Math. Proc. Camb. Phil. Soc. (1976), **80**, 71
MPCPS 80–8
Printed in Great Britain

Syntopogenous preordered spaces

By D. C. J. BURGESS and M. FITZPATRICK
Queen's University, Belfast and Northern Ireland Polytechnic, Jordanstown

(*Received* 1 *March* 1975, *revised* 6 *June* 1975)

1. *Introduction*

The investigation of a topological, uniform or proximity space endowed with a preordering is by now well established. A topological space along with a preordering related to the topology in varying degrees has been studied, for example, in (3), (4), (5) and (6). 'Uniform preordered spaces' and 'Proximity preordered spaces' have been considered in (6) and (8) respectively. In each of these cases generalizations of varying proportions of the classical theories have been obtained.

A generalization, in another direction, of topological, uniform and proximity spaces, namely syntopogenous spaces, has been given by Császár in (2). The aim of the present paper is to introduce a theory of syntopogenous preordered spaces which will:

(i) contain, as a particular case, the uniform preordered spaces of Nachbin (6) and the proximity preordered spaces of Singal and Lal (8);

(ii) yield a method of relating a topology and a preordering so as to give a natural definition of a 'topological preordered space'; and

(iii) generalize some of the classical theory of syntopogenous spaces.

With reference to (i), some results from (6) and (8) are strengthened and a preordering generalization of the classical proximity–uniformity relationship, not dealt with in (6) or (8), is established.

2. *Preliminaries*

If τ is a classical topology on a set E, we denote the τ-closure (τ-interior) of $A \subset E$ by $C^\tau(A)$ $(I^\tau(A))$; if τ_1 and τ_2 are two such topologies then, as usual (1), $\tau_1 \vee \tau_2$ denotes the coarsest such topology finer than τ_1 and τ_2.

We assume familiarity with the notation and results of Császár ((2); in particular chapter 8) and with the correspondence between biperfect syntopological, topogenous or perfect topogenous structures and quasi-uniformities, quasi-proximities ((7); 19.2) and topologies respectively, as usually defined. (A (quasi-) uniformity, as defined by Császár, is what is more usually called a (quasi-) uniform base; but there is no difficulty in translating from one terminology to the other.) We note in particular that if $\mathscr{S}$ is a biperfect syntopology (topogenous structure) on E defining a (classical) quasi-uniformity $\mathscr{W}$ (quasi-proximity δ), then $\mathscr{S}^{sb} \sim (\mathscr{S} \vee \mathscr{S}^c)^b$ $(\mathscr{S}^s = (\mathscr{S} \vee \mathscr{S}^c)^t)$ defines the coarsest (classical) uniformity finer than $\mathscr{W}$ (proximity finer than δ), which we call the uniformity (proximity) generated by $\mathscr{W}$ (δ). If $\mathscr{S}$ is a syntopogenous structure on E, we denote by $\tau(\mathscr{S}) = \tau(\mathscr{S}^{tp})$ the classical topology corresponding with $\mathscr{S}^{tp}$; it is easily seen that $x \in C^{\tau(\mathscr{S})}(A)$ if and only if, for every $<$ in $\mathscr{S}$, $x \nleq E - A$.

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

In addition we use the following notation and terminology relating to preorderings. A preordering $\leqslant$ on a (non-empty) set E is a reflexive transitive relation; an ordering is an anti-symmetric preordering; the pair $(E, \leqslant)$ is called a preordered or ordered set, as the case may be. The graph $G(\leqslant)$ of the preordering $\leqslant$ is the set of all pairs (x, y) for which $x \leqslant y$ and is denoted by $G(\leqslant)$; $\leqslant^d$ is used to denote the preordering on E whose graph is $(G(\leqslant))^d$. If $x \leqslant y$ only when $x = y$, then $\leqslant$ is called the discrete ordering on E. In the general case, given $x \in E$, $[\leftarrow, x]$ ($[x, \rightarrow]$) is the set of all $y \in E$ such that $y \leqslant x$ ($x \leqslant y$). A subset A of E is said to be $\leqslant$-decreasing ($\leqslant$-increasing) if

$$[\leftarrow, x] \subset A ([x, \rightarrow] \subset A)$$

for each $x \in A$. A is said to be $\leqslant$-convex if $A = B \cap C$, where B is $\leqslant$-decreasing and C is $\leqslant$-increasing. Where there is no possibility of confusion the $\leqslant$ in these terms may be omitted.

Consider now the triple $(E, \tau, \leqslant)$, where τ is a classical topology on E and $\leqslant$ is a preordering on E. If $A \subset E$ then there is a smallest decreasing (increasing) τ-closed superset of A, denoted by $C^\tau_{\geqslant}(A)$ ($C^\tau_{\leqslant}(A)$) and a largest decreasing (increasing) τ-open subset of A, denoted by $I^\tau_{\geqslant}(A)$ ($I^\tau_{\leqslant}(A)$). Note also that the family of decreasing (increasing) τ-open subsets of E forms a topology (on E) denoted by $\tau^l(\tau^u)$. The triple $(E, \tau, \leqslant)$ is called:

(a) convex if $\tau = \tau^u \vee \tau^l$;

(b) T_1-preordered if $[x, \rightarrow]$ and $[\leftarrow, x]$ are τ-closed for each $x \in E$;

(c) T_2-preordered if, given $x \not\leqslant y$ in E, $\exists$ τ-neighbourhoods G_x and G_y of x and y respectively such that $G_x \cap G_y = \varnothing$, G_x is increasing and G_y is decreasing. (Clearly (b) is implied by (c).)

Finally note that throughout this paper, unless otherwise stated, E is a non-empty set, $\leqslant$ is a preordering on E and S is a syntopogenous structure on E.

3. *The preordering generated by a syntopogenous structure*

3·1. DEFINITION. A syntopogenous structure $\mathscr{S}$ on E is said to *generate*, or to *be compatible with*, the preordering $\leqslant$ if and only if $x \leqslant y \Leftrightarrow x \in C^{\tau(\mathscr{S})}(\{y\})$ (this relation is clearly a preordering, for any $\mathscr{S}$); in this case $G(\leqslant)$ is often denoted by $G(\mathscr{S})$.

(We note here that a (classical) topology, quasi-proximity or quasi-uniformity will be called $\leqslant$-compatible if it corresponds to a $\leqslant$-compatible syntopogenous structure.)

3·2. THEOREM. *Let $\leqslant$ be the preordering generated by $\mathscr{S}$.*

Then:

(i) *$\leqslant$ is an ordering if and only if $\mathscr{S}$ is T_0,*

(ii) *$\leqslant$ is the discrete ordering on E if and only if $\mathscr{S}$ is T_1;*

(iii) *if $\mathscr{S}$ is symmetrical, then $\leqslant$ is an equivalence relation on E.*

3·3. THEOREM. *Let $\mathscr{S}$, $\mathscr{S}_1$ and $\mathscr{S}_2$ be syntopogenous structures on E, Then:*

(i) *if $\mathscr{S}_1 < \mathscr{S}_2$, $G(\mathscr{S}_2) \subset G(\mathscr{S}_1)$;*

(ii) *if $\mathscr{S}_1 \sim \mathscr{S}_2$, $G(\mathscr{S}_1) = G(\mathscr{S}_2)$;*

(iii) $G(\mathscr{S}) = G(\mathscr{S}^b) = G(\mathscr{S}^p) = G(\mathscr{S}^t)$;

(iv) $G(\mathscr{S}^c) = (G(\mathscr{S}))^d$;

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

(v) $G(\mathscr{S}_1 \vee \mathscr{S}_2) = G(\mathscr{S}_1) \cap G(\mathscr{S}_2)$;

(vi) $G(\mathscr{S}^s) = G(\mathscr{S}) \cap (G(\mathscr{S}))^d$.

Proof. (i) and (ii). Immediate.

(iii) Since $\mathscr{S}^{tp} = (\mathscr{S}^t)^{tp} = (\mathscr{S}^p)^{tp}$ it is immediate that $\mathscr{S}$, $\mathscr{S}^t$ and $\mathscr{S}^p$ have the same corresponding topologies and hence generate the same preordering $\leqslant$. Applying ((2), 5.38) it follows that $x <^b E - y$, for some $<$ in $\mathscr{S}$, if and only if $x < E - y$, and hence $\mathscr{S}^b$ also generates $\leqslant$.

(iv) Immediate.

(v) By ((2), 8.100, 54, 90) $(\mathscr{S}_1 \vee \mathscr{S}_2)^{tp}$ is the coarsest (Császár) topology finer than $\mathscr{S}_1^{tp}$ and $\mathscr{S}_2^{tp}$; thus $\tau(\mathscr{S}_1 \vee \mathscr{S}_2) = \tau(\mathscr{S}_1) \vee \tau(\mathscr{S}_2)$, implying easily that $x \in C(\{y\})$ in $\tau(\mathscr{S}_1 \vee \mathscr{S}_2)$ if and only if $x \in C(\{y\})$ in both $\tau(\mathscr{S}_1)$ and $\tau(\mathscr{S}_2)$.

(vi) Since $\mathscr{S}^s \sim \mathscr{S} \vee \mathscr{S}^c$ ((2), 8.107) then the result follows from (ii,) (iv) and (v).

3·4. Corollary. *Let $\mathscr{S}$ be $\leqslant$-compatible and let*

$$\tau_1 = \tau(\mathscr{S}), \quad \tau_2 = \tau(\mathscr{S}^c) \quad \text{and} \quad \tau = \tau(\mathscr{S}^s) = \tau_1 \vee \tau_2.$$

Then:

(*a*) $\tau_1 = \tau_1^u$ *and* $\tau_2 = \tau_2^l$;

(*b*) *for subsets A and B of E and $<$ in $\mathscr{S}$*:

(i) *if $A < B$, then $A <' I^{\tau_1}_{\leqslant}(B)$ for some $<'$ in $\mathscr{S}$*;

(ii) *if $A < B$, then $C^{\tau_2}_{\leqslant}(A) <' B$ for some $<'$ in $\mathscr{S}$*;

(iii) *if $A < B$, then $A <' I^{\tau}_{\leqslant}(B)$ for some $<'$ in $\mathscr{S}$*;

(iv) *if $A < B$, then $C^{\tau}_{\leqslant}(A) <' B$ for some $<'$ in $\mathscr{S}$.*

Proof. (*a*) Since $\mathscr{S}$ is $\leqslant$-compatible and, by 3·3 (iv), $\mathscr{S}^c$ is $\leqslant^d$-compatible, then every τ_1-open subset of E is $\leqslant$-increasing and every τ_2-open subset of E is $\leqslant^d$-increasing and hence $\leqslant$-decreasing.

(*b*) (i) If $A < B$, then, by ((2), (S_2)), $A <' C <' B$ for some $<'$ in $\mathscr{S}$ and some subset C of E. Since $C <' B$, then $C \subset I^{\tau_1}(B)$ and hence $A <' I^{\tau_1}(B)$. Applying (*a*) $I^{\tau_1}(B)$ is $\leqslant$-increasing and hence coincides with $I^{\tau_1}_{\leqslant}(B)$.

(ii) If $A < B$ then $E - B <^c E - A$ and, as in (i), $E - B <'' I^{\tau_2}(E - A) = E - C^{\tau_2}(A)$ for some order $<''$ in $\mathscr{S}^c$. Since $<'' = <'^c$ for some $<'$ in $\mathscr{S}$, then $C^{\tau_2}(A) <' B$. By (*a*) above $C^{\tau_2}(A) = C^{\tau_2}_{\leqslant}(A)$.

(iii) and (iv). It suffices to note that $I^{\tau_1}_{\leqslant}(B) \subset I^{\tau}_{\leqslant}(B)$, $C^{\tau}_{\leqslant}(A) \subset C^{\tau_2}_{\leqslant}(A)$.

The final result in this section shows that the preordered set $(E, \leqslant)$ can be endowed with at least one $\leqslant$-compatible syntopogenous structure.

3·5. Theorem. *Let $<$ be the relation on the subsets of E defined as follows:*

$$A < B \text{ if and only if } [x, \rightarrow] \subset B \text{ for every } x \text{ in } A.$$

Then $\mathscr{S}(= \{<\})$ is a biperfect topogenous structure on E and is $\leqslant$-compatible. $\mathscr{S}$ is called the biperfect topogenous structure generated by $\leqslant$.

Proof. It is almost immediate that $\{<\}$ is the (Császár) topology corresponding to the topology whose open sets are exactly the increasing sets of $(E, \leqslant)$, and hence is a perfect topogenous structure; it is easy to check that it is in fact biperfect. For the $\leqslant$-compatibility of $\mathscr{S}$ note that, given x, y in E, $x \not< E - y$ if and only if $[x, \rightarrow] \not\subset E - y$; thus $x \not< E - y$ if and only if $x \leqslant y$.

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

4. *The definition of a syntopogenous preordered space*

4·1. DEFINITION. *The triple* $(E, \mathscr{S}, \leqslant)$ *is called a syntopogenous preordered space* (SPS) *if there exist syntopogenous structures* $\mathscr{S}_1, \mathscr{S}_2$ *on* E *such that*

(i) $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^a$ *for some ordinary operation* a *in* $\{i, p, b\}$;

(ii) $(\mathscr{S}_1, \mathscr{S}_2)$ *is an* $\leqslant$*-defining pair (that is,* $G(\leqslant) = G(\mathscr{S}_1) = G(\mathscr{S}_2^c)$*).*

If, in addition, $\mathscr{S}$ *is a topogenous structure (syntopology) on* E*, then* $(E, \mathscr{S}, \leqslant)$ *is called a topogenous (syntopological) preordered space, denoted TPS (SyPS).*

The following theorem shows that there is no loss of generality in restricting the ordinary operation a as indicated above.

4·2. THEOREM. *If* $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^t$ *or* $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^s$*, for some* $\leqslant$*-defining pair of syntopogenous structures* $(\mathscr{S}_1, \mathscr{S}_2)$ *on* E*, then* $(E, \mathscr{S}, \leqslant)$ *is a* SPS*, and, in the second case,* $\mathscr{S} \sim \mathscr{S}_0^s$ *for some* $\leqslant$*-compatible syntopogenous structure* $\mathscr{S}_0$ *on* E.

Proof. $(\mathscr{S}_1 \vee \mathscr{S}_2)^t \sim (\mathscr{S}_1^t \vee \mathscr{S}_2^t)$ ((2), 8.101), $G(\mathscr{S}_1^t) = G(\mathscr{S}_1)$ and $G(\mathscr{S}_2^{tc}) = G(\mathscr{S}_2^{ct})$ ((2), 8.61) $= G(\mathscr{S}_2^c)$; so if $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^t$, then $(E, \mathscr{S}, \leqslant)$ is a SPS.

Also $(\mathscr{S}_1 \vee \mathscr{S}_2)^s \sim (\mathscr{S}_1 \vee \mathscr{S}_2) \vee (\mathscr{S}_1 \vee \mathscr{S}_2)^c \sim \mathscr{S}_1 \vee \mathscr{S}_2^c \vee \mathscr{S}_2 \vee \mathscr{S}_1^c \sim (\mathscr{S}_1 \vee \mathscr{S}_2^c) \vee (\mathscr{S}_1 \vee \mathscr{S}_2^c)^c$ and $G(\mathscr{S}_1 \vee \mathscr{S}_2^c) = G(\mathscr{S}_1) \cap G(\mathscr{S}_2^c) = G(\leqslant)$. Thus $\mathscr{S}_0 (= \mathscr{S}_1 \vee \mathscr{S}_2^c)$ is $\leqslant$-compatible and $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^s$ implies $\mathscr{S} \sim \mathscr{S}_0^s$.

4·3. EXAMPLES.

(i) *Suppose* $\mathscr{S}$ *is symmetrical and let* $\leqslant$ *be the preordering on* E *generated by* $\mathscr{S}$*. Since* $G(\mathscr{S}) = G(\leqslant) = (G(\mathscr{S}^c))^d$ *(by* 3·3 (iv)*) and* $\mathscr{S}^s \sim (\mathscr{S} \vee \mathscr{S}^c)^i$*, then* $(E, \mathscr{S}, \leqslant)$ *is a SPS.*

(ii) *If* $(E, \mathscr{S})$ *is* T_1*, then, by* 3·2, $G(\mathscr{S}) = G(\leqslant)$*, where* $\leqslant$ *is the discrete ordering on* E*. Since, trivially,* $\mathscr{S} \sim (\mathscr{S} \vee \mathscr{S})^i$*, then* $(E, \mathscr{S}, \leqslant)$ *is a SPS.*

(iii) *Let* $\mathscr{S}$ *be constructed from* $\leqslant$ *as in* 3·5*. By* 3·3

$$G(\mathscr{S}) = G(\leqslant) = (G(\mathscr{S}^c))^d = G(\mathscr{S}^t) = (G(\mathscr{S}^{tc}))^d.$$

Since $\mathscr{S}^s \sim (\mathscr{S} \vee \mathscr{S}^c)^i$ ((2), 8.107), $\mathscr{S}^{st} = (\mathscr{S} \vee \mathscr{S}^c)^t$ ((2), 8.36), $\mathscr{S}^{sb} \sim (\mathscr{S} \vee \mathscr{S}^c)^b$ and $\mathscr{S}^{stb} = \mathscr{S}^{tsb}$ ((2), 8.51) $\sim (\mathscr{S}^t \vee \mathscr{S}^{tc})^b$*, then each of* $(E, \mathscr{S}^s, \leqslant)$, $(E, \mathscr{S}^{st}, \leqslant)$, $(E, \mathscr{S}^{sb}, \leqslant)$ *and* $(E, \mathscr{S}^{stb}, \leqslant)$ *is a SPS.*

4·4. THEOREM. *Let* $(E, \mathscr{S}, \leqslant)$ *be a SPS. Then* $\mathscr{S}$ *is* T_0 *if and only if* $\leqslant$ *is an ordering on* E.

Proof. In the notation of 4·1 suppose $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^a$. If $\leqslant$ is an ordering, then, by 3·2 (i), $\mathscr{S}_1$ is T_0 and hence $\mathscr{S}$ is also T_0 ((2), 14.8).

Conversely, if $\mathscr{S}$ is T_0, then, by ((2), 14.8–9), $\mathscr{S}_1 \vee \mathscr{S}_2$ is T_0 and, by (3·2) (i), the preordering, $\leqslant_0$, generated by $\mathscr{S}_1 \vee \mathscr{S}_2$ is an ordering. Since

$$G(\leqslant) \cap (G(\leqslant))^d = G(\mathscr{S}_1) \cap G(\mathscr{S}_2) = G(\mathscr{S}_1 \vee \mathscr{S}_2) = G(\leqslant_0)$$

then $\leqslant$ is also an ordering.

A result of Nachbin ((6), p. 59) is reflected in the following corollary.

4·5. COROLLARY. *Let* $(E, \mathscr{S}, \leqslant)$ *be a symmetrical SPS. Then* $\mathscr{S}$ *is* T_2 *if and only if* $\leqslant$ *is an ordering on* E.

Proof. Immediate from 4·4 and ((2), 14.15).

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

4·6. EXAMPLE. *Let $(E, \leqslant)$ be an ordered set and let $\mathscr{S}$ be as in* 4·3 (iii). *Since $(E, \mathscr{S}^{stb}, \leqslant)$ is a SPS, then, by* 4·4 *and* ((2), 14.9), *$\mathscr{S}$ is T_0. It follows from* ((2), 14.7) that *$\mathscr{S}^{stb} = \mathscr{D}_E$, the discrete syntopogenous structure on E. Hence $(E, \mathscr{D}_E, \leqslant)$ is a SPS.*

5. *Perfect topogenous preordered spaces*

5·1. DEFINITION. *A SPS $(E, \mathscr{S}, \leqslant)$ is called a tp-preordered space (TPPS) if $\mathscr{S}$ is perfect and topogenous.*

Motivation for the 'tp' symbol is provided below.

5·2. THEOREM. *The following statements are equivalent:*

(i) *$(E, \mathscr{S}, \leqslant)$ is a TPPS;*

(ii) *$\mathscr{S} = \mathscr{S}_0^{tp}$, where $(E, \mathscr{S}_0, \leqslant)$ is a SPS;*

(iii) *$\mathscr{S} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{tp}$ for some $\leqslant$-defining pair of syntopogenous structures $(\mathscr{S}_1, \mathscr{S}_2)$ on E;*

(iv) *$\mathscr{S} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{tp}$ for some $\leqslant$-defining pair of topologies $(\mathscr{S}_1, \mathscr{S}_2)$ on E.*

Proof. (i) implies (ii): Immediate (by taking $\mathscr{S} = \mathscr{S}_0$).

(ii) implies (iii): For $\mathscr{S}_0 \sim (\mathscr{S}_1 \vee \mathscr{S}_2)$ or $\mathscr{S}_0 \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^p$, in the notation of 4·1, this is trivial as $(\mathscr{S}_1 \vee \mathscr{S}_2)^{ptp} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{tp}$ ((2), 8.50). If $\mathscr{S}_0 \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^b$, then $\mathscr{S} = (\mathscr{S}_1^{btp} \vee \mathscr{S}_2^{btp})^{tp}$ ((2), 8.103) and it suffices to note that $G(\mathscr{S}_i^{btp}) = G(\mathscr{S}_i)$ for $i = 1, 2$.

(iii) implies (iv): $(\mathscr{S}_1 \vee \mathscr{S}_2)^{tp} = (\mathscr{S}_1^{tp} \vee \mathscr{S}_2^{tp})^{tp}$ and $G(\mathscr{S}_i^{tp}) = G(\mathscr{S}_i)$ for $i = 1, 2$.

(iv) implies (i): $(\mathscr{S}_1 \vee \mathscr{S}_2)^{tp} = (\mathscr{S}_1^t \vee \mathscr{S}_2^t)^p = (\mathscr{S}_1 \vee \mathscr{S}_2)^p$ since $\mathscr{S}_1$ and $\mathscr{S}_2$ are topogenous.

5·3. THEOREM. *If $\mathscr{S}$ is perfect and topogenous and τ is the classical topology associated with $\mathscr{S}$, then the following statements are equivalent:*

(i) *$(E, \mathscr{S}, \leqslant)$ is a TPPS.*

(ii) *$\tau = \tau_1 \vee \tau_2$ for some $\leqslant$-defining pair of classical topologies (τ_1, τ_2) on E (that is, $x \leqslant y$ if and only if $x \in C^{\tau_1}(\{y\})$ if and only if $y \in C^{\tau_2}(\{x\})$* (3)).

Proof. Immediate from 5·2 and the equality $\tau(\mathscr{S}_1 \vee \mathscr{S}_2) = \tau(\mathscr{S}_1) \vee \tau(\mathscr{S}_2)$ in the proof of 3·3 (v).

Accordingly we make the following definition:

5·4. DEFINITION. *If τ is a classical topology on E, then the triple $(E, \tau, \leqslant)$ is called a topological preordered space if $\tau = \tau_1 \vee \tau_2$ for some $\leqslant$-defining pair of classical topologies (τ_1, τ_2) on E.*

5·5. THEOREM. *If τ is a classical topology on E, then $(E, \tau, \leqslant)$ is a topological preordered space if and only if $(E, \tau, \leqslant)$ is convex and T_1-preordered.*

Proof. In the notation of 5·4 suppose $\tau = \tau_1 \vee \tau_2$. By 3·4 (*a*) $\tau_1 = \tau_1^u$ and $\tau_2 = \tau_2^l$. Hence $\tau = \tau_1^u \vee \tau_2^l = \tau^u \vee \tau^l$ and so $(E, \tau, \leqslant)$ is convex. If $x \in E$, then $[\leftarrow, x] = C^{\tau_1}(\{x\})$, $[x, \rightarrow] = C^{\tau_2}(\{x\})$, so that both intervals are τ-closed.

Suppose conversely that $(E, \tau, \leqslant)$ is convex and T_1-preordered. From convexity $\tau = \tau^u \vee \tau^l$. The remainder follows directly from the definitions of τ^u, τ^l and that fact that $(E, \tau, \leqslant)$ is T_1-preordered.

We observe now that, by 5·2 and 5·3, the conditions of this theorem are satisfied by $\tau = \tau(\mathscr{S})$, whenever $(E, \mathscr{S}, \leqslant)$ is a SPS.

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

6. *Symmetrical syntopological preordered spaces*

6·1. DEFINITION. *A SPS $(E, \mathscr{S}, \leqslant)$ is called a sb-preordered space (SBPS) if $\mathscr{S}$ is a symmetrical syntopology, that is, $\mathscr{S}$ is symmetrical and biperfect.*

Motivation for the 'sb' symbol in this definition is provided below.

6·2. THEOREM. *If $\mathscr{S}$ is a symmetrical syntopology on E, then the following statements are equivalent:*

(i) *$(E, \mathscr{S}, \leqslant)$ is a SBPS;*

(ii) *$\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^b$ for some $\leqslant$-defining pair of syntopogenous structures $(\mathscr{S}_1, \mathscr{S}_2)$ on E;*

(iii) *$\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^{sb}$ for some $\leqslant$-defining pair of syntopogenous structures $(\mathscr{S}_1, \mathscr{S}_2)$ on E;*

(iv) *$\mathscr{S} \sim \mathscr{S}_0^{sb}$ for some $\leqslant$-compatible syntopogenous structure $\mathscr{S}_0$ on E;*

(v) *$\mathscr{S} \sim \mathscr{S}_0^{sb}$ for some $\leqslant$-compatible biperfect syntopology $\mathscr{S}_0$ on E.*

Proof. (i) implies (iii): If, in the notation of 4·1, $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^a$ ($a = i$, p or b) we have at once $\mathscr{S} < (\mathscr{S}_1 \vee \mathscr{S}_2)^{sa} < (\mathscr{S}_1 \vee \mathscr{S}_2)^{sb}$. However, $(\mathscr{S}_1 \vee \mathscr{S}_2) < \mathscr{S}$, so that, $\mathscr{S}$ being symmetrical and biperfect, $(\mathscr{S}_1 \vee \mathscr{S}_2)^{sb} < \mathscr{S}$, yielding $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^{sb}$.

(iii) implies (iv): Note that $(\mathscr{S}_1 \vee \mathscr{S}_2)^s \sim (\mathscr{S}_1 \vee \mathscr{S}_2^c)^s$ by ((2), 8.102,107) and $G(\mathscr{S}_1 \vee \mathscr{S}_2^c) = G(\mathscr{S}_1) \cap (G(\mathscr{S}_2))^d = G(\leqslant)$.

(iv) implies (v): Suppose $\mathscr{S} \sim \mathscr{S}_0^{sb}$ and $\mathscr{S}_0$ is $\leqslant$-compatible. Since

$$\mathscr{S}_0^{sb} = \mathscr{S}_0^{sbb} = \mathscr{S}_0^{bspb} = \mathscr{S}_0^{bsb} \quad \text{and} \quad G(\mathscr{S}_0^b) = G(\mathscr{S}_0),$$

then (v) is true.

(v) implies (ii): Note that $\mathscr{S}_0^{sb} \sim (\mathscr{S}_0 \vee \mathscr{S}_0^c)^b$ and $G(\mathscr{S}_0) = G(\leqslant)$.

(ii) implies (i): Immediate.

The condition of 6·2 (v) is equivalent to saying that the (classical) uniformity defined by $\mathscr{S}$ is generated by the quasi-uniformity defined by $\mathscr{S}_0$, which is $\leqslant$-compatible. It follows that our definition of a SBPS corresponds to Nachbin's definition (6) of a uniform preordered space.

We note that every uniform space $(E, \mathscr{U})$ can, in view of 4·3 (i), be endowed with a preordering $\leqslant$ so that $(E, \mathscr{U}, \leqslant)$ is a uniform preordered space. The preordering $\leqslant$ is defined by: $x \leqslant y$ if and only if $(x, y) \in \cap \{U; U \in \mathscr{U}\}$.

7. *Symmetrical topogenous preordered spaces*

7·1. THEOREM. *$(E, \mathscr{S}, \leqslant)$ is a symmetrical TPS if and only if one of the following conditions is satisfied:*

(i) *$\mathscr{S} = \mathscr{S}_0^s$ for some $\leqslant$-compatible topogeneous structure $\mathscr{S}_0$ on E;*

(ii) *$\mathscr{S} = \mathscr{S}_0^{sb}$ for some $\leqslant$-compatible topogenous structure $\mathscr{S}_0$ on E.*

Proof. Let $\mathscr{S} = \mathscr{S}^{st}$. In the notation of 4·1 suppose $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^a$. If $a = i$, then $\mathscr{S} = \mathscr{S}^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2^c)^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2^c)^{ts}$ ((2), 8.51) and so $\mathscr{S} = \mathscr{S}_0^s$, where $\mathscr{S}_0 = (\mathscr{S}_1 \vee \mathscr{S}_2^c)^t$ and $G(\mathscr{S}_0) = G(\mathscr{S}_1) \cap (G(\mathscr{S}_2))^d = G(\leqslant)$. Hence (i) is true.

If $a = b$, then $\mathscr{S}^b \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^{bb} = (\mathscr{S}_1 \vee \mathscr{S}_2)^b \sim \mathscr{S}$. Since $\mathscr{S}$ and $\mathscr{S}^b$ are both topogenous then $\mathscr{S} = \mathscr{S}^b$, implying that $\mathscr{S}$ is biperfect and, in particular, perfect. With $\mathscr{S}$ also being symmetrical it follows that

$$\mathscr{S} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{btbsp} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{tbsp} \text{ ((2), 8.50)} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{tsb}.$$

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

Now $(\mathscr{S}_1 \vee \mathscr{S}_2)^{ts} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{st}$ and, just as in the first part of this proof, this can be written $\mathscr{S}_0^s$, where $\mathscr{S}_0$ is topogenous and generates $\leqslant$. That is, $\mathscr{S} = \mathscr{S}_0^{sb}$, so that (ii) is satisfied. For $a = p$, then, by a similar argument to that for 'b', $\mathscr{S}$ is perfect and, being symmetrical, is also biperfect ((2), 7.6). Hence $\mathscr{S} = \mathscr{S}^b \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^{pb} = (\mathscr{S}_1 \vee \mathscr{S}_2)^b$ and, as above, (ii) is true.

Suppose, conversely, that either (i) or (ii) is satisfied. $\mathscr{S}$ is clearly a symmetrical topogenous structure on E. In addition, since $\mathscr{S}_0^s \sim (\mathscr{S}_0 \vee \mathscr{S}_0^c)$ and $G(\mathscr{S}_0) = G(\leqslant)$,then $(E, \mathscr{S}, \leqslant)$ is a TPS.

7·2. Two alternative forms of 7·1 (ii) are established below:

(ii)′ $\mathscr{S} = \{<\}$, *where* $A < B$ *if and only if, given* $x \in A$ *and* $y \in E - B$, *either* $x \not\leqslant y$ *or* $y \not\leqslant x$;

(ii)″ $\mathscr{S} = \mathscr{S}_0^{sb}$, *where* $\mathscr{S}_0$ *is the biperfect topogenous structure on* E *generated by* $\leqslant$.

Proof. If 7·1 (ii) holds then denote $\mathscr{S}$ by $\{<\}$ and $\mathscr{S}_0$ by $\{<_0\}$. Since

$$< \; = \; <_0^{sb} = (<_0 \cup <_0^c)^b \text{ ((2), 5.30)},$$

then $A < B$ if and only if, given $x \in A$ and $y \in E - B$, either $x <_0 E - y$ or $y <_0 E - x$ ((2), 5.38), that is, either $x \not\leqslant y$ or $y \not\leqslant x$, since $\mathscr{S}_0$ is $\leqslant$-compatible.

To see that (ii)′ implies (ii)″ let $\mathscr{S}_0$ $(= \{<_0\})$ be the biperfect topogenous structure generated by $\leqslant$ (see 3·5 above). Then $A <_0^{sb} B$ if and only if, given $x \in A$ and $y \in E - B$, either $x <_0 E - y$ or $y <_0 E - x$, that is, either $x \not\leqslant y$ or $y \not\leqslant x$. Hence $\mathscr{S}_0^{sb} = \mathscr{S}$.

That (ii)″ implies (ii) is immediate.

7·3. Theorem. *Let* $\leqslant$ *be an ordering on* E. *Then* $(E, \mathscr{S}, \leqslant)$ *is a symmetrical TPS if and only if one of the following conditions is satisfied:*

(i) $\mathscr{S} = \mathscr{S}_0^s$ *for some* $\leqslant$*-compatible topogenous structure* $\mathscr{S}_0$ *on* E;

(ii) $\mathscr{S} = \mathscr{D}_E$.

Proof. Denote $\mathscr{S}$ by $\{<\}$. It suffices to note that, in statement 7·2 (ii)′, $A < B$ if and only if, given $x \in A$ and $y \in E - B$, $x \neq y$, that is, $A < B$ if and only if $A \subset B$.

It follows from 7·2 that condition 7·1 (ii) is satisfied by exactly one syntopogenous structure $\mathscr{S}$ on E and that $\mathscr{S}$ is completely determined by $\leqslant$; moreover, when $\leqslant$ is an ordering, $\mathscr{S}$ reduces to the discrete syntopogenous structure on E. Accordingly attention will, henceforth, be focused on those syntopogenous structures on E which satisfy condition 7·1 (i).

7·4. Definition. *A SPS is said to be a st-preordered space (STPS) if* $\mathscr{S} = \mathscr{S}_0^s$ *for some* $\leqslant$*-compatible topogenous structure* $\mathscr{S}_0$ *on* E.

Motivation for the '*st*' symbol is provided in 7·6 below.

7·5. Theorem. *If* $(E, \mathscr{S}, \leqslant)$ *is a totally bounded SPS, then* $(E, \mathscr{S}^{st}, \leqslant)$ *is a STPS.*

Proof. Suppose $\mathscr{S}$ is totally bounded and that, in the notation of 4·1, $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^a$. Then $\mathscr{S}^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{ast}$. If $a = i$, then $\mathscr{S}^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{st}$. For $a = b$ then, since $\mathscr{S}_1 \vee \mathscr{S}_2$ is totally bounded ((2), 19.6), $(\mathscr{S}_1 \vee \mathscr{S}_2) \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^b$ ((2), 19.3). Hence

$$\mathscr{S}^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{bst} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{st}.$$

If $a = p$, a similar application of ((2), 19.6, 13) yields $\mathscr{S}^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{st}$.

Since in all cases $\mathscr{S}^{st} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{st}$, the result follows as in the first stage of the proof of 7·1.

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

7·6. COROLLARY. *The following statements are equivalent:*

(i) $(E, \mathscr{S}, \leqslant)$ *is a STPS;*

(ii) $\mathscr{S} = \mathscr{S}_0^{st}$ *for some* $\leqslant$*-compatible syntopogenous structure* $\mathscr{S}_0$ *on* E*;*

(iii) $\mathscr{S} = \mathscr{S}_0^{st}$ *for some* $\leqslant$*-compatible totally bounded biperfect syntopology* $\mathscr{S}_0$ *on* E*;*

(iv) $\mathscr{S} = \mathscr{S}_0^{sbt}$ *for some* $\leqslant$*-compatible totally bounded biperfect syntopology* $\mathscr{S}_0$ *on* E*;*

(v) $\mathscr{S} = \mathscr{S}_0^{sbt}$ *for some* $\leqslant$*-compatible totally bounded syntopogenous structure* $\mathscr{S}_0$ *on* E.

Proof. The equivalence of (i) and (ii) is an immediate consequence of the fact that $\mathscr{S}_0^{st} = \mathscr{S}_0^{ts}$ and $G(\mathscr{S}_0) = G(\mathscr{S}_0^t)$.

(ii) implies (iii): Suppose $\mathscr{S} = \mathscr{S}_0^{st}$ and that $\mathscr{S}_0$ is $\leqslant$-compatible. Now

$$\mathscr{S}_0^{st} = \mathscr{S}_0^{ts} = \mathscr{S}_1^{ts} = \mathscr{S}_1^{st}$$

for some totally bounded biperfect syntopology $\mathscr{S}_1$ on E such that $\mathscr{S}_1^t = \mathscr{S}_0^t$ ((2), 19.38). From 3·3(iii), $\mathscr{S}_1$ is clearly $\leqslant$-compatible.

(iii) implies (iv): If $\mathscr{S}_0$ is totally bounded then $\mathscr{S}_0^s$ is also totally bounded and hence $\mathscr{S}_0^s \sim \mathscr{S}_0^{sb}$ ((2), 19.8, 13). It follows that $\mathscr{S}_0^{st} = \mathscr{S}_0^{sbt}$.

(iv) implies (v): Immediate.

(v) implies (ii): If $\mathscr{S}_0$ is totally bounded, then, as in the proof of '(iii) implies (iv)', $\mathscr{S}_0^{st} = \mathscr{S}_0^{sbt}$.

The condition of 7·4 is equivalent to saying that the proximity defined by $\mathscr{S}$ is generated by a quasi-proximity which is $\leqslant$-compatible, so that our definition of a STPS corresponds to Singal and Lal's definition (8) of a proximity preordered space.

We note here that every proximity space (E, ∂) can, in view of 4·3 (i), be endowed with a preordering $\leqslant$ so that $(E, \partial, \leqslant)$ is a proximity preordered space. The preordering $\leqslant$ is defined by: $x \leqslant y$ if and only if $x \,\partial\, y$.

8. *Relationships between sb- and st-preordered spaces*

A preordering analogue of the major part of ((2), 19.38) is given below.

8·1. THEOREM. $(E, \mathscr{S}, \leqslant)$ *is a STPS if and only if* $\mathscr{S} = \mathscr{S}_0^t$, *where* $(E, \mathscr{S}_0, \leqslant)$ *is a totally bounded SBPS.*

Proof. Suppose that $(E, \mathscr{S}, \leqslant)$ is a STPS. By 7·6 (v) $\mathscr{S} = \mathscr{S}_1^{sbt}$ for some $\leqslant$-compatible totally bounded syntopogenous structure $\mathscr{S}_1$ on E. It suffices to set $\mathscr{S}_0 = \mathscr{S}_1^{sb}$ and apply 6·2 (iv).

The converse follows from 7·5.

8·1 A. THEOREM. $(E, \partial, \leqslant)$ *is a proximity preordered space if and only if* ∂ *is the proximity deduced from a totally bounded uniformity* $\mathscr{U}$ *on* E, *where* $(E, \mathscr{U}, \leqslant)$ *is a uniform preordered space.*

For the combination of 12.61, 12.62 and 13.54 in (2), and noting Császár's definition of 'uniformizable' in ((2), p. 174), the following preordering analogue is obtained.

8·2. THEOREM. *If* $(E, \mathscr{S}, \leqslant)$ *is a SBPS (STPS), then* $\mathscr{S}^{tp} = \mathscr{S}_0^{tp}$, *where* $(E, \mathscr{S}_0, \leqslant)$ *is a STPS (SBPS).*

Proof. Suppose $\mathscr{S} \sim \mathscr{S}_1^{sb}$ for some $\leqslant$-compatible biperfect syntopology $\mathscr{S}_1$ on E. If $\mathscr{S}_0 = \mathscr{S}_1^{st}$, then $(E, \mathscr{S}_0, \leqslant)$ is a STPS by 7·6 and $\mathscr{S}_0^{tp} = \mathscr{S}_1^{sttp} = \mathscr{S}_1^{stp} = \mathscr{S}_1^{sptp}$ ((2), 8·50) $= \mathscr{S}_1^{bsptp} = \mathscr{S}_1^{sbtp}$ ((2), 8.52) $= \mathscr{S}^{tp}$.

The result in parentheses follows immediately from 8·1.

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press

8·2A. THEOREM. *If τ is a classical topology on E then the two statements below are equivalent:*

τ is deduced from a uniformity $\mathscr{U}$ (proximity ∂) on E for which $(E, \mathscr{U}, \leqslant)$ ($(E, \partial, \leqslant)$) is a uniform (proximity) preordered space.

In view of 4·3 (i), Theorems 8·1 and 8·2 are also generalizations of the corresponding results in (2).

8·3. THEOREM. *If $(E, \mathscr{S}, \leqslant)$ is either a STPS or a SBPS and τ is the classical topology associated with $\mathscr{S}^{tp}$, then $(E, \tau, \leqslant)$ is T_2-preordered.*

Proof. In view of 8·2 it suffices to consider the case where $(E, \mathscr{S}, \leqslant)$ is a STPS. Suppose $\mathscr{S} = \mathscr{S}_0^s$ in the notation of 7·4. If $x \not\leqslant y$ then $x <_0 E - y$, where $\mathscr{S}_0 = \{<_0\}$. By ((2), (S_2)) $x <_0 A <_0 B <_0 E - y$ for some subsets A and B of E. Set $G_x = I^\tau_{\leqslant}(A)$ and $G_y = E - C^\tau_{\leqslant}(B)$. Then $x <_0 G_x$ and $y <_0 G_y$ by 3·4 (b). Clearly $G_x(G_y)$ is a $\leqslant$-increasing (-decreasing) τ-neighbourhood of $x(y)$ and $G_x \cap G_y = \varnothing$.

The above theorem, combined with 5·2, gives the following result:

8·4. THEOREM. *If $(E, \mathscr{U}, \leqslant)$ ($(E, \partial, \leqslant)$) is a uniform (proximity) preordered space and τ is the classical topology deduced from $\mathscr{U}(\partial)$, then:*

(i) *$(E, \tau, \leqslant)$ is a topological preordered space;*

(ii) *$(E, \tau, \leqslant)$ is T_2-preordered.*

Since conclusion (i) of 8·4 implies, by 5·5, that $(E, \tau, \leqslant)$ is convex, it strengthens Nachbin's result ((6), Proposition 9) that the classical topology deduced from a uniform preordered space is locally convex ((6), p. 26). The same may be said for the analogous result of Singal and Lal in (8). Conclusion (ii) of 8·6 echoes the corresponding results of Nachbin ((6), Proposition 8) and Singal and Lal (8).

In closing, the authors wish to express their sincere gratitude to the referee for his many helpful suggestions.

REFERENCES

(1) BURGESS, D. C. J. *Analytical topology*. (London, D. Van Nostrand Company, Ltd, 1966).

(2) CSÁSZÁR, A. *Foundations of general topology* (Oxford, Pergamon Press, 1963).

(3) MCCALLION, T. Compactifications of ordered topological spaces. *Proc. Cambridge Philos. Soc.* **71** (1972), 463–73.

(4) MCCARTAN, S. D. Separation axioms for topological ordered spaces. *Proc. Cambridge Philos. Soc.* **64** (1968), 965–73.

(5) MCCARTAN, S. D. Bicontinuous preordered topological spaces. *Pacific J. Math.* **38** (1971), 523–29.

(6) NACHBIN, L. *Topology and order* (Princeton, New Jersey, D. Van Nostrand Company, Inc., 1965).

(7) NAIMPALLY, S. A. and WARRACK, B. D. *Proximity spaces* (Cambridge Tracts. No. 59, Cambridge University Press, 1970).

(8) SINGAL, M. K. and LAL, SUNDER. Proximity ordered spaces. To appear in *Indian J. Math.*

https://doi.org/10.1017/S0305004100052683 Published online by Cambridge University Press