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An Inverse Diffraction Problem: Shape Reconstruction

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Abstract. An inverse diffraction problem is considered. Both classical Tikhonov regularisation and a slow-evolution-from-the-continuation-boundary (SECB) method are used to solve the ill-posed problem. Regularisation error estimates for the two methods are compared, and the SECB method is seen to be an improvement on the classical Tikhonov method. Two numerical examples demonstrate their feasibility and efficiency.

AMS subject classifications: 65J20, 65R35
Key words: Inverse diffraction problem, ill-posed problems, Tikhonov regularisation, stability estimate, error estimate, SECB.

1. Introduction

The reconstruction problem for an aperture in the plane from the resulting diffraction pattern arises in acoustics and optics, and in this article we consider the following inverse diffraction problem.

Let D be a bounded aperture in an infinite perfectly soft screen located in the plane $z=0$ in $\mathbb{R}^3$. An harmonic plane wave with wavenumber k propagates along with the positive z direction, and on hitting the screen it escapes through the aperture D. The measured data at the receiving screen at $z=d>0$ is given. The problem is to reconstruct the shape (domain) of the aperture D— cf. Fig. 0. This problem has been considered by Sondhi [1], Bertero [2] and Magnanini [3]. Under the Kirchhoff approximation, the mathematical modelling of this phenomenon can be formulated as follows:

Problem 1. *If $u(x,y,z)$ denotes the solution of the boundary value problem*

$$u_{xx}+u_{yy}+u_{zz}+k^2u(x,y,z)=0\,,\qquad (x,y)\in\mathbb{R}^2, z>0\,, \tag{1.1}$$

$$u(x,y,d)=g(x,y)\,,\qquad (x,y)\in\mathbb{R}^2\,, \tag{1.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}-iku\right)=0\,,\ \textit{where } r=\sqrt{x^2+y^2+z^2}\,, \tag{1.3}$$

we seek to determine $u(x,y,0)=\chi_D(x,y)$ from the data $g(x,y)$ where $\chi_D(x,y)$ is the characteristic function of the domain D.

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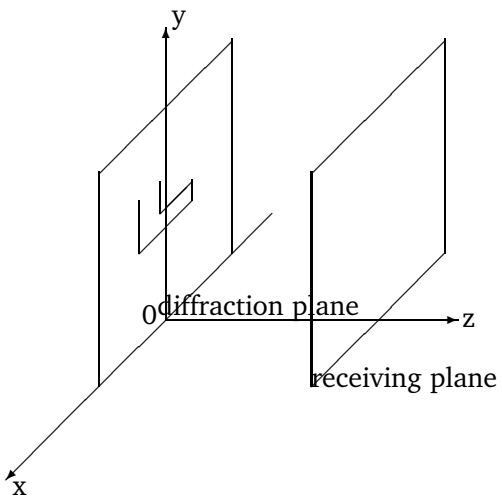


Figure 1: Inverse diffraction problem.

However, here we will consider that the characteristic function χ_D is replaced by an $\mathbb{R}^2$-square integrable function $f(x,y)$, in the following generalisation of Problem 1:

Problem 2. *If $u(x,y,z)$ denotes the solution to the boundary value problem*

$$u_{xx}+u_{yy}+u_{zz}+k^2u(x,y,z)=0\,, \qquad (x,y)\in\mathbb{R}^2, z>0\,, \tag{1.4}$$

$$u(x,y,d)=g(x,y)\,, \qquad (x,y)\in\mathbb{R}^2\,, \tag{1.5}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}-iku\right)=0\,,\ \textit{where } r=\sqrt{x^2+y^2+z^2}\,, \tag{1.6}$$

we seek to determine $u(x,y,0)=f(x,y)$ from the data $g(x,y)$.

Before investigating the inverse problem, let us briefly discuss the corresponding forward problem of Problem 2 as follows.

Forward problem of Problem 2. *Find the solution $u(x,y,z)$ of the boundary value problem*

$$u_{xx}+u_{yy}+u_{zz}+k^2u(x,y,z)=0\,, \qquad (x,y)\in\mathbb{R}^2, z>0\,, \tag{1.7}$$

$$u(x,y,0)=f(x,y), \qquad (x,y)\in\mathbb{R}^2\,, \tag{1.8}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r}-iku\right)=0\,,\textit{where } r=\sqrt{x^2+y^2+z^2}\,. \tag{1.9}$$

We wish to find the $u(x,y,z)=g(x,y)$ from the data $f(x,y)$ for a fixed $0<z\le d$. It is well known that if $f(x,y)$ has bounded support, then the forward problem admits a unique solution

$$u(x,y,z)=\int_{\mathbb{R}^2} H_z(x-\mu,y-\nu)f(\mu,\nu)d\mu d\nu := A(z)f(x,y)\,, \tag{1.10}$$

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where

$$H_z(x,y) = -\frac{1}{2\pi}\left.\frac{\partial\left(\frac{e^{ik\sqrt{\tau^2+x^2+y^2}}}{\sqrt{\tau^2+x^2+y^2}}\right)}{\partial\tau}\right|_{\tau=z}$$

and $A(z) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the solution operator.

The problem can also be re-posed in the Fourier domain. On writing

$$\hat{h}(\xi,\eta) = \frac{1}{2\pi}\int_{\mathbb{R}^2} h(x,y)e^{-i(\xi x+\eta y)}dxdy \tag{1.11}$$

for the Fourier transform of the function $h(x,y) \in L^2(\mathbb{R}^2)$, the corresponding inverse Fourier transform of the function $\hat{h}(\xi,\eta)$ is given by

$$h(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{h}(\xi,\eta)e^{i(\xi x+\eta y)}d\xi d\eta\,. \tag{1.12}$$

Then on taking Fourier transform of (1.7)-(1.9), the solution of the forward problem of Problem 2 in the frequency domain is

$$\hat{u}(\xi,\eta,z) = e^{-z\sqrt{\xi^2+\eta^2-k^2}}\hat{f}(\xi,\eta) = \widehat{A(z)f}\,, \tag{1.13}$$

and from formula (1.13) we obtain

$$\hat{f}(\xi,\eta) = e^{d\sqrt{\xi^2+\eta^2-k^2}}\hat{g}(\xi,\eta)\,, \tag{1.14}$$

and we can formulate the problem as an operator equation in the frequency domain:

$$\hat{f}(\xi,\eta) = \hat{B}(d)\,\hat{g}(\xi,\eta)\,, \tag{1.15}$$

where $\hat{B}(d) = e^{d\sqrt{\xi^2+\eta^2-k^2}} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a multiplication operator. Obviously, if $\xi^2+\eta^2 \geq k^2$, it is an unbounded operator; and if $\xi^2+\eta^2 \leq k^2$, it is a bounded operator. Since the factor $e^{d\sqrt{\xi^2+\eta^2-k^2}} \to +\infty$ as $\xi,\eta \to \infty$, it is easy to see that the ill-posedness of Problem 2 is due to the perturbation of high frequencies in the noisy data. To stabilise the ill-posed Problem 1 or Problem 2, *a priori* information about the unknown solution is needed [4].

Problem 2 is similar to the Cauchy problem for the Helmholtz equation. Regularisation methods have been suggested by various authors to solve Cauchy problems — including the potential function method [5], the modified Tikhonov regularisation method [6–8], the truncation method [9], the quasi-reversibility method [10], the method of approximate solutions [11], the wavelet moment method [12], the method of fundamental solutions [13], the conjugate gradient methods with the boundary element method [14–17], and the alternating boundary element method [18].

A priori information about an unknown solution has often proven to be essential in the analysis of ill-posed problems in mathematical physics, for otherwise the convergence

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rate of the constructed regularisation method is arbitrarily slow [4, 19–26]. The classical regularisation method is based on prescribed bounds on the derivatives of the unknown solution — cf. (2.1). However, derivatives may fail to exist in many problem of practical interest. To overcome this, Carasso [27–32] introduced an additional constraint called slow evolution from the continuation boundary (SECB), an *a priori* statement about the rate of change of the unknown solution near the continuation boundary (at $z = 0$ here). We mainly concentrate on the error estimates for the SECB method, which requires new *a priori* information on the solution. Unlike the classical Tikhonov method, the SECB method requires no *a priori* smooth assumption on the exact solution to serve as a supplementary stabilisation information. In Ref. [33], an up-to-date discussion of the SECB method can be found, where it was shown that *a priori* information on the solution is redundant.

However, no error estimate for the SECB solution against the exact solution was provided, and in this article we discuss the SECB method for Problem 2 without using the *a priori* information $\|u(\cdot,\cdot,0)\| \leq \tilde{M}$ and prove an error estimate. We stress that the SECB constraint is distinct from prescribing bounds on derivatives of the exact solution, and the SECB approach does not imply and is not implied by differentiability of the exact solution — cf. Ref. [27]. For comparison, *a priori* information on the derivatives of the unknown solution is first given, and we investigate the Tikhonov method under this *a priori* information. A conditional stability estimate for Problem 2 is provided in Section 2, and the error estimate for the Tikhonov regularisation method based on the established stability estimate is given in Section 3. (A logarithmic-type estimate is made at the continuation boundary $z = 0$.) In order to compare with Tikhonov method, we present the SECB method for solving Problem 2 in Section 4, and prove an improved error estimate under the SECB constraint. In Section 5, some numerical results are reported, showing that the Tikhonov method and SECB method are effective for the inverse diffraction problem. Our theoretical and numerical results demonstrate that the SECB method is more robust.

2. Stability Estimate under Prescribed Bounds

A conditional stability estimate for ill-posed problems defines how much any two solutions differ from each other when some error exists. Since Problem 2 is linear, a stability estimate can be derived by examining the solutions to the corresponding homogenous problem.

For Problem 2, we assume that there exists an *a priori* bound:

$$\|f(\cdot,\cdot)\|_p \leq M\,, \tag{2.1}$$

where $\|\cdot\|_p$ denotes the norm in Sobolev space $H^p(\mathbb{R}^2)$ with $p \geq 0$ and M is a positive constant. At the same time, the given data g satisfies

$$\|g(\cdot,\cdot)\| \leq \epsilon\,, \tag{2.2}$$

where $\|\cdot\|$ denotes $L^2(\mathbb{R}^2)$ norm, ϵ is a small positive constant.

We can establish the stability estimate for Problem 2 in a Theorem.

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Theorem 2.1. *Suppose $z + d \geq 1/4$, $u(x, y, z)$ is the solution of Problem 2, and conditions (2.1) and (2.2) are satisfied. Then the following estimate holds for $0 < z \leq d$:*

$$\|u(\cdot,\cdot,z)\| \leq M^{1-\frac{z}{d}} \epsilon^{\frac{z}{d}} \left[\frac{1}{d} \ln \frac{M}{\epsilon}\right]^{-p(1-\frac{z}{d})} (1 + o(1)), \quad \text{for } \frac{\epsilon}{M} \to 0. \tag{2.3}$$

Proof. Condition (2.1) can be rewritten as

$$\|f(\cdot,\cdot)\|_p^2 = \int_{\mathbb{R}^2} \left(1 + \xi^2 + \eta^2\right)^p |\widehat{f(\cdot,\cdot)}|^2 d\xi d\eta \leq M^2, \tag{2.4}$$

and from (1.13) condition (2.2) can be rewritten as:

$$\|g(\cdot,\cdot)\|^2 = \int_{\mathbb{R}^2} e^{-2d\sqrt{\xi^2+\eta^2-k^2}} |\widehat{f(\cdot,\cdot)}|^2 d\xi d\eta \leq \epsilon^2. \tag{2.5}$$

We define two sets in $\mathbb{R}^2 = I_1 \bigcup I_2$:

$$I_1 = \left\{(\xi,\eta) | \xi^2 + \eta^2 \leq k^2\right\},$$
$$I_2 = \left\{(\xi,\eta) | \xi^2 + \eta^2 \geq k^2\right\}.$$

Then for $\xi^2 + \eta^2 - k^2 < 0$, Problem 2 is 'well-posed' for $d > 0$; and from (1.13) and (2.2),

$$\int_{I_1} |\hat{u}(\cdot,\cdot,z)|^2 \, d\xi d\eta = \int_{I_1} \left| e^{zi\sqrt{k^2-\xi^2-\eta^2}} \right|^2 |\hat{g}(\cdot,\cdot)|^2 \, d\xi d\eta \leq \epsilon^2. \tag{2.6}$$

Obviously,

$$\|u(\cdot,\cdot,z)\|^2 = \int_{I_1} |\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta + \int_{I_2} |\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta, \tag{2.7}$$

so we only need to estimate the second term on the right-hand side in (2.7). Let us denote a function $\phi(\lambda)$ in parameter representation implicitly — i.e.

$$\phi(\xi,\eta) = (1 + \xi^2 + \eta^2)^{-p} e^{-2d\sqrt{\xi^2+\eta^2-k^2}},$$
$$\lambda(\xi,\eta) = (1 + \xi^2 + \eta^2)^{-p} e^{-2z\sqrt{\xi^2+\eta^2-k^2}}, \quad \xi^2 + \eta^2 - k^2 > 0. \tag{2.8}$$

The function $\phi(\lambda)$ is convex — cf. Proposition A.1 in the Appendix. From Jensen's inequal-

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ity, for the convex function $\phi(\lambda)$ we therefore have

$$\begin{aligned}
&\phi\left(\frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}\right)\\
=&\phi\left(\frac{\int_{I_2} e^{-2z\sqrt{\xi^2+\eta^2-k^2}}|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}\right) \quad \text{[from (1.13)]}\\
=&\phi\left(\frac{\int_{I_2} e^{-2z\sqrt{\xi^2+\eta^2-k^2}}(1+\xi^2+\eta^2)^{-p}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}\right)\\
\le&\frac{\int_{I_2}\phi(e^{-2z\sqrt{\xi^2+\eta^2-k^2}}(1+\xi^2+\eta^2)^{-p})(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}\\
=&\frac{\int_{I_2}(1+\xi^2+\eta^2)^{-p}e^{-2d\sqrt{\xi^2+\eta^2-k^2}}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}\\
=&\frac{\int_{I_2} e^{-2d\sqrt{\xi^2+\eta^2-k^2}}|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}\\
\le&\frac{\epsilon^2}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta},
\end{aligned} \tag{2.9}$$

where we have used (2.5) and (2.8). Noting that $\phi(\lambda)/\lambda$ increases with λ, as

$$\frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta} \ge \frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{M^2}$$

we have

$$\frac{\phi\left(\frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{M^2}\right)}{\frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{M^2}} \le \frac{\phi\left(\frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}\right)}{\frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{\int_{I_2}(1+\xi^2+\eta^2)^p|\hat{f}(\cdot,\cdot)|^2 d\xi d\eta}}. \tag{2.10}$$

Combining this result with (2.9) then yields

$$\phi\left(\frac{\int_{I_2}|\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta}{M^2}\right) \le \frac{\epsilon^2}{M^2}. \tag{2.11}$$

The inverse function of $\phi(\lambda)$ can be expressed asymptotically as $\lambda \to 0$ (cf. the Appendix):

$$\phi^{-1}(\lambda) = \lambda^{\frac{z}{d}}\left(\frac{1}{2d}\ln\frac{1}{\lambda}\right)^{-2p\frac{d-z}{d}}(1+o(1)). \tag{2.12}$$

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From (2.10), we then get

$$\int_{I_2} |\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta \le M^2 \phi^{-1}\left(\frac{\epsilon^2}{M^2}\right) \tag{2.13}$$

— i.e.

$$\left(\int_{I_2} |\hat{u}(\cdot,\cdot,z)|^2 d\xi d\eta\right)^{1/2} \le M^{1-\frac{z}{d}} \epsilon^{\frac{z}{d}} \left[\frac{1}{2d}\ln\frac{M}{\epsilon}\right]^{-p(1-\frac{z}{d})} (1+o(1)), \text{ for } \frac{\epsilon}{M} \to 0\,. \tag{2.14}$$

According to (2.6) and (2.13), this produces the estimate (2.3).

Remark 2.1. If any two functions $u_1(\cdot,\cdot,z)$, $u_2(\cdot,\cdot,z)$ satisfy $\|g_1(\cdot,\cdot)-g_2(\cdot,\cdot)\| \le \epsilon$, and $\|f_1(\cdot,\cdot)-f_2(\cdot,\cdot)\|_p \le M$, where $u_1(\cdot,\cdot,z)$ and $u_2(\cdot,\cdot,z)$ are the corresponding solutions, then $\|u_1(\cdot,\cdot,z)-u_2(\cdot,\cdot,z)\| \le M^{1-\frac{z}{d}} \epsilon^{\frac{z}{d}} (\frac{1}{2d}\ln\frac{M}{\epsilon})^{-p(1-\frac{z}{d})}(1+o(1))$, for $\epsilon/M \to 0$.

Remark 2.2. Let $u(x,y,z)$ be the solution of Problem 2, if $p=0$; then $\phi(\lambda)=\lambda^{z/d}$, and we can easily obtain the following estimate for $0<z\le d$:

$$\|u(\cdot,\cdot,z)\| \le \|f(\cdot,\cdot)\|^{1-\frac{z}{d}} \|g(\cdot,\cdot)\|^{\frac{z}{d}}. \tag{2.15}$$

Remark 2.3. Let the source set $\mathcal{M}_{p,M} = \{u(\cdot,\cdot,z) | \|f(\cdot,\cdot)\|_p \le M\}$, with the modulus of continuity for $\hat{A}_z$ given by

$$\omega(\epsilon, \hat{A}_z, \mathcal{M}_{p,M}) := \sup\left\{\|\hat{u}(\cdot,\cdot,z)\| \,\middle|\, f(\cdot,\cdot) \in \mathcal{M}_{p,M},\ \|\hat{g}(\cdot,\cdot)\| \le \epsilon\right\}. \tag{2.16}$$

From (2.3), we can then conclude that

$$\omega(\epsilon, \hat{A}_z, \mathcal{M}_{p,M}) \le M^{1-\frac{z}{d}} \epsilon^{\frac{z}{d}} \left[\frac{1}{2d}\ln\frac{M}{\epsilon}\right]^{-p(1-\frac{z}{d})} (1+o(1)), \quad \text{for } \frac{\epsilon}{M} \to 0\,. \tag{2.17}$$

Remark 2.4. According to Ref. [20], the $\omega(\epsilon, \hat{A}_z, \mathcal{M}_{p,M})$ is the best possible error bound (or optimal error bound) for all regularisation methods.

Remark 2.5. Usually, the conditional stability result cannot guarantee the stability of the numerical computation, because only noisy data are available for ill-posed problems, so some regularisation methods are needed for solving Problem 2.

3. Tikhonov Regularisation and Error Estimate

Many authors have invoked a conditional stability estimate and modulus of continuity to deduce error estimates for the Tikhonov method — e.g. [34–37]. Here we focus on the regularisation problem. Assume that the approximate noisy data $g_\delta(\cdot,\cdot)$ satisfies

$$\|g(\cdot,\cdot) - g_\delta(\cdot,\cdot)\| \le \delta\,, \tag{3.1}$$

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where $\delta > 0$ is the noise level. The Tikhonov regularisation method [38] for the solution of Problem I involves minimising the quadratic functional

$$\|A(z)f(\cdot,\cdot) - g_\delta(\cdot,\cdot)\|^2 + \alpha\|f(\cdot,\cdot)\|_p^2 , \tag{3.2}$$

where $\alpha = \delta^2/M^2$, $A(z)f = g$. From Parseval's identity and (1.10) applied to (3.2), the variational problem becomes

$$\left\| e^{-(d-z)\sqrt{\xi^2+\eta^2-k^2}}\hat{u}(\cdot,\cdot,z) - g_\delta \right\|^2 + \alpha \left\| (1+\xi^2+\eta^2)^{p/2} e^{z\sqrt{\xi^2+\eta^2-k^2}}\hat{u}(\cdot,\cdot,z) \right\|^2 , \tag{3.3}$$

so the solution $\hat{u}_\delta^\alpha(\cdot,\cdot,z)$ of this problem satisfies the Euler equation

$$\begin{aligned} &\left(e^{-2(d-z)\sqrt{\xi^2+\eta^2-k^2}} + \alpha(1+\xi^2+\eta^2)^p e^{2z\sqrt{\xi^2+\eta^2-k^2}} \right) \hat{u}_\delta^\alpha(\xi,\eta,z) \\ =& e^{-(d-z)\sqrt{\xi^2+\eta^2-k^2}} \hat{g}_\delta(\xi,\eta) . \end{aligned} \tag{3.4}$$

The Tikhonov regularisation solution $\hat{u}_\delta^\alpha(\xi,\eta,z)$ in the frequency domain is then

$$\hat{u}_\delta^\alpha(\xi,\eta,z) = \frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}} \hat{g}_\delta(\xi,\eta) , \tag{3.5}$$

and the inverse Fourier transform of $\hat{u}_\delta^\alpha(\xi,\eta,z)$ produces the regularisation solution $u_\delta^\alpha(x,y,z)$ in the physical domain.

Let us now use the stability estimate (2.3) to determine the error estimate between the regularisation solution and the exact solution. From Parseval's identity,

$$\begin{aligned} &\left\| f_\delta^\alpha(\cdot,\cdot) - f(\cdot,\cdot) \right\|_p \\ =& \left[\int (1+\xi^2+\eta^2)^p \left| \frac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}} \hat{g}_\delta(\xi,\eta) - e^{d\sqrt{\xi^2+\eta^2-k^2}} \hat{g}(\xi,\eta) \right|^2 d\xi d\eta \right]^{1/2} \\ \le& \left[\int (1+\xi^2+\eta^2)^p \left(\frac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}} \right)^2 |\hat{g}_\delta(\xi,\eta) - \hat{g}(\xi,\eta)|^2 d\xi d\eta \right]^{1/2} \\ &+ \left[\int (1+\xi^2+\eta^2)^p \left| \frac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}} \hat{g}(\xi,\eta) - e^{d\sqrt{\xi^2+\eta^2-k^2}} \hat{g}(\xi,\eta) \right|^2 d\xi d\eta \right]^{1/2} \\ \le& \sup \left| \frac{(1+\xi^2+\eta^2)^{p/2} e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}} \right| \|\hat{g}_\delta - \hat{g}\| \\ &+ \sup \left| \frac{\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}} \right| \|f(\cdot,\cdot)\|_p . \end{aligned}$$

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Since $h(\eta) := \eta(1+\eta^2)^{-1}$, the maximum value of $\eta > 0$ is $1/2$ such that

$$\sup\left|\frac{(1+\xi^2+\eta^2)^{p/2}e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\right| \le \frac{1}{2}(\sqrt{\alpha})^{-1} = \frac{1}{2}\frac{M}{\delta},$$
$$\sup\left|\frac{\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\right| < 1,$$

hence

$$\|f_\delta^\alpha(\cdot,\cdot) - f(\cdot,\cdot)\|_p \le \frac{3}{2}M. \tag{3.6}$$

Similarly

$$\begin{aligned}
&\|\hat{g}_\delta^\alpha(\cdot,\cdot) - \hat{g}(\cdot,\cdot)\| \\
=&\left(\int\left|\frac{1}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\hat{g}_\delta - \hat{g}\right|^2 d\xi d\eta\right)^{1/2} \\
\le&\left[\int\left(\frac{1}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\right)^2 |\hat{g}_\delta - \hat{g}|^2 d\xi d\eta\right]^{1/2} \\
&+\left[\int\left|\frac{1}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\hat{g} - \hat{g}\right|^2 d\xi d\eta\right]^{1/2} \\
\le&\delta + \left[\int\left|\frac{\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\hat{g}\right|^2 d\xi d\eta\right]^{1/2} \\
=&\delta + \left[\int\left|\frac{\alpha(1+\xi^2+\eta^2)^p e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}e^{d\sqrt{\xi^2+\eta^2-k^2}}\hat{g}(\xi,\eta)\right|^2 d\xi d\eta\right]^{1/2} \\
\le&\delta + \sup\left|\frac{\alpha(1+\xi^2+\eta^2)^{p/2}e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha(1+\xi^2+\eta^2)^p e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\right| \|f(\cdot,\cdot)\|_p \le \delta + (\sqrt{\alpha}/2)M = \frac{3}{2}\delta,
\end{aligned}$$

such that

$$\left\|\hat{g}_\delta^\alpha(\cdot,\cdot) - \hat{g}(\cdot,\cdot)\right\| \le \frac{3}{2}\delta. \tag{3.7}$$

Let us now define

$$\begin{aligned}
&\omega_1(\delta, \hat{A}_z, \mathscr{M}_{p,M}) \\
:=&\sup\left\{\|\hat{u}_\delta^\alpha(\cdot,\cdot,z) - \hat{u}(\cdot,\cdot,z)\| \,\middle|\, \|\hat{f}_\delta^\alpha(\cdot,\cdot) - \hat{f}(\cdot,\cdot)\|_p \le \frac{3}{2}M,\ \|\hat{g}_\delta^\alpha(\cdot,\cdot) - \hat{g}(\cdot,\cdot)\| \le \frac{3}{2}\delta\right\}.
\end{aligned} \tag{3.8}$$

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On noting the definition of $\omega(\delta,\hat{A}_z,\mathscr{M}_{p,M})$ in (2.16), we have

$$\omega_1(\delta,\hat{A}_z,\mathscr{M}_{p,M}) = \frac{3}{2}\omega(\delta,\hat{A}_z,\mathscr{M}_{p,M}) \tag{3.9}$$

such that

$$\|u_\delta^\alpha(\cdot,\cdot,z) - u(\cdot,\cdot,z)\| \le \frac{3}{2}\omega(\delta,\hat{A}_z,\mathscr{M}_{p,M})\,. \tag{3.10}$$

Now from (2.17), we may summarise the above in the following theorems.

Theorem 3.1. *Suppose that $u(x,y,z)$ is the exact solution of Problem 2 with exact data $g(x,y)$,* (3.5) *is the Tikhonov regularisation solution with noisy data $g_\delta(x,y)$, and* (2.1) *and* (3.1) *hold. Then for $0 < z \le d$,*

$$\|u_\delta^\alpha(\cdot,\cdot,z) - u(\cdot,\cdot,z)\| \le \frac{3}{2}M^{1-\frac{z}{d}}\delta^{\frac{z}{d}}\left[\frac{1}{2d}\ln\frac{M}{\delta}\right]^{-p(1-\frac{z}{d})}(1+o(1))\,, \quad \text{for } \frac{\delta}{M}\to 0\,. \tag{3.11}$$

Theorem 3.2. *If $z = 0$, the error estimate* (3.11) *is*

$$\|f_\delta^\alpha(\cdot,\cdot) - f(\cdot,\cdot)\| \le \frac{3}{2}M\left(\frac{1}{2d}\ln\frac{M}{\delta}\right)^{-p}(1+o(1))\,, \quad \text{for } \delta\to 0. \tag{3.12}$$

Although (3.12) is a very weak convergence estimate, Theorem 3.2 provides logarithmic stability estimates for boundary error estimation in ill-posed problems. However, in the next section we show there is a notable improvement from the SECB constraint.

4. SECB Method and Error Estimate

The SECB method has previously been used for solving the backward heat conduction equation, but here we apply it to solve Problem 2. Let δ be the noise level such that

$$\|g(\cdot,\cdot) - g_\delta(\cdot,\cdot)\| \le \delta\,. \tag{4.1}$$

For any constant $K > 0$ such that $K \ll \tilde{M}/\delta$, we define s^* by

$$s^* = \frac{d\ln(\frac{\tilde{M}}{\tilde{M}-K\delta})}{\ln\frac{\tilde{M}}{\delta}} \tag{4.2}$$

The slow evolution constraint applied to Problem 2 requires that there exist a known constant K and a known fixed s where $d > s > s^*$ such that

$$\|u(\cdot,\cdot,s) - u(\cdot,\cdot,0)\| \le K\delta\,. \tag{4.3}$$

Let $A(z)$ be the solution operator corresponding to the forward problem — cf. (1.10). Given two regularisation parameters K and s, the SECB solution to the Problem 2 is defined as the minimiser of the functional

$$\|u(\cdot,\cdot,d) - g_\delta(\cdot,\cdot)\|^2 + K^{-2}\|u(\cdot,\cdot,s) - u(\cdot,\cdot,0)\|^2\,. \tag{4.4}$$

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In terms of $A(z)$, we may express the SECB solution $f^\dagger(x,y)$ for Problem 2 at $z=0$ as

$$f^\dagger(x,y) = \mathrm{Argmin}_{f\in L^2}\left\{\|A(d)f - g_\delta\|^2 + K^{-2}\|(A(s)-I)f\|^2\right\}, \tag{4.5}$$

leading to the following Euler equation for $f^\dagger(x,y)$:

$$\left[A^*(d)A(d) + K^{-2}(A(s)-I)^*(A(s)-I)\right]f^\dagger = A^*(d)g_\delta\,. \tag{4.6}$$

From (1.13) and $\widehat{A(z)f} = e^{-z\sqrt{\xi^2+\eta^2-k^2}}\hat{f}(\xi,\eta)$, we can find the SECB solution $f^\dagger$ in closed form in the Fourier transform domain. Let $\bar{z}$ denote the complex conjugate of z such that

$$\hat{f}^\dagger(\xi,\eta) = \frac{e^{-d\sqrt{\xi^2+\eta^2-k^2}}}{e^{-2d\sqrt{\xi^2+\eta^2-k^2}} + K^{-2}(1-e^{-s\sqrt{\xi^2+\eta^2-k^2}})^2}\hat{g}_\delta(\xi,\eta)\,, \tag{4.7}$$

leading to $f^\dagger(x,y)$ by inverse Fourier transform.

We derive the following error estimate for the SECB method — cf. also Refs. [27, 28].

Theorem 4.1. *Suppose that the solution $f(x,y) := u(x,y,0)$ satisfies* (4.1) *and* (4.3)*, and $f^\dagger(x,y)$ denotes the SECB solution. Let $P = P(d,K,s)$ be the positive definite self-adjoint operator in $L^2(\mathbb{R}^2)$ given by*

$$P = A^*(d)A(d) + K^{-2}(A(s)-I)^*(A(s)-I)\,. \tag{4.8}$$

Then $f^\dagger$ is the unique solution of $Pf^\dagger = A^*(d)g_\delta$, and $f^\dagger$ satisfies

$$\|A(d)f^\dagger - g_\delta\|^2 + K^{-2}\|(A(s)-I)f^\dagger\|^2 \le 2\delta^2\,, \tag{4.9}$$

$$\|A(d)(f-f^\dagger)\|^2 + K^{-2}\|(A(s)-I)(f-f^\dagger)\|^2 \le 2\delta^2\,, \tag{4.10}$$

$$\|f^\dagger - f\| \le \sqrt{2}\delta\|P^{-\frac{1}{2}}\|\,. \tag{4.11}$$

Proof. Let $\mathscr{H}$ denote the Hilbert space direct sum $L^2(\mathbb{R}^2)\bigoplus L^2(\mathbb{R}^2)$ with elements $[u,v]$, the scalar product of a pair of elements $([u_1,v_1],[u_2,v_2]) := \langle u_1,u_2\rangle + \langle v_1,v_2\rangle$ and the corresponding norm $|||[u_1,v_1]|||^2 := \|u_1\|^2 + \|v_1\|^2$. Let $L : L^2(\mathbb{R}^2) \to \mathscr{H}$ be defined by $Lh = [A(d)h, K^{-1}(A(s)-I)h]$ and $\tilde{g} = [g_\delta, 0]$. Minimising the $|||Lh - g_\delta|||$ over all $h \in L^2(\mathbb{R}^2)$, we get the normal equation $L^*Lf^\dagger = L^*g_\delta$ yielding $Pf^\dagger = A^*(d)g_\delta$. From (4.1) and (4.3), we have $|||Lf - g_\delta|||^2 \le 2\delta^2$. The minimising element $f^\dagger$ is such that $Lf^\dagger$ is the orthogonal projection in $\mathscr{H}$ of $\tilde{g}$, and from the Pythagorean theorem

$$|||Lf^\dagger - \tilde{g}|||^2 + |||L(f-f^\dagger)|||^2 = |||Lf - \tilde{g}|||^2 \le 2\delta^2\,.$$

Hence $|||Lf^\dagger - \tilde{g}|||^2 \le 2\delta^2$ and $|||L(f-f^\dagger)|||^2 \le 2\delta^2$, yielding (4.9) and (4.10). Since P is a self-adjoint operator in $L^2(\mathbb{R}^2)$, we therefore obtain

$$\|P^{\frac{1}{2}}(f-f^\dagger)\|^2 = \langle P(f-f^\dagger),(f-f^\dagger)\rangle = |||L(f-f^\dagger)|||^2 \le 2\delta^2$$

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such that

$$|f-f^{\dagger}\| = \|P^{-\frac{1}{2}}P^{\frac{1}{2}}(f-f^{\dagger})\| \le \|P^{-\frac{1}{2}}\|\|P^{1/2}(f-f^{\dagger})\| \le \sqrt{2}\delta\|P^{-\frac{1}{2}}\| .$$

□

To deduce the error estimate for SECB, we need to estimate the $\|P^{-1/2}\|$. Using Parseval's identity in Fourier analysis, we have $\|P^{-1/2}h\| = \|\widehat{P^{-1/2}h}\|$ such that

$$\begin{aligned}\|P^{-\frac{1}{2}}\| &= \sup_{\xi,\eta}\left[\left|e^{-d\sqrt{\xi^2+\eta^2-k^2}}\right|^2 + K^{-2}\left|1-e^{-s\sqrt{\xi^2+\eta^2-k^2}}\right|^2\right]^{-1/2} \\ &= \left[\inf_{x\ge 0}\left(e^{-2dx}+K^{-2}(1-e^{-sx})^2\right)\right]^{-1/2} . \end{aligned} \tag{4.12}$$

The key point in the next step is to minimise the function $z(x)$ with $x \ge 0$, where

$$\upsilon(x) = e^{-2dx} + K^{-2}(1-e^{-sx})^2 .$$

The following lemma shows why we choose $s > s^*$ in the SECB constraint (4.3).

Lemma 4.1. *Let s^* be given as in* (4.2). *If $s \le s^*$, then*

$$\tilde{M}/(\sqrt{2}\delta) \le \|P^{-\frac{1}{2}}\| \le \tilde{M}/\delta .$$

Proof. Let $x_0 = (1/d)\ln(\tilde{M}/\delta)$, when $e^{-2dx_0} = \delta^2/\tilde{M}^2$, and $1-e^{-sx_0} \le 1-e^{-s^*x_0} = K\delta$ and $\delta^2/\tilde{M}^2 \le \upsilon(x_0) \le 2\delta^2/\tilde{M}^2$ if $s \le s^*$. The result is then straightforward from (4.12). □

Remark 4.1. If $s \le s^*$, from (4.11) the SECB method has no convergence for $\delta \to 0$.

Lemma 4.2. *If $0 < s < d$, then $z(x)$ has a unique minimum at $x = \tilde{x} > 0$, where the $\tilde{x}$ satisfy*

$$e^{-2d\tilde{x}} = sK^{-2}(e^{-s\tilde{x}} - e^{-2s\tilde{x}}) . \tag{4.13}$$

Proof. By simple calculation for $\upsilon'(x) = 0$, we can find the $\tilde{x}$ satisfy $e^{-2d\tilde{x}} = sK^{-2}(e^{-s\tilde{x}} - e^{-2s\tilde{x}})$, and similarly $\upsilon''(\tilde{x}) = 4d^2e^{-2d\tilde{x}} - 2s^2K^{-2}e^{-s\tilde{x}} + 4s^2K^{-2}e^{-2s\tilde{x}}$. Due to $2s^2K^{-2}e^{-s\tilde{x}} = 2sde^{-2d\tilde{x}} + 2s^2K^{-2}e^{-2s\tilde{x}}$, we find that $\upsilon''(\tilde{x}) = 2d(2d-s)e^{-2d\tilde{x}} + 2s^2K^{-2}e^{-2s\tilde{x}}$. Since $0 < s < d$, we have $\upsilon''(\tilde{x}) > 0$, so the $\tilde{x}$ is unique. □

Lemma 4.3. *If $K/s > 1$, then the $\tilde{x}$ satisfy*

$$\frac{2e}{1+2de}\ln\frac{K}{s} \le \tilde{x} \le \frac{1}{2d-s}\ln\frac{K^2}{s(1-(s/K)^{(2es)/(1+2de)})}. \tag{4.14}$$

Proof. Since $e^{-2d\tilde{x}} = [s^2K^{-2}\tilde{x}(e^{-s\tilde{x}} - e^{-2s\tilde{x}})]/(s\tilde{x})$ and the $\theta(y) := (e^{-y} - e^{-2y})/y$ is a monotone decreasing function with the maximum value of 1 at $y = 0$, we find $e^{-2d\tilde{x}} \le s^2K^{-2}\tilde{x}$. Hence $\tilde{x}e^{2d\tilde{x}} \ge K^2/s^2$, and $\tilde{x}(d + \frac{\ln\tilde{x}}{2\tilde{x}}) \ge \ln\frac{K}{s}$. For $\varsigma(y) := d + \frac{\ln y}{2y}$ with $y >$

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0, $\varsigma(y)$ attains its maximum value at $y = e$ — i.e. $\varsigma(y) \le d + 1/(2e) = \frac{1+2ed}{2e}$. Thus $\tilde{x}\frac{1+2ed}{2e} \ge \tilde{x}(d + \frac{\ln \tilde{x}}{2\tilde{x}}) \ge \ln \frac{K}{s}$ — i.e., $\tilde{x} \ge \frac{2e}{1+2ed} \ln \frac{K}{s}$. To get the upper bound of $\tilde{x}$, first we have $1 - e^{-s\tilde{x}} \ge 1 - e^{-s(2e)/(1+2de)\ln(K/s)} = 1 - (s/K)^{(2es)/(1+2de)}$, and from (4.13) we have $e^{-(2d-s)\tilde{x}} = sK^{-2}(1 - e^{-s\tilde{x}})$ such that $sK^{-2}(1 - e^{-s\tilde{x}}) \ge sK^{-2}[1 - (s/K)^{(2es)/(1+2de)}]$ — i.e. $e^{-(2d-s)\tilde{x}} \ge sK^{-2}[1-(s/K)^{(2es)/(1+2de)}]$. Taking the logarithm then produces the result. □

Lemma 4.4. *If $0 < K/s \le 1$, then $\|P^{-1/2}\| \le d + 1$.*

Proof. From (4.13), $\tilde{x}e^{2d\tilde{x}} = (K^2 s^{-2})[(s\tilde{x})/(e^{-s\tilde{x}} - e^{-2s\tilde{x}})] \le (s\tilde{x})/(e^{-s\tilde{x}} - e^{-2s\tilde{x}})$ if $K/s < 1$, hence $\tilde{x}e^{(2d-s)\tilde{x}} \le (s\tilde{x})/(1-e^{-s\tilde{x}})$. Noting that $\ell(y) := y/(1-e^{-y})$ is a monotone increasing function for $y \ge 0$, for $0 < s < d$ we get $\tilde{x}e^{(2d-s)\tilde{x}} \le (d\tilde{x})/(1 - e^{-d\tilde{x}})$, therefore $e^{d\tilde{x}} \le e^{(2d-s)\tilde{x}} \le d/(1 - e^{-d\tilde{x}})$, i.e., $\tilde{x} \le \frac{1}{d}\ln(d+1)$. Thus $e^{-2d\tilde{x}} \ge (d+1)^{-2}$, and from (4.12) we have the result. □

Theorem 4.2. *Suppose that the solution $f(x, y) := u(x, y, 0)$ satisfies (4.1) and (4.3), and $f^\dagger(x, y)$ denotes the SECB solution. If $s > s^*$ and $0 < K/s < 1$, then $\|f - f^\dagger\| \le \sqrt{2}(d+1)\delta$. If $s > s^*$ and $K/s > 1$ denote two constants such that*

$$\mathbf{M} := \left[sK^{-2}(1 - (s/K)^{\frac{2es}{1+2de}})\right]^{\frac{2d}{2d-s}}, \tag{4.15}$$

$$\mathbf{L} := K^{-2}\left[1 - (s/K)^{\frac{2es}{1+2de}}\right]^2, \tag{4.16}$$

then

$$\|f^\dagger - f\| \le \sqrt{2}\delta(\mathbf{M} + \mathbf{L})^{-\frac{1}{2}}. \tag{4.17}$$

Proof. If $0 < K/s \le 1$, the result is obvious from Lemma 4.4. If $K/s > 1$, we need to estimate the $\upsilon(\tilde{x})$. Using the upper and lower bound for $\tilde{x}$ in Lemma 4.3, we get $\upsilon(\tilde{x}) \ge e^{-2d\frac{1}{2d-s}\ln\frac{K^2}{s(1-(s/K)^{(2es)/(1+2de)})}} + K^{-2}(1 - e^{-s\frac{2e}{1+2de}\ln\frac{K}{s}})^2 = \mathbf{M} + \mathbf{L}$. The result follows from Theorem 4.1 and $\|P^{-1/2}\| = [\upsilon(\tilde{x})]^{-1/2}$. □

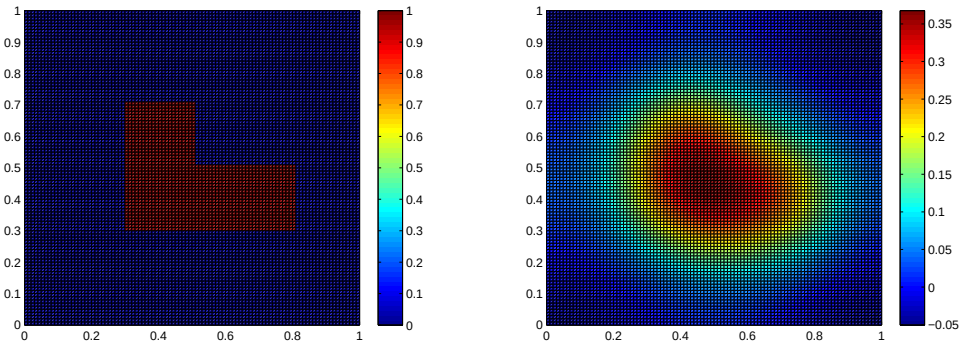
5. Numerical Examples

We now describe an experiment to illustrate the ability of the generalised Tikhonov regularisation and the SECB method to solve Problem 2. Although Problem 2 involves an unbounded domain in (x, y), we are interested in the domain $(x, y) \in [0, 1] \times [0, 1]$ since there can be an immediate periodic extension in (x, y) plane. Let us define the discrete L^2 norm of $f(x, y)$ by

$$\|f\| = \left(n^{-2}\sum_{j,k=1}^{n} f(x_j, y_k)^2\right)^{1/2}, \tag{5.1}$$

where n is the total number of sampled points (we take $n = 100$ here).

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(a) (b)

Figure 2: The original D (left) and data g (right).

In order to measure the accuracy of numerical results, we defined the discrete L^2-norm error for the exact solution f between the approximate solution f_a as

$$E = \left(n^{-2} \sum_{j,k=1}^{n} |f(x_j, y_k) - f_a(x_j, y_k)|^2 \right)^{1/2}. \tag{5.2}$$

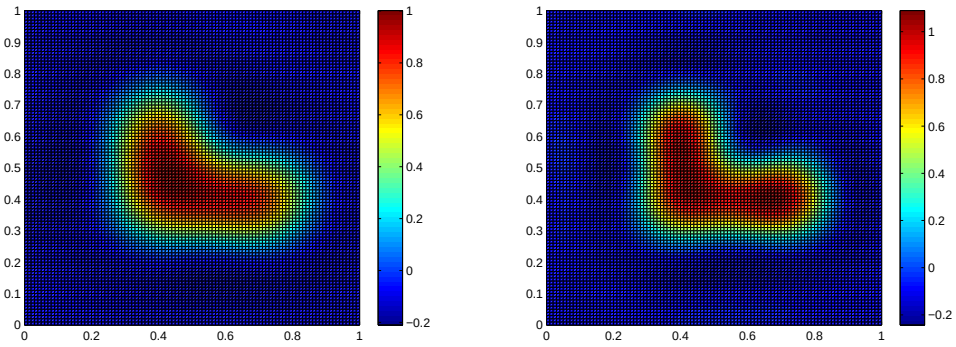
The noise was added to $g(x, y)$ by setting $\delta(x_j, t_k) = \sigma\, r_{jk} \max\{g(x_j, y_k)\})$, where σ denotes the noise level and r_{jk} a random number drawn from a uniform distribution in the range $[0, 1]$. In the numerical experiment, we used (5.2) to compute $\delta = \|g - g_\delta\|$; and since it is very hard to find the H^p-norm M of the exact solution $f(x, y)$, in the computation the regularisation parameter of the Tikhonov method was taken to be $\alpha = \delta^2$.

We first solved the forward problem (1.7)-(1.9) using the formula (1.13) to get the data g, and then generated the noisy data g_δ. The regularised solutions given by the formulas (3.5) and (4.7) can be obtained via the FFT algorithm. Although we used some "inverse crimes" in the reconstruction, our main goal in this numerical experiment was to show the feasibility of the SECB method, and we illustrate the reconstruction methods, (3.5) and (4.7) with different numerical examples.

Example 5.1. The aperture D had the shape of the letter 'L' for our test, and the characteristic function shown in Fig. 2 was constructed on the domain D. Fig. 3 displays the data g, as a very blurred picture of the original aperture. In this example, a 1% relative error was added to the data $g(x, y)$ at the sampled points in the simulation. Fig. 3 shows the results from the Tikhonov and SECB methods, where the distance from the receiving plane to the screen was $d = 0.2$ for the wavenumber $k = 5$.

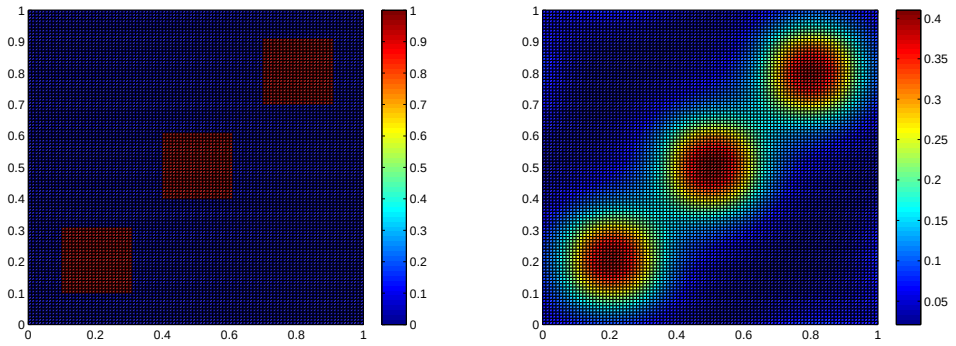
Example 5.2. The aperture D consisted of three independent squares, and the characteristic function and the data g are displayed in Fig. 4. In this example, a 1% relative error was added to the data $g(x, y)$ at the sampled points in the simulation (as in Example 5.1). Fig. 5 shows the results from the Tikhonov and SECB methods, where the distance from receiving plane to the screen was $d = 0.1$ for the wavenumber $k = 5$.

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(a) (b)

Figure 3: (a): Tikhonov regularisation with $\alpha = 4.6*10^{-4}$, $p = 0.1$. (b): SECB method with $s = 0.05$, $s^* = 0.02$ and $K = 83$.



(a) (b)

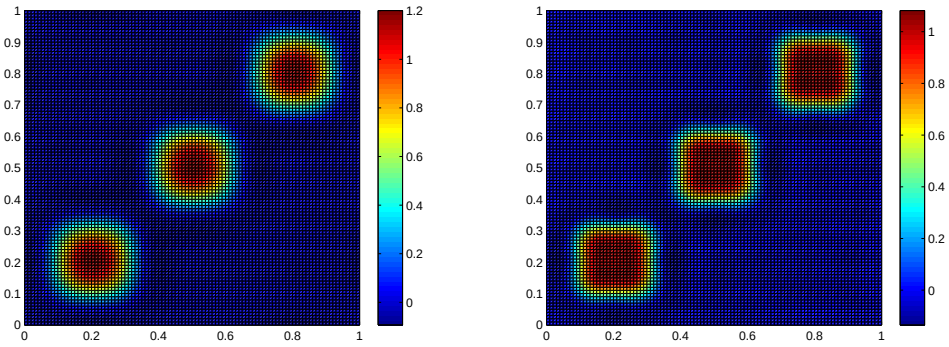
Figure 4: (a): the original D. (b): the data g.

We noted that the reconstruction generally became more severe as the receiving plane was moved further away from the aperture and as the wavenumber k decreased, which can be understood from the formula (1.14). When d increases or k decreases, the degree of ill-posedness of the Problem 2 increases. The SECB method shows better results than the Tikhonov method, as has previously been observed in the solution of backward heat conduction problems [27, 33].

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(a) (b)

Figure 5: (a): Tikhonov regularisation with $\alpha = 5.7 * 10^{-4}$, $p = 0.1$. (b): SECB method with $s = 0.05$, $s^* = 0.01$ and $K = 87$.

A. Properties of the function ϕ

The function $\phi(\lambda)$ in parameter ξ, η representation for $\xi^2 + \eta^2 - k^2 > 0$ is

$$\phi(\xi,\eta) = (1+\xi^2+\eta^2)^{-p} e^{-2d\sqrt{\xi^2+\eta^2-k^2}},$$
$$\lambda(\xi,\eta) = (1+\xi^2+\eta^2)^{-p} e^{-2z\sqrt{\xi^2+\eta^2-k^2}}.$$

Let $r = \sqrt{\xi^2+\eta^2-k^2} > 0$ such that $\xi^2+\eta^2 = r^2+k^2$, and we can rewrite the function $\phi(\lambda)$ in parameter r as

$$\phi(r) = (1+r^2+k^2)^{-p} e^{-2dr}, \qquad \lambda(r) = (1+r^2+k^2)^{-p} e^{-2zr}. \tag{A.1}$$

Proposition A.1. *The $\phi(\lambda)$ defined by* (A.1) *has the following properties:*

(i) The function $\phi(\lambda)/\lambda$ increases as λ increases;

(ii) $\phi^{-1}(\lambda)$ is strictly monotonically increasing and has the parameter representation

$$\lambda(r) = (1+r^2+k^2)^{-p} e^{-2dr}, \qquad \phi^{-1}(r) = (1+r^2+k^2)^{-p} e^{-2zr}. \tag{A.2}$$

(iii) For the inverse function $\phi^{-1}(\lambda)$ of $\phi(\lambda)$,

$$\phi^{-1}(\lambda) = \lambda^{\frac{z}{d}} \left(\frac{1}{2d}\ln\frac{1}{\lambda}\right)^{-2p\frac{d-z}{d}} (1+o(1)), \text{ for } \lambda \to 0. \tag{A.3}$$

Proof. (i) Since $d\lambda/dr = e^{-2zr}(1+r^2+k^2)^{-p-1}(-p-2z(1+r^2+k^2)) < 0$, then

$$\frac{d(\frac{\phi(\lambda)}{\lambda})}{d\lambda} = \frac{d}{d\lambda}(e^{-2(d-z)r}) = -2ze^{-2(d-z)r}/(d\lambda/d\xi) > 0.$$

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(ii) The result follows directly.

(iii) In order to prove (A.3), we need to show that $\lim_{\lambda\to 0} G(\lambda) = 1$, where $G(\lambda)$ is

$$G(\lambda) = \phi^{-1}(\lambda)\left(\frac{1}{\lambda}\right)^{\frac{z}{d}}\left(\frac{1}{2d}\ln\frac{1}{\lambda}\right)^{2p(\frac{d-z}{d})}. \tag{A.4}$$

Taking into account (A.2), we have

$$\begin{aligned}
&\lim_{\lambda\to 0} G(\lambda)\\
&= \lim_{r\to+\infty}(1+r^2+k^2)^{-p}e^{-2zr}\left[(1+r^2+k^2)^p e^{2dr}\right]^{\frac{z}{d}}\cdot\left[\frac{1}{2d}\ln((1+r^2+k^2)^p e^{2dr})\right]^{2p\frac{d-z}{d}}\\
&= \lim_{r\to+\infty}(1+r^2+k^2)^{-p(1-z/d)}\left[r+p\ln(1+r^2+k^2)\right]^{2p\frac{d-z}{d}}\\
&= \lim_{r\to+\infty}\left(1+\frac{1+k^2}{r^2}\right)^{-p(1-z/d)}\left[1+\frac{p}{r}\ln(1+r^2+k^2)\right]^{2p\frac{d-z}{d}} = 1\,. \quad \square
\end{aligned}$$

Proposition A.2. *$\phi(\lambda)$ is strictly convex if and only if the following inequality holds:*

$$p^2\varphi^2(r) + p\left[(d+z)\varphi(r) + \frac{1}{2}\dot{\varphi}(r)\right] + zd > 0 \qquad \forall r > 0\,, \tag{A.5}$$

where the function $\varphi(r)$ is defined by

$$\varphi(r) = \frac{r}{1+r^2+k^2}\,, \quad r > 0\,. \tag{A.6}$$

Proof. Let $\dot{a}(r)$ denote $da(r)/dr$ and $\ddot{a}(r)$ denote $d^2a(r)/dr^2$, respectively. Then from $\phi'' = (\ddot{\phi}\dot{\lambda} - \dot{\phi}\ddot{\lambda})/\dot{\lambda}^3$ and $\dot{\lambda} < 0$, we have that $\phi'' > 0$ is equivalent to $\ddot{\phi}\dot{\lambda} < \dot{\phi}\ddot{\lambda}$. Noting that $\lambda(r) = \phi(r)q(r)$ and $q(r) = e^{2(d-z)r}$, we have $\phi'' > 0$ equivalent to

$$\phi\ddot{\phi} - 2\dot{\phi}^2 < \phi\dot{\phi}\frac{\ddot{q}}{\dot{q}}\,. \tag{A.7}$$

A direct computation yields

$$\begin{aligned}
&\phi\ddot{\phi} = (1+r^2+k^2)^{-2p-2}e^{-4dr}\left\{4d^2(1+r^2+k^2)^2 + (8dpr-2p)(1+r^2+k^2) + 4p^2r^2 + 4pr^2\right\},\\
&2\dot{\phi}^2 = (1+r^2+k^2)^{-2p-2}e^{-4dr}\left\{8p^2r^2 + 8d^2(1+r^2+k^2)^2 + 16dpr(1+r^2+k^2)\right\},\\
&\frac{\ddot{q}}{\dot{q}} = 2(d-z), \qquad \phi\dot{\phi} = (1+r^2+k^2)^{-2p-1}e^{-4dr}\left\{-2pr - 2d(1+r^2+k^2)\right\},
\end{aligned}$$

whence

$$\begin{aligned}
\phi\ddot{\phi} - 2\dot{\phi}^2 &= (1+r^2+k^2)^{-2p-2}e^{-4dr}\left\{-4d^2(1+r^2+k^2)^2 - (8dpr+2p)(1+r^2+k^2)\right.\\
&\qquad \left. -4p^2r^2 + 4pr^2\right\},\\
\phi\dot{\phi}\frac{\ddot{q}}{\dot{q}} &= (1+r^2+k^2)^{-2p-2}e^{-4dr}\left\{(4dz-4d^2)(1+r^2+k^2)^2 - 4pr(d-z)(1+r^2+k^2)\right\}.
\end{aligned}$$

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Consequently, (A.7) is equivalent to

$$4dz(1+r^2+k^2)^2+(2p+4dpr+4prz)(1+r^2+k^2)+4p^2r^2-4pr^2>0$$

or

$$4p^2r^2+p\left\{(2+4r(d+z))(1+r^2+k^2)-4r^2\right\}+4dz(1+r^2+k^2)^2>0$$

on simplification, so we get (A.5). □

Remark on Proposition A.2. Since $p^2\varphi^2+zd>0$, the inequality (A.5) is satisfied if $(d+z)\varphi+\frac{1}{2}\dot{\varphi}>0$. Now $-\frac{\dot{\varphi}}{\varphi}(r)=\frac{r^2-a}{r(r^2+a)}$ with $a=1+k^2$.

Then since $\frac{r^2-a}{r(r^2+a)}=\frac{1-a/r^2}{r+a/r}\leq\frac{1-a/r^2}{2\sqrt{a}}\leq\frac{1}{2\sqrt{a}}\leq 1/2$, the inequality holds with $d+z\geq 1/4$.

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