Q,R\}}$ are conjugate in G, it follows that G_P contains an involution exchanging Q and R. This implies that $G_P \cong G_2(2)$, and then wA^d is such an involution, so $(vgwA^d)^3 \in G_{P,Q,R}$. Since $C_{G_{P,Q,R}}(wA^d) = \langle bc\rangle$, we have

$$(vab^\beta wA^d)^3 = (bc)^\epsilon \tag{3·7}$$

with $\epsilon = 0$ or 1. Conjugating $A^w = A^{-1}$ by v, we find that $m = 0$. Let $\tilde{G}_{n,\beta,\epsilon}$ denote the group generated by A, B, C, d, w and v and defined by (3·1), Case (1) of (3·4), (3·5) with $m = x = 0$ and (3·7). By coset enumeration, $\tilde{G}_{n,\beta,\epsilon} = 1$ for all $n, \beta, \epsilon = 0$ or 1. ∎

Since $G_{P,Q}$ acts transitively on Y_3, condition (∗) implies that $\Delta(P) \cap \Delta(Q) = Y_1 \cup Y_3$. Let $e = d^{Ad}$ and $S = Q^e$. Then $S \in Y_3$, so $S^v \in Y_3$ as well, and $G_{P,Q,S} = \langle AC, Aw\rangle$. Since $\langle a, B, C\rangle \leqslant G_{P,Q}$ acts transitively on Y_3, we have $vg \in G_S$ for some $g \in \langle a, B, C\rangle$. Since $|\langle a, B, C\rangle| = 32$ and $w \notin \langle A, B, C\rangle$, it follows that

$$(vg)^2 \in G_{P,Q,S} \cap \langle a, B, C\rangle = \langle AC, Aw\rangle \cap \langle a, B, C\rangle = 1.$$

Let $h = vge$. Then $h^3 \in G_{P,Q,S} = \langle AC, Aw\rangle$ and, since vg and e are both involutions, $h^e = h^{-1}$. Since $[AC, e] = 1$, it follows that $h^6 = 1$.

Suppose $h^3 \neq 1$. Then $(AC)^h = (AC)^{-1}$, and so $(AC)^{vg} = (AC)^{-1}$ as well. It follows that $g = fB^p b^q$ for $p, q = 0$ or 1, where $f = C^{-1}$ in Case 1 of (3·4), $f = c$ in Case 2, $f = aC^{-1}$ in Case 3 and $f = aC$ in Case 4. Since $(vg)^2 = 1$, we must have Case 2 with $p = 1$ or Case 3. In Case 2,

$$(Aw)^{vg} = ACAwB^{m+1}b^{n+1}c^{m+1},$$

so $m = 1$ and $n = 0$, and in Case 3,

$$(Aw)^{vg} = ACAwB^m b^{n+p+1},$$

so $m = 0$ and $n \equiv p + 1 \pmod 2$. We conclude that

$$(vgd^{Ad})^3 = wC \tag{3·8}$$

in both cases. Let $\tilde{G}_{i,p,q}$ be the group generated by elements A, B, C, d, w and v and defined by the relations (3·1), Case (i) of (3·4) for $i = 2$ or 3, (3·5) with $m = p = 1$ and $x = n = 0$ if $i = 2$ and $m = 0$, $x = 1$ and $n \equiv p + 1 \pmod 2$ if $i = 3$ and (3·8). By coset enumeration, we find that $\tilde{G}_{i,p,q} = 1$ for all i, p and q.

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Thus $h^3 = 1$, so h centralizes AC. Therefore vg centralizes AC as well. Since $(vg)^2 = 1$, we have either Case 1 of (3·4) with $g \in \langle a, b\rangle$ or Case 3 with $g \in \langle B\rangle$. The first case is ruled out by coset enumeration. In the second case, we can assume $g = 1$ by substituting vg for v. Let $\tilde{G}$ be the group generated by elements A, B, C, d and v and defined by those relations in (3·1) not involving w, the relations (3) of (3·4) and $(ve)^3 = 1$. By coset enumeration, we find that $|\tilde{G}| = |J_2|$. Since it is known that J_2 acts on an extended generalized hexagon with point-residues isomorphic to $\mathscr{H}_1(2)$ and point-stabilizers isomorphic to $G_2(2)'$, it follows that $\tilde{G} \cong J_2$ and Γ is isomorphic to the geometry with points the left-cosets of $\langle A, B, C, d\rangle$, lines the left-cosets of $\langle A, B, C, v\rangle$ and circles the left-cosets of $\langle a, B, C, d, v\rangle$ in $\tilde{G}$, in which two of these left-cosets are incident whenever their intersection contains a coset of the intersection of the corresponding two subgroups. If $G_P \cong G_2(2)$ then $G \cong \mathrm{Aut}(J_2)$.

4. *The case* $\mathscr{H}_1(2)^\circ$

In this section we suppose that $\Gamma_P \cong \mathscr{H}_1(2)^\circ$. As in the previous section, $G_P \cong G_2(2)'$ or $G_2(2)$, and G_P has generators A, B, C, d and, if $G_P \cong G_2(2)$, also w such that the relations (3·1) hold, however this time $G_{P,Q} = \langle a, B, C, d, w\rangle$ and $G_{P,\gamma} = \langle A, B, C, w\rangle$ for some $Q \in \Delta(P)$ and some $\gamma \in \Gamma_{\mathscr{G}}(P) \cap \Gamma_{\mathscr{G}}(Q)$. We have $N_\gamma = \langle a, B, c, w\rangle$. Since $G_\gamma^\circ \cong S_4$, there exists an element $v \in G_\gamma$ inducing the transposition (P, Q) on γ. Then

$$v^2 \equiv (vA)^3 \equiv [v, C] \equiv 1 \,(\mathrm{mod}\, N_\gamma). \tag{4·1}$$

In fact we have

LEMMA 4·2. *The element v can be chosen so that $a^v = aBc$, $B^v = B$, $C^v = C$, $d^v = d$, $v^2 = c$ and $(vA)^3 = Bb^s$ for $s = 0$ or 1.*

Proof. We show first that G_γ must normalize $\langle B\rangle$. If $G_P \cong G_2(2)'$, then $\langle B\rangle = Z(N_\gamma)$, and the claim follows. If $G_P \cong G_2(2)$, then there are precisely six cyclic subgroups of order four in N_γ, and the commuting graph on these subgroups is a complete bipartite graph with $\langle B\rangle$, $\langle acw\rangle$ and $\langle aBcw\rangle$ on one side and $\langle aB\rangle$, $\langle Bc\rangle$ and $\langle ac\rangle$ on the other. Since both A and C normalize $\langle B\rangle$, it follows that G_γ does too.

Since $A \in C_{G_\gamma}(B)$, the subgroup $C_{G_\gamma}(B)$ induces S_4 on γ. By replacing v with an element of $C_{G_\gamma}(B)$ which induces the transposition (P, Q) on γ, we may assume that $B^v = B$. We can also assure that $v^4 = 1$ by selecting v conjugate to A.

The element C normalizes $\langle Bc\rangle$ and exchanges $\langle aB\rangle$ and $\langle ac\rangle$. By (4·1), it follows that v has the same action on these subgroups, so

$$a^v = aBb^n c \quad \text{and} \quad c^v = b^p c$$

for $n, p = 0$ or 1. We have $v^2 \in C_{N_\gamma}(B) = \langle a, B, c\rangle$. Suppose $p = 1$. Then $C_{\langle a,B,c\rangle}(v) = \langle B\rangle$. Since $v^4 = 1$, we must have $v^2 = b$. But then vB is an involution in $C_{G_\gamma}(B)$ inducing a transposition on γ. The existence of such an element contradicts the fact that the only involutions in $C_{G_{P,Q,\gamma}}(B) = \langle a, B, C\rangle$ lie in N_γ. Thus $p = 0$, $C_{\langle a,B,c\rangle}(v) = \langle B, c\rangle$ and $v^2 = b^k c$ for $k = 0$ or 1. Substituting vc for v if necessary, we may assume that $n = 0$. Substituting vB for v if necessary, we may assume that $k = 0$. Since $[v, C] \in C_{N_\gamma}(B) = \langle a, B, c\rangle$ and $[v, c] = 1$, we have $C^v = b^z C c^m$ for $z, m = 0$ or 1. Conjugating the relation $[a, C] = B^{-1}c$ by v, we find that $m = 0$. Since $(vA)^3 \in C_{N_\gamma}(B) = \langle a, B, c\rangle$ and vA acts fixed point freely on $\langle a, B, c\rangle/\langle B\rangle$, it follows that $(vA)^3 \in \langle B\rangle$.

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To summarize, we now have

$$(vA)^3 = B^r b^s, \quad a^v = aBc, \quad B^v = B, \quad C^v = Cb^z \quad \text{and} \quad v^2 = c \tag{4·3}$$

for $r, s, z = 0$ or 1.

Since v exchanges P and Q, it normalizes $G_{P,Q} = \langle a, B, C, d, w\rangle$. Thus,

$$d^v \equiv d \quad \text{or} \quad d^a \,(\text{mod}\, \langle B, C, w\rangle).$$

Note that $B^{d^v d^a} = (Bb^{z+1}C)^{ada} = BCbc^z$. Thus $d^v d^a \notin \langle B, C, w\rangle$, since $\langle B\rangle$ is normal in $\langle B, C, w\rangle$. Hence $d^v d \in \langle B, C, w\rangle$. Then $B^{d^v d} = Bb^z c^z$, so $z = 0$. In particular, we have $d^v d \in C_{\langle B,C,w\rangle}(B) = \langle B, C\rangle$. Thus $d^v = d(bC)^q c^j$ for $j, q = 0$ or 1 since $d^2 = 1$. Then $d = d^c = (d^v)^v = dc^q$, so $q = 0$. Substituting vb for v if necessary, we can assume that $j = 0$ as well. Finally, we note that by coset enumeration, the group generated by A, B, C, d and v and defined by the relations (3·1), $[v, d] = 1$, and (4·3) with $z = 0$ is trivial unless $r = 1$. ▮

LEMMA 4·4. *If* $G_P \cong G_2(2)$, *then* $w^v = wc$.

Proof. Since v normalizes both N_γ and $\langle a, B, c\rangle$, we have $[v, w] \in \langle a, B, c\rangle$. Since v normalizes both $O_2(G_{P,Q})$ and $\langle B, C\rangle$, it follows that $[v, w] \in \langle B, C\rangle$. Thus $w^v = wB^\alpha b^\beta c^\delta$ for $\alpha, \beta, \delta = 0$ or 1. Then $w = (w^v)^v = wb^\alpha$ so $\alpha = 0$. Since $[d, v] = [d, w] = 1$, also $[d, b^\beta c^\delta] = 1$, and hence $\beta = 0$. Finally, $a^{v^{-1}wvw} = (aB^{-1}c)^{wvw} = ab$ so $\delta = 1$. ▮

LEMMA 4·5. *If* $Y_2 \subseteq \Delta(Q)$, *then* $|\mathscr{P}| = 128$ *and* $G \cong 2^7 : G_2(2)'$ *or* $2^7 : G_2(2)$ *or* $|\mathscr{P}| = 64$ *and* $G \cong 2^6 : G_2(2)'$ *or* $2^6 : G_2(2)$.

Proof. Let $R = Q^{A^{-1}dA}$. Then $R \in Y_2 \cap \Delta(Q^A)$ and $G_{P,Q,R} = \langle aBc, b, aw\rangle \leqslant G_{P,Q,Q^A}$. If $R \in \Delta(Q)$, then $R^v \in \Delta(P)$. Since $|G_{P,Q,R}| = |G_{P,Q,R^v}|$, it follows that $R^v \in Y_2$. Moreover, $G_{P,Q,R^v} \leqslant G_{P,Q,Q^A}$ since $v \in G_{Q^A}$. Since $\langle B, C\rangle$ acts transitively on

$$\{T \in Y_2 \mid G_{P,Q,T} \leqslant G_{P,Q,Q^A}\},$$

we have $vg \in G_R$ for some $g \in \langle B, C\rangle$. This implies that vg normalizes $G_{P,Q,R} = \langle aBc, b, aw\rangle$, so $gC \in \langle B, c\rangle$. Substituting gb for g if necessary, we can assume that $g = B^i C^j$ for $i, j = 0$ or 1. We have $(vg)^2 = b^i$.

Let $K = \langle aBc, b\rangle$. Then $G_{P,\{Q,R\}} = \langle aBc, b, aw, d^A\rangle$ and $C_{G_{P,\{Q,R\}}}(K) = G_{P,Q,R}$. Since vgd^A normalizes K and maps Q to P to R to Q, we conclude that $C_{G_{R,\{P,Q\}}}(K) = G_{P,Q,R}$. Since $vg \in G_{R,\{P,Q\}}$, it follows that vg does not centralize K. Hence $j = 1$. Then $C_{G_{P,Q,R}}(vgd^A) = \langle (aB^{-1}c)^{i+1}aw\rangle$, so

$$(vgd^A)^3 = ((aB^{-1}c)^{i+1}aw)^\epsilon \tag{4·6}$$

for $\epsilon = 0$ or 1. Let $\tilde{G}_{s,i,\epsilon}$ denote the group generated by elements A, B, C, d and, if $\epsilon = 1$, also w and defined by (3·1), (4·2), (4·4) and (4·6). By coset enumeration, $\tilde{G}_{s,i,\epsilon} = 1$ unless $s = i = \epsilon = 0$. Let $\tilde{G} = \tilde{G}_{0,0,0}$. Then, using CAYLEY, we find that $|\tilde{G}| = 2^7 . |G_2(2)'|$, $|O_2(\tilde{G})| = 2^7$ and $|Z(\tilde{G})| = 2$. Thus G and Γ are as claimed. ▮

LEMMA 4·7. *If* $Y_3 \cap \Delta(Q) \neq \varnothing$, *then* $Y_2 \subseteq \Delta(Q)$.

Proof. The set Y_3 consists of two $\langle B, C\rangle$-orbits. If $G_P \cong G_2(2)'$, then these are also $G_{P,Q}$-orbits; if $G_P \cong G_2(2)$, then $\langle B, C, w\rangle$ acts transitively on Y_3. Let

$$e = A^{dA}, \quad f = e^{-1}, \quad S = Q^e \quad \text{and} \quad T = Q^f.$$

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Then S and $T \in Y_3$. Also $[e, aBc] = 1$ and $(aBd)^e = aB^{-1}Cd$, so

$$G_{P,Q,S} = \langle aBc, aB^{-1}Cd \rangle \cong S_3 \quad \text{and} \quad G_{P,Q,T} = \langle aBc, aBd \rangle.$$

Since $(G_{P,Q,T})^B = G_{P,Q,S}$ and $eB^{-1}e \notin G_{P,Q}$, the points S and T lie in different $\langle B, C \rangle$-orbits.

We claim that for some $g \in \langle B, C, w \rangle$, the element vg maps S to T. If $G_P \cong G_2(2)$, then $\langle B, C, w \rangle$ acts transitively on Y_3, so the claim is of course valid. Suppose $G_P \cong G_2(2)'$. Then $G_{P,\{Q,V\}} = G_{P,Q,V}$ for each $V \in Y_3$. Since there is just one G_P-orbit of collinear pairs $\{U, V\}$ in $\Delta(P)$ not co-circular with P, the group $G_{\{P,Q,V\}}$ acts transitively on $\{P, Q, V\}$ and hence $G_{\{P,Q\},V} = G_{P,Q,V}$ for each $V \in Y_3 \cap \Delta(Q)$. It follows that $Y_3 \subseteq \Delta(Q)$ and v switches the two orbits of $\langle B, C \rangle$ in Y_3. Thus the claim holds in this case too.

We have $g = B^\alpha b^\beta C^\delta c^\epsilon w^\lambda$ for $\alpha, \beta, \delta, \epsilon, \lambda = 0$ or 1. If $h = vge$, then h maps Q to P to S to Q, so h normalizes $G_{P,Q,S}$. Thus $(aBc)^h \in G_{P,Q,S}$, so

$$(aBc)^{vg} \in G_{P,Q,T} = \langle aBc, aBd \rangle.$$

We have $(aBc)^{vg} = aB^\delta b^{1+\epsilon+\delta+\lambda\delta} c^\delta$ so $\delta = 1$ and $\epsilon = \lambda$. Similarly,

$$(aB^{-1}Cd)^{vg} = aBb^{\alpha+1}C^{\alpha+1}c^{\beta+\lambda+\lambda\alpha}d \in G_{P,Q,T},$$

so $\alpha = \beta = 1$ and h centralizes $G_{P,Q,S}$. Since $h^3 \in G_{P,Q,S}$ and $Z(G_{P,Q,S}) = 1$, we conclude that $h^3 = 1$. Thus

$$(vB^{-1}Cc^\lambda w^\lambda A^{dA})^3 = 1. \tag{4·8}$$

Let $\tilde{G}_{s,\lambda}$ be the group generated by elements A, B, C, d, v and, if $\lambda = 1$, also w and defined by the relations (3·1), (4·2), (4·4) and (4·8). By coset enumeration, $\tilde{G}_{s,\lambda} = 1$ unless $s = \lambda = 0$ and $|\tilde{G}_{0,0}| = 64 . |G_2(2)'|$. ∎

5. *The case* $\mathscr{H}_1(4)$

In this section we suppose that $\Gamma_P \cong \mathscr{H}_1(4)$. Then $G_P \cong G_2(4)$ or $\mathrm{Aut}(G_2(4))$, so G_P has generators $x_0, x_1, \ldots, x_7, h, k$ and, if $G_P \cong \mathrm{Aut}(G_2(4))$, also σ fulfilling the defining relations (3·2) for $G_2(2)$ and

$$\left.\begin{aligned} &h^3 = k^3 = [h,k] = (hx_0)^3 = (hx_0x_6)^5 = [h,x_1] = 1, \quad h^r = h^{-1}, \quad h^s = h, \\ &[k,x_0] = (kx_1)^3 = 1, \quad (kx_1x_7)^5 = h, \quad k^r = k, \quad k^s = hk^{-1}, \\ &\sigma^2 = 1, \quad [\sigma, x_i] = 1 \quad \text{for} \quad 0 \leqslant i \leqslant 6, \quad h^\sigma = h^{-1} \quad \text{and} \quad k^\sigma = k^{-1}, \end{aligned}\right\} \tag{5·1}$$

where, as in (3·2), $r = x_0^{x_6}$ and $s = x_1^{x_7}$. These are the Steinberg relations for $G_2(4)$, respectively, $\mathrm{Aut}(G_2(4))$.

Since $\Gamma_P \cong \mathscr{H}_1(4)$, there exist $Q \in \Gamma_6(P)$ and $\gamma \in \Gamma_6(P) \cap \Gamma_6(Q)$ such that

$$G_{P,Q} = \langle x_0, \ldots, x_6, h, k, \sigma \rangle \quad \text{and} \quad G_{P,\gamma} = \langle x_1, \ldots, x_7, h, k, \sigma \rangle,$$

where σ is to be ignored if $G_P \cong G_2(4)$. Let $g = hk$. Then x_1, x_7 and g satisfy the following defining relations for A_5:

$$x_1^2 = x_7^2 = (x_1x_7)^3 = (gx_1)^3 = (gx_1x_7)^5 = (gx_1^{x_7})^2 = 1.$$

Since $G_\gamma^\circ \cong A_6$ or S_6, we can choose $v \in G_\gamma$ such that

$$v^2 \equiv [v, x_1] \equiv (vx_7)^3 \equiv (vg)^2 \equiv 1 \,(\mathrm{mod}\, N_\gamma). \tag{5·2}$$

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In fact we have

LEMMA 5·3. *The element v can be chosen so that*

$$v^2 = (vx_7)^3 = [h, v] = [x_0, v] = [x_6, v] = 1, \quad x_1^v = x_1 x_3, \quad x_2^v = x_4 \quad \text{and} \quad k^v = k^{-1}.$$

Proof. We have $\langle h\rangle \in \mathrm{Syl}_3(N_\gamma)$ and $C_{G_{P,\gamma}}(h) N_\gamma / N_\gamma \cong A_5$, so $C_{G_\gamma}(h) N_\gamma / N_\gamma \cong A_6$. Thus we can choose $v \in C_{G_\gamma}(h)$ conjugate to x_1, so $v^2 = [h, v] = 1$. Let $M = O_2(G_\gamma)$, $y_i = x_i^k$ for $i = 1, 2, 3$ and 5, $U = \langle x_3, y_3, x_5, y_5\rangle$ and $V = M/U$. Then $C_{N_\gamma}(h) = \langle U, h\rangle$, so

$$[x_1, v] \equiv (vx_7)^3 \equiv (vk)^2 \equiv 1 \,(\mathrm{mod}\, \langle U, h\rangle).$$

Since x_1, x_7 and v are all involutions, both $[x_1, v]$ and $(vx_7)^3$ are inverted by v; thus, in fact

$$[x_1, v] \equiv (vx_7)^3 \equiv 1 \,(\mathrm{mod}\, U).$$

Since $\langle h\rangle$ acts fixed point freely on V, we can consider V as a GF(4)-module for $C_{G_\gamma}(h)$. With respect to the basis $x_2 U$, $x_4 U$, $x_6 U$, the elements x_1, x_7, k and h are represented by

$$X_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where λ and μ are the two elements of GF(4) different from 0 and 1, and

$$H = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

since $x_2^h = x_2^k$. If $V^2 = [V, X_1] = (VX_7)^3 = 1$ and $VKV \equiv K^{-1} (\mathrm{mod}\, \langle H\rangle)$ for $V \in \mathrm{GL}_3(4)$, then

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, in fact, $VKV = K^{-1}$. Thus $x_2^v \equiv x_4 \,(\mathrm{mod}\, U)$, $x_6^v \equiv x_6 \,(\mathrm{mod}\, U)$ and $k^v \equiv k \,(\mathrm{mod}\, U)$.

Now

$$x_3^v = [x_2^v, x_4^v] = [x_4, x_2] = x_3 \quad \text{and} \quad x_5^v = [x_4^v, x_6^v] = [x_2, x_6] = x_3 x_5.$$

Hence

$$y_3^v = (x_3^v)^{k^{-1}} = x_3 y_3 \quad \text{and} \quad y_5^v = (x_3^v x_5^v)^{k^{-1}} = x_3 y_3 x_5 y_5.$$

Note that $[y_3, x_7] = x_5 y_5$. Then $(vx_7)^3 = 1$ since vx_7 acts fixed point freely on U. Since $v^2 = 1$,

$$x_1 x_1^v \in C_U(v) = \langle x_3, y_3 x_5\rangle.$$

Since $x_1^2 = 1$, it follows that $x_1^v = x_1 x_3^i$ for $i = 0$ or 1. Similarly, $x_6^v = x_6 x_3^j y_3^m x_5^m$ for j, $m = 0$ or 1. Then

$$kk^v \in C_U(vk^{-1}) = \langle x_3 y_5, y_3\rangle,$$

so $k^v = k^{-1} x_3^n y_3^p y_5^n$ for $n, p = 0$ or 1. By substituting $vx_3^{n+p} y_3^n x_5^n$ for v, we can assume that $n = p = 0$.

Since v exchanges P and Q and normalizes $\langle h, k\rangle$, it normalizes $N_{G_{P,Q}}(\langle h, k\rangle) = \langle h, k, r\rangle$. Thus $[r, v] = 1$. Conjugating the relation $x_1^r = x_5$ by v, we obtain $x_3 = x_3^i$ and

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hence $i = 1$. We have $x_2^v = x_4 x_3^\alpha y_3^\beta x_5^\delta y_5^\epsilon$ for α, β, δ and $\epsilon = 0$ or 1. Conjugating by r, we obtain $x_4^v = x_2 x_3^\alpha y_3^\beta x_1^\delta y_1^\epsilon$, so $\delta = \epsilon = 0$. Then $x_2 = (x_2^v)^v = x_2 x_3^\beta$, so $\beta = 0$. Next, we have

$$x_2 = x_2^{(vx_7)^3} = x_2 x_3^{\alpha+j} y_3^m x_5^{\alpha+j} y_5^m,$$

so $m = 0$ and $j = \alpha$. It follows that $x_0^v = (x_6^v)^r = x_0 x_3^\alpha$, so

$$r = r^v = (x_0^{x_6})^v = (x_0 x_3^\alpha)^{x_3^\alpha x_6} = r x_3^\alpha$$

and hence $\alpha = 0$. ∎

LEMMA 5·4. $Y_2 \not\subseteq \Delta(Q)$.

Proof. Let $R = Q^{srs}$. Then $R \in Y_2 \cap \Delta(Q^s)$ and

$$G_{P,Q,R} = \langle x_4, x_5, x_6, h, k, \sigma \rangle \leqslant G_{P,Q,Q^s}.$$

If $R \in \Delta(Q)$, then $R^v \in \Delta(P)$. Since $|G_{P,Q,R}| = |G_{P,Q,R^v}|$, we have $R^v \in Y_2$. Moreover, $G_{P,Q,R^v} \leqslant G_{P,Q,Q^s}$ since $v \in G_{Q^s}$. Let $H = \langle x_2, y_2, x_3, y_3 \rangle$. Since H acts transitively on

$$\{T \in Y_2 \mid G_{P,Q,T} \leqslant G_{P,Q,Q^s}\},$$

we have $vg \in G_R$ for some $g \in H$. Thus vg normalizes $G_{P,Q,R}$. But $x_4^{vg} = x_2$ for all $g \in H$. ∎

Condition (*) now implies that $Y_3 \subseteq \Delta(Q)$. Let $e = s^{rs}$ and $S = Q^e$. Then $S \in Y_3$, so $S^v \in Y_3$ as well, and $G_{P,Q,S} = \langle x_0, x_6, h, k, \sigma \rangle$. Since $O_2(G_{P,Q})$ acts transitively on Y_3, we have $vg \in G_S$ for some $g \in O_2(G_{P,Q})$. The element vg normalizes $G_{P,Q,S}$. Since both vg and v normalize $\langle k \rangle = O_3(G_{P,Q,S})$, so does g. Since k acts fixed point freely on $O_2(G_{P,Q})$, it follows that $g = 1$, and so $(ve)^3 \in C_{G_{P,Q,S}}(\langle x_0, x_6, h \rangle) = \langle k \rangle$. Thus

$$(vs^{rs})^3 = k^\epsilon \tag{5·5}$$

for $\epsilon = 0$, 1, or 2. Let $\tilde{G}_\epsilon$ be the group generated by elements $x_0, \ldots, x_7$, h, k, v and defined by the relations (3·2), (5·1), (5·3) and (5·5). By coset enumeration, $\tilde{G}_\epsilon = 1$ unless $\epsilon = 0$ and $|\tilde{G}_0| = |\mathrm{Suz}|$. Since it is known that Suz acts on an extended generalized hexagon fulfilling condition (*) with $G_P \cong G_2(4)$, Γ must be isomorphic to this geometry.

We remark that by omitting the relation $k^v = k^{-1}$ in the definition of $\tilde{G}_0$, we obtain a presentation of $3 \cdot \mathrm{Suz}$.

6. *The case* ${}^3D_4(2)$

Suppose that $\Gamma_P \cong \mathscr{H}_2(2)$. Then $G_P \cong {}^3D_4(2)$ or $\mathrm{Aut}({}^3D_4(2))$, so G_P has generators $x_0, x_1, \ldots, x_7, h$ and, if $G_p \cong \mathrm{Aut}({}^3D_4(2))$, also σ fulfilling the relations (3·2) and

$$\left.\begin{aligned} &h^7 = [x_1, h] = [x_7, h] = [x_0, x_0^h] = [x_0, x_0^{h^2}] = 1, \\ &h^r = h^{-1}, \quad (h^{x_0 h^{-2}} x_0 x_6)^2 = 1, \quad x_0^{h^3} = x_0 x_0^h, \\ &\sigma^3 = 1, \quad h^\sigma = h^2 \quad \text{and} \quad [x_i, \sigma] = 1 \quad \text{for} \quad 0 \leqslant i \leqslant 7, \end{aligned}\right\} \tag{6·1}$$

where, as before, $r = x_0^{x_6}$. There exist $Q \in \Delta(P)$ and $\gamma \in \Gamma_{\mathscr{G}}(P) \cap \Gamma_{\mathscr{G}}(Q)$ such that

$$G_{P,Q} = \langle x_0, \ldots, x_6, h, \sigma \rangle \quad \text{and} \quad G_{P,\gamma} = \langle x_1, \ldots, x_7, h, \sigma \rangle,$$

where σ is to be ignored if $G_P \cong {}^3D_4(2)$.

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Let $y_i = x_i^h$ and $z_i = x_i^{h^2}$ for $i = 2$, 4 and 6 and let $M = O_2(G_\gamma)$. Then

$$M = \langle x_2, y_2, z_2, x_3, x_4, y_4, z_4, x_5, x_6, y_6, z_6 \rangle,$$

$M' = \langle x_3, x_4, y_4, z_4, x_5 \rangle$ and $Z(M) = \langle x_3, x_5 \rangle$. The element h acts fixed point freely on both M/M' and $M'/Z(M)$, so both of these quotient groups can be considered as modules for $C_{G_\gamma}(h)$ over GF(8).

Since $\langle h \rangle \in \mathrm{Syl}_7(N_\gamma)$, we have $C_{G_\gamma}(h) N_\gamma / N_\gamma \cong S_4$. Let v be an element of $C_{G_\gamma}(h)$ inducing the transposition (P, Q) on γ. Then

$$v^2 = [v, h] = 1 \quad \text{and} \quad [v, x_1] \equiv (ve_7)^3 \equiv 1 \,(\mathrm{mod}\, N_\gamma). \tag{6·2}$$

In fact we have

LEMMA 6·3. *$v^2 = [v, h] = [v, x_2] = [v, x_3] = [v, x_4] = (vx_7)^3 = 1$, $x_0^v = x_0 x_1 x_2 x_3 x_4$, $[v, x_1] = [v, x_5] = x_3$ and $x_6^v = x_2 x_3 x_4 x_5 x_6$.*

Proof. Since $|Z(M)| = 4$ and $\langle x_1, x_7, v \rangle N_\gamma / N_\gamma \cong S_4$, the element $x_1 v$ must centralize $Z(M)$. Thus $[v, x_3] = 1$ and $x_5^v = x_3 x_5$. The element v must act trivially on the 1-dimensional $C_{G_\gamma}(h)$-module $M'/Z(M)$, so $x_4^v = x_4 x_3^a x_5^b$ for $a, b = 0$ or 1. Then $x_4 = (x_4^v)^v = x_4 x_3^b$ so $b = 0$. Since v, x_1 and x_7 are involutions centralizing h, it follows by (6·2) that $(vx_7)^3$ and $x_1^v x_1$ both lie in $C_M(h) = \langle x_3, x_5 \rangle$. Since vx_7 acts fixed point freely on $\langle x_3, x_5 \rangle$, it follows that $(vx_7)^3 = 1$. We have $x_1^v = x_1 x_3^A x_5^B$ for $A, B = 0$ or 1. Then $x_1 = (x_1^v)^v = x_1 x_3^B$, so $B = 0$.

Since $C_{G_{P,\gamma}}(h)$ induces $7.S_3$ on M/M' and the 2-Sylow subgroups of $\mathrm{GL}_2(8)$ are elementary abelian, the element $x_1 v$ must centralize M/M'. Thus

$$x_2^v = x_2 x_3^i x_4^j y_4^l z_4^m x_5^n \quad \text{and} \quad x_6^v = x_2 x_3^\alpha x_4^\beta y_4^\delta z_4^\epsilon x_5^\lambda x_6$$

for $i, j, \ldots, \lambda = 0$ or 1. We have $C_{\langle x_4, y_4, z_4 \rangle}(\langle x_2, x_6 \rangle) = \langle y_4, z_4 \rangle$. Thus $(x_2^v)^2 = (x_6^v)^2 = 1$ implies that $j = 0$ and $\beta = 1$. Then

$$(x_6^v)^v = x_3^{1+i+a+a\delta+a\epsilon+\lambda} y_4^l z_4^m x_5^n x_6,$$

so

$$l = m = n = 0 \quad \text{and} \quad \lambda \equiv 1 + i + a + a\delta + a\epsilon \,(\mathrm{mod}\, 2).$$

Then $x_2^{(x_7 v)^3} = x_2 x_3^{\alpha+a+1}$, so $\alpha \equiv a + 1 \,(\mathrm{mod}\, 2)$.

Since v exchanges P and Q, it normalizes $N_{G_{P,Q}}(h) = \langle r, h, x_1, x_3, x_5 \rangle$ and hence $r^v = r h^I x_1^J x_3^K x_5^L$ for $I, J, K, L = 0$ or 1. Then $(r^v)^v = r h^{2I} x_2^{AJ+L}$, so $I = 0$ and $L = AJ$, and then $(r^v)^2 = x_1^{J+L} x_3^{JL+J} x_5^{J+L}$, so $J = L = AJ$. Conjugating the relations $x_2^r = x_4$ and $x_1^r = x_5$ by v, we find that $x_3^i = x_3^a$, so $i = a$, and $x_3 = x_3^{A+J}$. Since $AJ = J$, it follows that $A = 1$ and $J = 0$. Thus

$$[v, x_1] = x_3, \quad [v, x_2] = [v, x_4] = x_3^a \quad \text{and} \quad x_6^v = x_2 x_3^{a+1} x_4 y_4^\delta z_4^\epsilon x_5^{1+a\delta+a\epsilon} x_6.$$

We have $[y_2, x_6] = z_4 x_5$ and $C_{\langle x_4, y_4, z_4 \rangle}(y_2) = \langle x_4 y_4, z_4 \rangle$. Thus

$$x_3^{a+1} z_4 x_5 = (z_4 x_5)^v = [y_2^v, x_6^v] = x_3^{\delta+1} z_4 x_5,$$

so $\delta = a$. Since $[z_2, x_6] = y_4 z_4 x_5$ and $C_{\langle x_4, y_4, z_4 \rangle}(z_2) = \langle x_4 y_4, y_4 z_4 \rangle$, it follows that

$$x_3 y_4 z_4 x_5 = (y_4 z_4 x_5)^v = [z_2^v, x_6^v] = x_3^{\epsilon+a+1} y_4 z_4 x_5,$$

so $\epsilon = a$. Since $x_2^{h^{-1}} = y_2 z_2$, we have $(x_4 y_4^a z_4^a)^r = x_2^{a+1} z_2^a$. Thus $x_0 = x_6^r$ implies

$$x_0^v = (x_2 x_3^{a+1} x_4 y_4^a z_4^a x_5 x_6)^{r x_3^K} = x_4 x_3^{a+1} x_2^{a+1} z_2^a x_1 x_0.$$

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Since $C_{\langle x_2, y_2, z_2\rangle}(\langle x_0, x_4\rangle) = \langle x_2 y_2, y_2 z_2\rangle$, it follows that $x_0^v = x_0 x_1 x_2^{a+1} z_2^a x_3^{a+1} x_4$. Using the identity $[x_0, y_4] = y_2 z_2 x_3$, we can calculate that $x_0^{vy_4 vy_4} = x_0 x_3^a$. Since $[x_0, [v, y_4]] = 1$, $a = 0$. ∎

LEMMA 6·4. $Y_2 \not\subseteq \Delta(Q)$.

Proof. Let $R = Q^{srs}$. Then $R \in Y_2 \cap \Delta(Q^s)$ and $G_{P,Q,R} = \langle x_4, x_5, x_6, h, \sigma\rangle \leqslant G_{P,Q,Q^s}$. If $R \in \Delta(Q)$, then $R^v \in \Delta(P)$. Since $|G_{P,Q,R}| = |G_{P,Q,R^v}|$, we have $R^v \in Y_2$. Moreover, $G_{P,Q,R^v} \leqslant G_{P,Q,Q^s}$ since $v \in G_{Q^s}$. Let $H = \langle x_1, x_2, y_2, z_2, x_3\rangle$. Since H acts transitively on

$$\{T \in Y_2 \mid G_{P,Q,T} \leqslant G_{P,Q,Q^s}\},$$

it follows that $vg \in G_R$ for some $g \in H$. Thus vg normalizes $G_{P,Q,R}$. Then

$$x_6^{vg} = (x_2 x_3 x_4 x_5 x_6)^g \in G_{P,Q,R}$$

implies that $x_1 g \in \langle x_2, y_2, z_2, x_3\rangle$ and hence $(vg)^2 = x_3$. This contradicts the fact that $(vg)^2 \in G_{P,Q,R}$. ∎

Condition (*) now implies that $Y_3 \subseteq \Delta(Q)$. Let $e = s^{rs}$ and $S = Q^e$. Then $S \in Y_3$, so $S^v \in Y_3$ as well, and $G_{P,Q,S} = \langle x_0, x_6, h, \sigma\rangle \cong L_2(8)$ or $L_2(8):3$. Since $O_2(G_{P,Q})$ acts transitively on Y_3, we have $vg \in G_S$ for some $g \in O_2(G_{P,Q})$. The element vg normalizes $G_{P,Q,S}$. Since

$$x_0^{vg} = (x_0 x_1 x_2 x_3 x_4)^g \in x_0 O_2(G_{P,Q}),$$

it follows that $g = x_5 t$ with $t \in C_{O_2(G_{P,Q})}(x_0) = \langle x_1, x_2 y_2, y_2 z_2, x_3\rangle$. Similarly, $x_6^{vg} = (x_2 x_3 x_4 x_5 x_6)^t$ implies that $t = x_1 p$ with $p \in C_{\langle x_1, x_2 y_2, y_2 z_2, x_3\rangle}(x_6) = \langle x_3\rangle$. Thus

$$(vge)^3 \in C_{G_{P,Q,S}}(\langle x_0, x_6, h\rangle) = 1.$$

Let $\tilde{G}_n$ be the group generated by elements $x_0, \ldots, x_7, h, v$ and defined by the relations (3·2), (6·1), (6·3) and

$$(vx_1 x_3^n x_5 s^{rs})^3 = 1$$

for $n = 0$ or 1. By coset enumeration, we find that $\tilde{G}_n = 1$ for $n = 0$ and 1.

7. *The remaining cases*

Choose $\gamma \in \Gamma_{\mathscr{C}}(P)$ and let $M = O_p(G_\gamma)$, where p is the prime dividing q.

LEMMA 7·1. $\Gamma_P \ncong \mathscr{H}_1(3)$.

Proof. Suppose $\Gamma_P \cong \mathscr{H}_1(3)$. Then $G_P \cong G_2(3)$, $|M/Z(M)| = 9$ and $G_{P,\gamma}$ induces $\mathrm{GL}_2(3)$ on $M/Z(M)$. On the other hand, $G_\gamma/M \cong 2.S_5$, so G_γ can only act trivially on $M/Z(M)$. ∎

LEMMA 7·2. $\Gamma_P \ncong \mathscr{H}_1(4)^\circ$.

Proof. Suppose $\Gamma_P \cong \mathscr{H}_1(4)^\circ$. Then $M = \langle x_1, x_1^k, x_2, x_2^k, \ldots, x_5, x_5^k\rangle$, where $x_1, \ldots, x_5, k$ are as in Section 5. The subgroup $C_{G_{P,\gamma}}(k)$ induces $3.A_5$ on $M/Z(M)$. Since k acts fixed point freely on M, the group $M/Z(M)$ can be considered as an orthogonal module for $C_{G_\gamma}(k)$ over GF(4). In particular, $C_{G_\gamma}(k)$ can induce at most $O_4^+(4)$ on $M/Z(M)$. But $C_{G_\gamma}(k) N_\gamma/N_\gamma \cong A_6$, and A_6 is not a composition factor of the group $O_4^+(4)$. ∎

LEMMA 7·3. $\Gamma_P \ncong \mathscr{H}_2(3)$.

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Proof. Suppose $\Gamma_P \cong \mathscr{H}_2(3)$. Then $G_P \cong {}^3D_4(3)$ or $\mathrm{Aut}({}^3D_4(3))$. If h is an element of order 26 in N_γ, then $|C_M(h)| = 9$ and $C_{G_{P,\gamma}}(h)$ induces $\mathrm{GL}_2(3)$ on $C_M(h)$. But $C_{G_\gamma}(h)N_\gamma/N_\gamma \cong S_5$. ∎

LEMMA 7·4. $\Gamma_P \ncong \mathscr{H}_2(4)$.

Proof. Suppose $\Gamma_P \cong \mathscr{H}_2(4)$. Then G_P is isomorphic to a subgroup of $\mathrm{Aut}({}^3D_4(4))$ containing ${}^3D_4(4)$ and $M/M' \cong 2^{12}$. If h is an element of order 63 in N_γ, then $\langle h \rangle$ acts regularly on M/M', so M/M' can be considered as a 2-dimensional module for $C_{G_\gamma}(h)$ over GF(64). The group $C_{G_{P,\gamma}}(h)$ induces $63.A_5$ on M/M'. But $C_{G_\gamma}(h)N_\gamma/N_\gamma \cong A_6$, and A_6 is not a composition factor of the group $\mathrm{GL}_2(64)$. ∎

By (2·2), the proof of Theorem 1·2 is now complete.

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