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Invariant jets of foliations along hyperbolic sets

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Abstract. It is shown that under certain conditions a hyperbolic set will possess an invariant structure in addition to the invariant submanifolds. This invariant structure is in the form of a 'jet of a foliation' along the leaves.

1. *Introduction*

In this paper we are interested in an invariant structure which is in addition to the invariant manifolds of a hyperbolic, or pseudohyperbolic, set. Let $f : M \to M$ be a C^r diffeomorphism ($r \geq 1$) of a finite-dimensional, C^∞ Riemannian manifold M without boundary. We assume there exists a compact invariant set $\Lambda \subset M$, i.e. $f(\Lambda) = \Lambda$, which is α-pseudohyperbolic for f. That is, there is a continuous, Tf-invariant splitting of the tangent bundle of M over Λ, $T_\Lambda M = E_1 \oplus E_2$, and an $\alpha > 0$, such that relative to some Riemannian metric and for some $\varepsilon > 0$ we have,

$$\|Tf(u)\| \geq (\alpha + \varepsilon)\|u\|, \quad \text{for all } u \in E_1,$$

and

$$\|Tf(v)\| \leq (\alpha - \varepsilon)\|v\|, \quad \text{for all } v \in E_2.$$

We will always assume $\alpha \geq 1$, in which case E_1 is the *strong unstable* direction and E_2 is the *pseudo-stable* direction. If $\alpha = 1$, then Λ is *hyperbolic*, and these are the unstable and stable directions, respectively.

By [**HPS**, p. 61], there is a unique immersed *strong unstable manifold* through each point $x \in \Lambda$, $W^{uu}(x)$, with the property that $W^{uu}(x)$ is tangent to $E_1(x)$ at x, and $f(W^{uu}(x)) = W^{uu}(f(x))$. Moreover, $W^{uu}(x)$ is as smooth as f, and varies continuously with x. For Λ hyperbolic, there are both unique stable and unstable manifolds. We show that under additional assumptions on the rates of expansion and contraction at the hyperbolic set, there is more invariant structure. Along each of the strong unstable manifolds there exists what can be thought of as an 'infinitesimal foliation', or invariant 'transverse derivatives'. Intuitively, these measure how nearby leaves would have to be arranged if a single isolated leaf were to be part of a larger invariant foliation.

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If the stable or unstable manifolds of a hyperbolic set happen to fill a region, then the uniqueness implies that they form an invariant foliation. For example, this is the case for Anosov systems [**A**], where the hyperbolic set is the entire manifold. The foliation need not be even C^1, although the leaves are very smooth. However, in [**H**] it is shown that bunching conditions on the expansion and contraction rates of an Anosov system will guarantee differentiability of the stable and unstable foliations arbitrarily close to C^2. (See also [**HPS**].)

In general, for non-Anosov systems, the stable or unstable manifolds need not fill an open set. In this case, they do not form a foliation, but rather a *lamination*. This is the situation we imagine here. Our results can be interpreted as a generalization of the above result from [**H**]. Under the same bunching conditions, even if the invariant manifolds do not fill out a foliation, there is still the 'ghost' of an invariant foliation in the form of these intrinsic transverse derivatives. If the lamination is really a foliation, then these are the actual derivatives of the foliation, and the level of differentiability agrees with [**H**] (although in [**H**] differentiability is only considered up to C^2) or what could be obtained from [**HPS**]. Moreover, these results apply to pseudo-hyperbolic as well as hyperbolic sets.

In a future paper [**Mc2**] we will show that these ghost derivatives can be used as a foundation for locally 'filling-in' the lamination to a true foliation via the Whitney extension theorem. This is similar to the result of Pöschel [**P**] for the variant tori of a Hamiltonian system. The smoothness of the filled-in foliations is controlled by the bunching conditions. The assertion that a lamination is locally the restriction of any foliation, let alone a differentiable foliation, is a non-trivial statement. For general laminations there can be topological obstructions to filling-in to even a C^0 foliation [**Mc1**]. In fact, an open question dating back to early Axiom A work is whether the stable and unstable laminations of a basic set can locally be filled-in to C^0 foliations. Therefore, the current results as extended in [**Mc2**] provide a partial answer to this open question. In addition, the concepts contained here were the direct impetus for the work in [**Mc3, Mc4**].

Since a foliation is a geometric object, derivatives do not make sense *per se*. Therefore, we begin in §2 by defining the entity that is to play the role of a transverse k-jet of a leaf, and then use this in the next section to state the main results. These include a theorem for fixed points, which is the basis for the proof of the theorem for more general invariant sets. The proof for the fixed point case is in §4, while the proof for general invariant sets is handled in §5.

2. *Jets of foliations*

Throughout this paper we will follow the common practice whereby for any real number $r \geq 0$ a function is said to be a class C^r if it possesses continuous derivatives up to order $[r]$, and its $[r]$th order derivative is (r)-Hölder. Here $[r]$ and (r) denote the integer and fractional parts of r, respectively. The case when the top derivative is 1-Hölder or Lipschitz does not fit this well. This case is written C^{l+Lip} ($l \in \mathbb{N}$), where $r = l + Lip$ is considered distinguishable from $r = l + 1$. For integer r, the C^r-norm of a function is defined by $\|f\|_{C^r} := \max_{0\leq i\leq r}\{\sup_x \|(D^i f)_x\|\}$, where the supremum is over the domain of the function. For non-integer differentiability classes, $r = l + \theta$ with $0 < \theta \leq Lip$, $\|f\|_{C^r} := \max\{\|f\|_{C^l}, H_\theta[(D^l f)]\}$. Here $H_\theta(\cdot)$ denotes the θ-Hölder constant of $(\cdot)$.

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For non-integer r, the topology induced by the C^r-norm is too strong. For example, translation $a \mapsto (x+a)^\theta$ is not continuous in the θ-Hölder functions under this topology. Consequently, unless stated otherwise, we will assume spaces of C^r functions are endowed with the weaker C^l-topology ($r = l + \theta$). This has its own drawbacks. A space of C^r functions is not complete under the C^l-topology. On the other hand, a 'C^r-ball', i.e. a set of the form $\{f : \|f\|_{C^r} \leq K\}$, is a closed (and hence complete) subset of the C^l functions under the C^l-topology.

We begin by defining a notion of an 'infinitesimal foliation' along a leaf. A principle feature that we need to incorporate is a difference in differentiability along leaves as compared to across leaves. In our theorems, the leaves will be as smooth as the underlying dynamical system, but transverse differentiability will typically be limited. A model for this is a function of two variables which has partial derivatives up to order r in the first variable, but only up to $k < r$ in the second. This, however, leaves open the question of how the mixed partials behave. We want mixed partials of order $\leq r$ to exist provided they are order at most k in the second variable.

One knows a foliation on a neighbourhood of a leaf if and only if one knows its holonomy maps along the leaf. We want this same information on an infinitesimal level. Suppose M has dimension n, and let $r \in \mathbb{R}$, $k \in \mathbb{Z}$ with $r \geq 1$ and $0 \leq k \leq r$.

Definition 2.1. A local (k, r)-jet of a foliation on M is an equivalence class of triples $[D, U, \mathcal{F}]_{(k,r)}$, where $U \subset M$ is a neighbourhood of a closed (embedded) disk D and $\mathcal{F}$ is a C^r foliation of U, with leaves of the same dimension as D, such that D is contained in a leaf of $\mathcal{F}$ (the principal leaf). The equivalence relation is given by $(D_1, U_1, \mathcal{F}_1) \sim_{(k,r)} (D_2, U_2, \mathcal{F}_2)$ provided:

(1) $D_1 = D_2$ (=: D), and

(2) for all $x, x' \in D$ and all (smooth) transversals Σ, Σ' through x, x', respectively, the corresponding holonomy maps for $\mathcal{F}_1$ and $\mathcal{F}_2$, $H_1, H_2 : \Sigma \mapsto \Sigma'$, have the same k-jet at x.

The dimension of the local (k, r)-jet is the dimension of the disk D.

These jets are local in the sense that if $\mathcal{F}$ is a foliation of a neighbourhood of some point $p \in M$, then D can be taken to be any (arbitrarily small) disk plaque of $\mathcal{F}$ through p. D is the *base disk*, and when we wish to specify the disk D explicitly, we will refer to a (k, r)-jet of a foliation along D.

In general, a holonomy map is defined in terms of a particular path in the leaf connecting the points x, x'. However, since D is simply connected, the holonomy maps are path independent [**CLN**]. Moreover, although the definition is given in terms of all possible transversals Σ, Σ', it is in fact sufficient to restrict attention to any collection which forms a (smooth) transverse foliation to D.

Denote the set of all local (k, r)-jets of foliations of a given dimension d on the manifold M by $\mathcal{F}_d^{(k,r)}(M)$. For $\xi_1, \xi_2 \in \mathcal{F}_d^{(k,r)}(M)$, we will write $\xi_1 \supset \xi_2$ if ξ_2 is just the restriction of ξ_1 to a smaller base disk. That is, for $\xi_i := [D_i, U_i, \mathcal{F}_i]_{(k,r)}$ $(i = 1, 2)$, we must have $D_1 \supset D_2$ and $\xi_1|_{D_2} := [D_2, U_1, \mathcal{F}_1]_{(k,r)} = \xi_2$.

Any diffeomorphism $f : M \to M$ will induce a natural action on $\mathcal{F}_d^{(k,r)}(M)$ defined by $f_*(\xi) = [f(D), f(U), f(\mathcal{F})]_{(k,r)}$ for $\xi = [D, U, \mathcal{F}]_{(k,r)}$. We say ξ is *f-overflowing*

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invariant if $f_*(\xi) \supset \xi$, i.e. $f(D) \supset D$, and $f(\xi)|_D := [D, f(U), f(\mathcal{F})]_{(k,r)} = \xi$. In general, we will say a set $S \subset \mathcal{F}_d^{(k,r)}(M)$ is f-overflowing invariant if for every $\xi \in S$, there exists $\xi' \in S$ with $f_*(\xi') \supset \xi$.

We need to be able to talk about when a sequence in $\mathcal{F}_d^{(k,r)}(M)$ converges. In [**Mc1**] it is shown that there is a natural topology on this space which is characterized as follows. Let B_1^d denote the closed unit ball in $\mathbb{R}^d$. If $\xi = [D, U, \mathcal{F}]_{(k,r)}$, we can always find (the inverse of) a foliation chart $\varphi : B_1^d \times \mathbb{R}^{n-d} \to M$ such that the horizontal d-planes are taken into leaves of $\mathcal{F}$ and $\varphi(B_1^d \times \{0\}) = D$. Likewise, for $(\xi_n)_{n\geq 1} \subset \mathcal{F}_d^{(k,r)}(M)$, we can find embeddings φ_n for each ξ_n. There are, of course, many ways to choose these embeddings, and so for arbitrary choices there would be no relationship between φ_n and φ. However, we say $\xi_n \to \xi$ if and only if it is possible to choose φ_n such that $\varphi_n \to \varphi$ in the following sense. First, we want $\varphi_n(B) \to \varphi(B)$ as sets, where $B := B_1^d \times \{0\}$; i.e. $D_n \to D$ as sets. Since D is topologically trivial, we can always find a single coordinate chart (ψ, V) on M such that $D \subset V$. Hence, for n sufficiently large we must have $D_n \subset V$, and $\psi \circ \varphi_n$ is defined. We then require that for $0 \leq i \leq k$ and $0 \leq j \leq [r] - i$, each of the partial derivatives $D_1^j D_2^i(\psi \circ \varphi_n)$ converge uniformly to $D_1^j D_2^i(\psi \circ \varphi)$ on $B_1^d \times \{0\}$.

This can be restated equivalently as follows. Dropping ψ for clarity, each of the partial derivatives $D_2^i \varphi_n$ $(i = 0, \ldots, k)$ is a C^{r-i} function, which we restrict to B_1^d (identified with $B_1^d \times \{0\}$). As a function on B_1^d, we require that $D_2^i \varphi_n$ converge to $D_2^i \varphi$ in the $C^{[r]-i}$ topology. Note that this is the same sort of convergence one uses for embedded disks (e.g. the base disks without the additional structure). A sequence of C^r disks is said to converge if and only if there exist embeddings for those disks which converge as C^r functions.

In practice, we will always have a canonical representative for a local jet of a foliation. We will typically be interested in the following situation. Relative to a fixed coordinate chart, so we may as well assume $M = \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$, we will only need to consider local jets of foliations whose base disks are graphs over a common set in $\mathbb{R}^d$, which we can take to be the unit ball. This subset of $\mathcal{F}_d^{(k,r)}(M)$ (which is *not* open) has a simple representation in terms of function spaces. If the base disk of $[D, U, \mathcal{F}]_{(k,r)}$ is a graph over B_1^d, then nearby leaves of $\mathcal{F}$ are also graphs. Therefore, we can express the leaves of $\mathcal{F}$ as the graphs of a parameterized family of functions, $g_y(x)$, where the parameter is the y-intercept of the leaf. In this way one obtains a C^r diffeomorphism $G : \mathbb{R}^n \to \mathbb{R}^n$ by $G(x, y) := (x, g_y(x))$, and G is a foliation chart for $\mathcal{F}$. We refer to such a G (unique to the given coordinates) as a *standard presentation* of $\mathcal{F}$. G is close to the identity near $x = 0$, so it will be convenient to instead write $G(x, y) = (x, y + g(x, y))$, where $g(x, y) := g_y(x) - y$ and satisfies $g(0, y) \equiv 0$.

Now define $g^{(i)}(x) := (D_2^i g)_{(x,y_0)} \in C^{r-i}(B_1^d, L^i(\mathbb{R}^{n-d}, \mathbb{R}^{n-d}))$ $(i = 0, \ldots, k)$, where y_0 is the y-intercept of the base disk, and set $\mathbf{g} := (g^{(0)}, g^{(1)}, \ldots, g^{(k)})$. It is clear that $\mathbf{g}$ depends only on the equivalence class $[D, U, \mathcal{F}]_{(k,r)}$ and not on the specific foliation $\mathcal{F}$. Moreover, since $g_y(x_0)$ is really just the holonomy map for $\mathcal{F}$ from the vertical plane $x = 0$ to $x = x_0$, two equivalence classes (both of whose base disks are graphs over B_1^d) will be equal if and only if they determine the same $(k + 1)$-tuple $\mathbf{g}$. From the above characterization of the topology, a sequence ξ_i converges to ξ in this restricted subset of $\mathcal{F}_d^{(k,r)}(M)$ if and only if the corresponding $\mathbf{g}_i \to \mathbf{g}$ in

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$C^r(B_1^d, \mathbb{R}^{n-d}) \times \cdots \times C^{r-k}(B_1^d, L^k(\mathbb{R}^{n-d}, \mathbb{R}^{n-d}))$. Therefore, it is this transverse k-jet, $\mathbf{g}$, with which we will primarily work.

3. *Statements of results*

The main results of this paper are contained in the following theorems. The flavour of both the statements and proofs are closely related to the stable manifold theorem and its generalizations. The first is the general fixed point case. Here, however, we do not assume that the spectrum of the two parts of the derivative are disjoint. The derivative may in fact not be able to be made block diagonal. On the other hand, we take as a hypothesis the existence of an invariant manifold. The reason for stating the theorem in this manner is to emphasize that the conditions on the existence, smoothness, and/or uniqueness of an invariant manifold, for example, those from the stable manifold theorem, are independent of the conditions for 'transverse smoothness' (the existence of a (k, r)-jet of a foliation). By combining the hypotheses of the stable manifold theorem with those of Theorem 3.1, we get Theorem 3.2. From the fixed point case we can deduce generalizations of the Lambda lemma (Theorems 3.3 and 3.4) and the case for invariant sets (Theorem 3.5). The proofs of Theorem 3.1, 3.2 and 3.3 are in §4, and the proof of 3.5 is in §5.

We consider only the case when $(Df)_0|_{\mathbb{E}_1}$ is expanding, since this is more natural to the proof. Analogous results follow for the case when $(Df)_0|_{\mathbb{E}_1}$ is contracting by replacing f with f^{-1}. This theorem says nothing about the case when $(Df)_0|_{\mathbb{E}_1}$ has mixed character.

For a function λ, we will use $gr(\lambda) = graph(\lambda)$ to denote the set $\{(x, \lambda(x)) : x \in \text{domain}(\lambda)\}$. Depending on the context, $gr(\lambda)$ may also denote the function $gr(\lambda)(x) = (x, \lambda(x))$. Let $sp(A)$ denote the spectrum of an operator A and $|sp(A)| = \{|z| : z \in sp(A)\}$.

THEOREM 3.1. *Let $\mathbb{E}_1, \mathbb{E}_2$ be Euclidean spaces, set $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$, and let $f : \mathbb{E} \to \mathbb{E}$ be a C^r diffeomorphism fixing the origin, $r > 1$. Assume $(Df)_0$ expands $\mathbb{E}_1$ so that relative to this splitting $(Df)_0 = \left[\begin{smallmatrix} A & * \\ 0 & N \end{smallmatrix}\right]$ with $|sp(A)| \geq \alpha > 1$. Let τ_m, τ_M be the inner and outer spectral radii of N respectively, $\tau_m \leq |sp(N)| \leq \tau_M$. $sp(A)$ and $sp(N)$ need not be disjoint. Suppose for some integer $k \geq 1$ and number θ, $0 \leq \theta \leq 1$, with $k + \theta \leq r$, τ_m, τ_M, and α satisfy:*

$$\frac{\tau_M}{\tau_m} < \alpha, \quad \frac{\tau_M}{\tau_m^k} < \alpha^\theta.$$

Then for any C^r function $\lambda : \mathbb{E}_1(1) \to \mathbb{E}_2(1)$ such that $gr(\lambda)$ is an f-overflowing invariant, expanding submanifold tangent to $\mathbb{E}_1$ at zero (if such exists), there exists a unique f-overflowing invariant local (k, r)-jet of a foliation along $gr(\lambda)$.

Moreover, if $\bar{f} \in \text{Diff}^r(\mathbb{E})$ satisfies the above hypotheses, there is a C^1 neighbourhood $\mathcal{N}$ of $\bar{f}$ in $\text{Diff}^r(\mathbb{E})$ such that for all $f \in \mathcal{N}$ with $f(gr(\lambda)) \supset gr(\lambda)$ and $f(0) = 0$, there is a unique f-invariant local (k, r)-jet of a foliation, ξ_f, along $gr(\lambda)$. If $r > k + Lip$, then the function $f \mapsto \xi_f$ is continuous.

In the above statement, by *expanding* we mean that the invariant disk can be regenerated by any arbitrarily small open subdisk about the origin in a finite number of iterates, i.e. given any $D \subset gr(\lambda)$, there exists an $n > 0$ such that $f^n(D) \supset gr(\lambda)$. Since $|sp(A)| > 1$, any invariant manifold tangent to $\mathbb{E}_1$ at zero must be expanding on a sufficiently small

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neighbourhood of zero. Therefore, this requirement can always be met by restricting to a smaller set. An alternate conclusion, avoiding the assumption that $gr\,\lambda$ is expanding, would be that for some $D \subset gr\,\lambda$, there exists an f-overflowing invariant local (k,r)-jet of a foliation with base disk D. However, we chose this statement to emphasize the uniqueness. Given any overflowing invariant disk tangent to $\mathbb{E}_1$ at zero (which is not so large that it fails to be expanding), there exists a unique invariant local (k,r)-jet of a foliation along that disk.

The hypotheses of the theorem are easiest to satisfy when $k,\theta = 1$. In this case the conditions on τ_{m} and τ_{M} require the minimum expansion in $\mathbb{E}_1$ to dominate the relative width of the annulus containing $sp(N)$, $\tau_{\mathrm{M}}/\tau_{\mathrm{m}} < \alpha$. If the spectrum of N lies nearly on a circle, so $\tau_{\mathrm{m}} \cong \tau_{\mathrm{M}}$, α can be very close to one. However, if $sp(N)$ is spread out, α will have to be large. In all cases, α must be greater than one since $\tau_{\mathrm{m}} \leq \tau_{\mathrm{M}}$. For $k > 1$, if $\tau_{\mathrm{m}} < 1$, i.e. if N is not purely expanding, then $\tau_{\mathrm{M}}(\tau_{\mathrm{m}})^{-k}$ increases with k. Therefore, $\tau_{\mathrm{M}}\tau_{\mathrm{m}}^{k} < \alpha^{\theta}$ becomes more difficult to satisfy with increasing k, requiring even larger α. In particular, if $\mathbb{E}_1 \oplus \mathbb{E}_2$ is a hyperbolic splitting for $(Df)_0$ with $|sp(N)| < 1$, one could attempt to apply the theorem to both f and f^{-1}. Although it is conceivable for the hypotheses to be satisfied for $k = 1$ in both the unstable and stable directions, a simple calculation shows it is impossible for them to be simultaneously satisfied for any integer $k > 1$.

THEOREM 3.2. *Let $f : M \to M$ be a C^r diffeomorphism, $r > 1$, with ρ-pseudo-hyperbolic fixed point $p \in M$ and corresponding splitting $T_pM = E_1 \oplus E_2$. For $i = 1, 2$, let $T_i = (Df)_p|_{E_i} : E_i \to E_i$, and define*

$$b_i = \min\{|z| : z \in sp(T_i)\}, \quad B_i = \max\{|z| : z \in sp(T_i)\}.$$

Assume these satisfy $b_1 > 1$, $(B_2/b_2) < b_1$ and $(B_2/b_2^k) < b_1^{\theta}$ for some $k \in \mathbb{N}$ and $0 \leq \theta \leq 1$, with $k + \theta \leq r$. Suppose one of the following two conditions holds:

(i) *$B_2 < b_1$ (strong unstable versus pseudo-stable);*

(ii) *$B_1^r < b_2$ (weak unstable versus strong unstable).*

Existence. *There exists an f-overflowing invariance $\xi \in \mathcal{F}_d^{(k,r)}(M)$ with a base disk tangent to E_1 at p, $d = \dim(E_1)$. In the strong unstable case (i), ξ is unique up to the size of the base disk.*

Continuity. *If $\bar{f}$ satisfies the above hypotheses with $r > k + Lip$, then there exists a C^1 neighbourhood $\mathcal{N}$ of $\bar{f}$ in $\mathrm{Diff}^r(M)$ and a continuous function $\xi : \mathcal{N}_p := \{f \in \mathcal{N} : f(p) = p\} \to \mathcal{F}_d^{(k,r)}(M)$, $f \mapsto \xi_f$, such that ξ_f is f-overflowing invariant and has base disk tangent to $E_1(f)$, where $E_1(f)$ is the generalized eigenspace of $(Df)_p$ corresponding to all eigenvalues $z \in sp[(Df)_p]$ satisfying $|z| > \rho$ in case (i), and $|z| < \rho$ in case (ii). If, in addition, p is an elementary fixed point for $\bar{f}$ $(1 \notin sp[(D\bar{f})_p])$, then the restriction $f(p) = p$ can be eliminated and ξ is a continuous function on all of $\mathcal{N}$.*

These local jets of foliations should be considered as natural extensions of the invariant manifolds themselves. They have the same sort of properties and behaviour. This is highlighted by the following theorems, which are analogues to the Lambda lemma. We give two versions. The first is in the flavour of Theorem 3.1, where we do not assume the spectra of the two parts are disjoint, and everything we need to know about the invariant

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manifold is taken as an assumption. The second is a more direct generalization of the usual Lambda lemma.

THEOREM 3.3. *Let $f : \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} \to \mathbb{R}^n$ and $\lambda : B_1^d \to B_1^{n-d}$ satisfy the hypotheses of 3.1 with $r > k + Lip$. Let ξ be the unique f-invariant local (k, r)-jet of a foliation along $gr(\lambda)$. Suppose there exist $(\lambda_n)_{n\geq 0} \subset C^r(B_1^d, \mathbb{R}^{n-d})$ such that $f(gr\,\lambda_n) \supset gr(\lambda_{n+1})$ and $\lambda_n \to \lambda$ in the $C^{[r]}$ topology. Let ξ_0 be any local (k, r)-jet of a foliation along $gr(\lambda_0)$, and for $n \geq 1$ define $\xi_n := f_*(\xi_{n-1})|_{gr(\lambda_n)}$. Then $\xi_n \to \xi$.*

COROLLARY 3.4. (Lambda lemma for local jets of foliations) *Let $f : M \to M$ satisfy the hypotheses of Theorem 3.2 for a hyperbolic fixed point p and $r > k + Lip$. Let ξ be the unique f-invariant local (k, r)-jet of a foliation along $W^u_{\text{loc}}(p)$. For any $\xi_0 \in \mathcal{F}_d^{(k,r)}(M)$ whose base disk intersects $W^s(p)$ transversely, the iterates $f_*^n(\xi_0)$ converge to ξ in the sense that for each n we can choose $\xi_n \subset f_*^n(\xi_0)$ such that $\xi_n \to \xi$.*

Proof. Choose local coordinates so that $W^u_{\text{loc}}(p)$ is the graph of a function $\lambda : B_1^d \to B_1^{n-d}$. The Lambda lemma says that for any C^r d-disk D_0 transverse to $W^s(p)$, if n is sufficiently large, $f^n(D_0) \cap [B_1^d \times B_1^{n-d}]$ will be the graph of a function $\lambda_n : B_1^d \to B_1^{n-d}$ satisfying the hypotheses of Theorem 3.3. Therefore, Corollary 3.4 follows immediately. □

Finally we state our theorem for invariant sets. This also is a natural extension of the general stable manifold theorem. Given a diffeomorphism f of M, for $x \in M$ and $\alpha > 1$, let $W^u(x; \alpha) := \{y \in M : d(f^{-n}(y), f^{-n}(x)) \cdot \alpha^n \to 0 \text{ as } n \to \infty\}$, and let $W^u_\rho(x; \alpha)$ denote the set of all $y \in W^u(x; \alpha)$ such that $d(f^{-n}(y), f^{-n}(x)) \leq \rho$ for all $n \geq 0$. The stable manifold theorem says that for a α-pseudohyperbolic set, $W^u_\rho(x; \alpha)$ is an overflowing invariant, C^r embedded disk that varies continuously with x in the invariant set.

THEOREM 3.5. *Let Λ be a compact invariant set for a C^r diffeomorphism f of M, $r > 1$, such that the tangent bundle of M restricted to Λ decomposes as a continuous, Tf-invariant direct sum, $T_\Lambda M = E_1 \oplus E_2$. Relative to this splitting write $T_x f = \left[\begin{smallmatrix} A_x & 0 \\ 0 & N_x \end{smallmatrix}\right]$. Assume there is a Riemannian metric on M and numbers $0 \leq \theta \leq Lip$, $k \in \mathbb{N}$ with $k + \theta \leq r$, and $\alpha \geq 1$, such that for each $x \in \Lambda$ we have,*

$$\|N_x\| < \alpha < m(A_x) := \|A_x^{-1}\|^{-1} \quad \text{and} \quad \|N_x\| \cdot \|N_x^{-1}\|^k \cdot \|A_x^{-1}\|^\theta < 1.$$

Then given ρ sufficiently small, there exists a unique continuous function $\xi : \Lambda \to \mathcal{F}_d^{(k,r)}(M)$, $d = \dim(E_1)$, such that ξ_x is a local (k, r)-jet of a foliation along $W^u_\rho(x; \alpha)$, and $\xi(\Lambda)$ is f-overflowing invariant.

Since the above theorem makes its comparisons at each point, in the special case where E_1 has codimension one, we have $\|N_x\| \cdot \|N_x^{-1}\| \equiv 1$. Therefore, the conditions of the theorems are automatically satisfied for $k = 1$ and any $\theta > 0$, from which we get the following.

COROLLARY 3.6. *If E_1 has codimension one and $Tf|_{E_1}$ is expanding, then the hypotheses of Theorem 4.4 are satisfied for $k = 1$ and any $\theta > 0$. Consequently, there exists a unique f-overflowing invariant family of local $(1, r)$-jets of foliations along the disks $W^u_\varepsilon(x; \alpha)$, $x \in \Lambda$.*

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4. *The fixed point case*

The proof of the fixed point case (Theorem 3.1) uses the same method as the graph transform proof of the stable manifold theorem. Here, however, we need to work with more than just a single graph. Intuitively we start with a $(k+1)$-tuple $\mathbf{g} = (g^{(0)}, \dots, g^{(k)})$, $g^{(0)} = \lambda$, representing a (k, r)-jet of a foliation along $gr(\lambda)$, and then apply f_* to it and show that under iteration this converges to a unique attracting fixed point of f_*. The action of f_* on $\mathbf{g}$ can be obtained as follows. First we must choose a function $G(x, y) = (x, y + g(x, y))$ with $(D_2^i g)|_{B_1^d} = g^{(i)}$. G is then a standard presentation for a foliation $\mathcal{F}$ representing the given local jet of a foliation. Therefore, $f \circ G$ is a foliation chart for $f(\mathcal{F})$, but it is not a standard presentation. Consequently, $f \circ G$ must be converted into a standard presentation, $\tilde{G}(x, y) = (x, y + \tilde{g}(x, y))$. The image of $\mathbf{g}$ is then $\tilde{\mathbf{g}} = (\tilde{g}^{(0)}, \dots, \tilde{g}^{(k)})$, where $\tilde{g}^{(i)} := (D_2^i \tilde{g})|_{B_1^d}$.

The above outline leaves two points that need to be addressed. The first is whether we can always find the function G for a given $\mathbf{g}$. So far we have only argued that given a local jet of a foliation along $gr(\lambda)$, we can associate to it a unique $\mathbf{g}$. We have not shown the reverse. Moreover, given such a G, we need to know when it will be a diffeomorphism (so that it defines a foliation). These are handled in Lemma 4.1 with the use of the Whitney extension theorem [**AR, St, W**]. Second, we need to know how to convert an arbitrary foliation chart into a standard presentation. This is done in Lemma 4.2. In what follows let $L^i_{\text{sym}}(V, W)$ denote the symmetric, i-linear operators from V to W, and give the function spaces the stronger fractional differentiability class topologies.

LEMMA 4.1. *Given any* $\mathbf{g} = (g^{(0)}, \dots, g^{(k)})$ *with* $g^{(i)} \in C^{r-i}(B_i^d, L^i_{\text{sym}}(\mathbb{R}^{n-d}, \mathbb{R}^{n-d}))$ $(k \le r)$, *one can find a* $g \in C^r(\mathbb{R}^n, \mathbb{R}^{n-d})$ *with* $(D_2^i g)|_{B_1^d} = g^{(i)}$, $i = 0, \dots, k$. *Moreover,* $\mathbf{g} \mapsto g$ *is a bounded linear operator from* $C^r(B_1^d, \mathbb{R}^{n-d}) \times \cdots \times C^{r-k}(B_1^d, L^k_{\text{sym}}(\mathbb{R}^{n-d}, \mathbb{R}^{n-d}))$ *into* $C^r(\mathbb{R}^n, \mathbb{R}^{n-d})$.

If for each i, $g^{(i)}(0) = 0$, *then* g *can be taken to satisfy* $g(0, y) \equiv 0$ *for* y *near zero. The map* $G(x, y) := (x, y + g(x, y)) : U \to \mathbb{R}^d \times \mathbb{R}^{n-d}$ *is an embedding for some* $U \supset B_1^d$ *if and only if* $I + g^{(1)}(x) \in L(\mathbb{R}^{n-d}, \mathbb{R}^{n-d})$ *is non-singular for all* $x \in B_1^d$.

Proof. Since the $g^{(i)}$ are arbitrary, we can assume, without loss of generality, that $r = k+\theta$, for if $r - k > 1$ then we can augment $\mathbf{g}$ by defining $g^{(i)} \equiv 0$ for $k < i \le r$.

Define $f^{(0)} = g^{(0)}$. To apply the Whitney extension theorem to $f^{(0)}$ we require maps $f^{(i)} : B_1^d \to L^i_{\text{sym}}(\mathbb{R}^d \times \mathbb{R}^{n-d}, \mathbb{R}^{n-d})$, $i = 1, \dots, k$ (the prospective total derivatives) that satisfy the Whitney conditions (see below). Since the $g^{(j)}$ are to be the partial derivatives of g with respect to y (i.e. the second factor), define what would correspond to the $D_1^{i-j} D_2^j$-component of $f^{(i)}$ to be $D^{i-j} g^{(j)}$. Define the remaining components by making $f^{(i)}$ symmetric. Formally, given $s \in \{1, 2\}^i$, define $|s| = s_1 + \cdots + s_i - i =$ number of two's in s, and for $(u_j, v_j) \in \mathbb{R}^d \times \mathbb{R}^{n-d}$,

$$f^{(i)}(x, 0)[(u_j, v_j)_{j=1,\dots,i}] = \sum_{s \in \{1,2\}^i} (D^{i-|s|} g^{(|s|)})_x [\{u_j : s_j = 1\}, \{v_j : s_j = 2\}].$$

Since $D^{i-|s|} g^{(|s|)}$ is symmetric with respect to both the first and second sets of vectors (the second by assumption, and the first because this is a higher-order derivative), $f^{(i)}$ is both

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well defined and symmetric. $g^{(|s|)}$ is $C^{r-|s|}$, so $(D^{i-|s|}g^{(|s|)})$ is C^{r-i}, and $f^{(i)}$ is C^{r-i} on B_1^d.

The Whitney conditions are the following. For $p, q \in B_1^d$, define $R_i(p,q)$ by $f^{(i)}(p) = \sum_{j\le k-i}(1/j!)f^{(i+j)}(q)[(p-q)^{(j)}] + R_i(p,q)$. There must exist $K > 0$ such that $\|f^{(i)}(p)\| \le K$ and $\|R_i(p,q)\| \le K\|p-q\|^{r-i}$ for all p and q.

To show $\mathbf{f} = (f^{(0)}, \ldots, f^{(k)})$ satisfies the Whitney conditions, we note that all points in B_1^d are of the form $(x,0)$. Hence, $f^{(i+j)}(q)[(p-q)^{(j)}] = f^{(i+j)}(x,0)[(x'-x,0)^{(j)}] = (D_1^j f^{(i)})_{(x,0)}[(x'-x)^{(j)}]$. Therefore, $R_i(p,q)$ reduces to the remainder term for a true Taylor polynomial for a C^{r-i} function, and so the second half of the Whitney condition must be satisfied. On the other hand, B_1^d is compact, and so the boundedness condition is satisfied.

The conclusion of Whitney's theorem is the existence of a C^r map $g : \mathbb{R}^n \to \mathbb{R}^{n-d}$ such that $(D^i g)_p = f^{(i)}(p)$, and that $\mathbf{f} \mapsto g$ is a bounded linear operator between these function spaces. Since $f^{(0)} = g^{(0)}$ it follows that g is an extension of $g^{(0)}$, and $(D_2^i g)_{(x,0)}[v_1, \ldots, v_i] = f^{(i)}(x,0)[(0,v_1), \ldots, (0,v_i)] = g^{(i)}(x)[v_1, \ldots, v_i]$. Therefore, $(D_2^i g)|_{B_1^d} = g^{(i)}$.

Since $g^{(i)} \mapsto D^j g^{(i)}$ is a bounded linear operator (from C^{r-i} to C^{r-i-j}), the map $\mathbf{g} \mapsto \mathbf{f}$, and hence $\mathbf{g} \mapsto g$, is a bounded linear operator.

If $g^{(i)}(0) = 0$ for all i, define $h : \mathbb{R}^{n-d} \to \mathbb{R}^{n-d}$ by $h(y) = y + g(0,y)$. Then h equals the identity up to order k at zero, which implies h^{-1} exists near zero and $h^{-1} = \mathrm{id}$ up to order k at zero. Define $\overline{g}(x,y) = (h^{-1}y - y) + g(x, h^{-1}y)$, where $\overline{g}$ is defined on a neighbourhood U of B_1^d chosen sufficiently small that h^{-1} exists. $\overline{g} = g$ up to order k along B_1^d. Therefore, $\overline{g}$ is another extension of $g^{(0)}$ with $(D_2^i \overline{g})|_{B_1^d} = g^{(i)}$. Moreover, $\overline{g}(0,y) \equiv 0$ since $\overline{g}(0,y) = [h^{-1}y + g(0, h^{-1}y)] - y = (h \circ h^{-1})(y) - y$.

Finally, G will be an embedding on a neighbourhood of B_1^d if and only if (DG) is non-singular at all points of B_1^d (since G is necessarily one-to-one on B_1^d). However,

$$(DG)_{(x,0)} = \begin{bmatrix} I & 0 \\ (Dg^{(0)})_x & I + g^{(1)}(x) \end{bmatrix}$$

and so the last assertion follows. □

LEMMA 4.2. *Let $U \subset \mathbb{R}^d$, $V \subset \mathbb{R}^{n-d}$ be open with $B_1^d \subset U$. Suppose $f : U \times V \to \mathbb{R}^d \times \mathbb{R}^{n-d}$ is C^r ($r \ge 1$), and $f_1(x, y_0) : B_1^d \to \mathbb{R}^d$ is a C^r embedding covering B_1^d for some $y_0 \in V$. Let x_0 be the unique point with $f_1(x_0, y_0) = 0$, define h implicitly by $f_1(h(y), y) \equiv 0$ with $h(y_0) = x_0$, and set $F(x,y) := [f_1(x,y), f_2(h(y),y)]$. If $(Df)_{(x_0,y_0)}$ is non-singular, then there is an open set $W \subset \mathbb{R}^{n-d}$ about y_0 such that $G := f \circ F^{-1} : B_1^d \times W \to \mathbb{R}^d \times \mathbb{R}^{n-d}$ is well defined. G is C^r and has the form $G(x,y) = (x, y + g(x,y))$ with $g(0,y) \equiv 0$. If $\mathcal{H}$ is the family of horizontal planes $\mathbb{R}^d \times \{y\}$, then $G(\mathcal{H}) = f(\mathcal{H})$ on the image of G. Moreover, if f is an embedding onto its image, then G is an embeddding onto its image, and $G(\mathcal{H}) = f(\mathcal{H})$ is a foliation.*

Proof. The assertion that $f_1(x, y_0)$ is an embedding says the image under f of $B_1^d \times \{y_0\}$ is a smooth graph over B_1^d. In particular, $(D_1 f_1)_{(x,y_0)}$ is non-singular for all $x \in B_1^d$, including $x = x_0$. By the implicit function theorem, there is a unique C^r function

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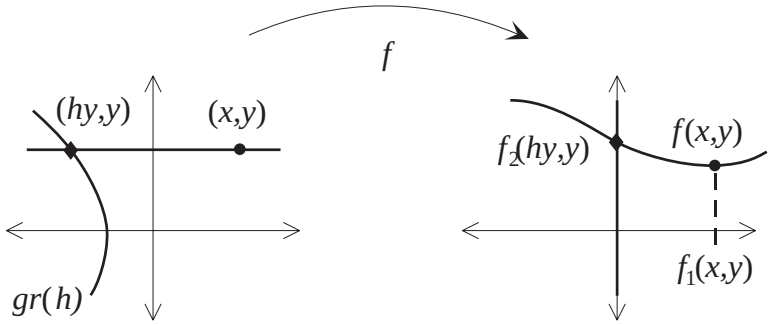


FIGURE 4.3.

$h : V' \to \mathbb{R}^d$ with $h(y_0) = x_0$ and satisfying $f_1(h(y), y) \equiv 0$ for some neighbourhood $V' \subset V$ of y_0. Therefore, F is well defined for y sufficiently small.

We want F to be invertible on a neighbourhood of $B_1^d \times \{y_0\}$ so that G will be well defined. Observe that since F_2 is a function of y only, (DF) is block triangular with diagonal blocks $(D_1 f_1)$ and $D_y[f_2(h(y), y)]$. $(D_1 f_1)_{(x,y_0)}$ is non-singular, and since $f(hy, y) = (0, f_2(hy, y))$, the rank of $D_y[f_2(h(y), y)]$ equals the rank of $D_y[f(hy, y)]$. $D_y[f(hy, y)]_{y_0} = (Df)_{(x_0,y_0)}[(Dh)_{y_0}\ I]^\top$, and $(Df)_{(x_0,y_0)}$ is non-singular and $[(Dh)_{y_0}\ I]$ has full rank. Therefore, $D_y[f_2(h(y), y)]_{y_0}$ has full rank, and (DF) is non-singular at all points in $B_1^d \times \{y_0\}$. Since $F(x, y_0) = (f_1(x, y_0), f_2(x_0, y_0))$, F is one-to-one on $B_1^d \times \{y_0\}$. Consequently, F is an embedding on a neighbourhood of $B_1^d \times \{y_0\}$, and F^{-1} exists and is C^r on a neighbourhood of the image of $B_1^d \times \{y_0\}$. Set $y_1 := f_2(x_0, y_0)$, then $F(B_1^d \times \{y_0\}) \supset B_1^d \times \{y_1\}$. Therefore, we can choose a sufficiently small neighbourhood W of y_1 such that $B_1^d \times W$ is contained in the domain of F^{-1}, and $f \circ F^{-1}$ will be well defined on $B_1^d \times W$. Since F^{-1} is a diffeomorphism on $B_1^d \times W$, clearly if f is an embedding, then so is G.

Since F_2 is a function of y only, F preserves horizontal planes. Hence, F^{-1} takes $\mathcal{H}$ to itself. Therefore, $G(\mathcal{H}) = f[F^{-1}(\mathcal{H})] = f(\mathcal{H})$ on the image of G. We are left with showing that G has the desired form. Observe that since (hy, y) is the unique point on the plane through (x, y) which is sent under f to the y-axis, $f_2(hy, y)$ is the y-intercept of the leaf of $f(\mathcal{H})$ which passes through $f(x, y)$, see Figure 4.3. $f_1(x, y)$ is the x-coordinate of $f(x, y)$. Therefore, we have F^{-1} assigning to any pair (x-coordinate, y-intercept) the pre-image under f of the point that has this x-coordinate and is on the leaf with this y-intercept. Hence, $G(u, v)$ is the unique point that has x-coordinate u and is on the leaf with y-intercept v. However, then $(\pi_1 \circ G)(u, v) \equiv u$, and clearly the y-intercept of the leaf through $(0, v)$ is $(0, v)$, so $(\pi_2 \circ G)(0, v) \equiv v$. □

We are now ready to prove Theorem 3.1. As with the proof of the stable manifold theorem presented in [**HP**], we use Theorem 3.1 to derive the general case of a non-trivial invariant set, Λ, by considering the Banach space of bounded sections $\Lambda \to T_\Lambda M$. This will necessitate treating the fixed point case in infinite dimensions. Fundamentally, there is no difference between proving Theorem 3.1 for Banach spaces as opposed to Euclidean space. Even the definition of a local jet of a foliation could be carried over into Banach

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spaces. However, we would loose the correspondence between the local jets of foliations along $gr(\lambda)$ and the $(k+1)$-tuples $(\lambda, g^{(1)}, \ldots, g^{(k)})$ that came from Lemma 4.1, because the Whitney extension theorem does not hold in Banach spaces.

Therefore, when working in a Banach space $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$, we only consider the special case when the principal leaf is given as the graph of a C^r function $\lambda : \mathbb{E}_1(1) \to \mathbb{E}_2$. By a local (k, r)-jet of a foliation along $gr(\lambda)$ we will simply mean a collection $g^{(i)} \in C^{r-i}(\mathbb{E}_1(1), L^i_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2))$ $(i = 1, \ldots, k)$, with $g^{(i)}(0) = 0$ and $I + g^{(1)}(x)$ non-singular for each $x \in V$. If f is a C^r diffeomorphism of $\mathbb{E}$ into itself and $f(gr\,\lambda)$ is still a graph over $\mathbb{E}_1(1)$, then by a purely formal calculation we can still define the action of f in taking a local jet of a foliation along $gr(\lambda)$ to one along $f(gr\,\lambda)$. That is, we proceed as though Lemma 4.1 held. In this way the calculations do not change from Euclidean space to Banach spaces. Of course, the conclusion for a Banach space is only that we have a unique $(\lambda, g^{(1)}, \ldots, g^{(k)})$ invariant with respect to this formal calculation. However, this will be enough for Theorem 3.5.

THEOREM 4.4. *If in Theorem 3.1 $\mathbb{E}_1$ and $\mathbb{E}_2$ are Banach spaces, there exists a unique f-overflowing invariant local (k, r)-jet of a foliation along $gr(\lambda)$.*

The proof of Theorem 4.4 will be by induction. For each $i = 1, \ldots, k$, we assume $(\lambda, g^{(1)}, \ldots, g^{(i-1)})$ exist and are smooth, and then show $g^{(i)}$ exists and is smooth. This will rely on the following lemma and its corollary.

LEMMA 4.5. [**HP**] (Fibre contraction theorem) *Let X be a topological space and Y a complete metric space. Let $F : X \times Y \to X \times Y$ be a continuous function of the form $F(x, y) = (f(x), F_x(y))$. Suppose f has a globally attracting fixed point $p \in X$, and for each $x \in X$, $F_x : Y \to Y$ is Lipschitz with $Lip(F_x) \leq k$ for some $k < 1$ independent of x. Then F has a globally attracting fixed point $(p, P) \in X \times Y$.*

COROLLARY 4.6. (Parameterized fibre contractions) *Let Λ, X be topological spaces and Y a complete metric space. Let $F : \Lambda \times X \times Y \to X \times Y$ be a continuous function of the form $F_\lambda(x, y) = (f_\lambda(x), F_{(\lambda,x)}(y))$. Suppose for each $\lambda \in \Lambda$, f_λ has a globally attracting fixed point $p_\lambda \in X$ which varies continuously with λ, and for some $k < 1$ and each $(\lambda, x) \in \Lambda \times X$, $F_{(\lambda,x)} : Y \to Y$ is Lipschitz with $Lip(F_{(\lambda,x)}) \leq k$. Then F_λ has a globally attracting fixed point $(p_\lambda, P_\lambda) \in X \times Y$ depending continuously on λ.*

Proof. The existence of the fixed point is the difficult part, and that follows from Lemma 4.5. We are left with showing $\lambda \mapsto P_\lambda$ is continuous. Given $\lambda_i \in \Lambda$, $i = 0, 1$, let $F_i = F_{(\lambda_i, p(\lambda_i))}$. Then $F_i(P_{\lambda_i}) = P_{\lambda_i}$. For any y we have $y, F_i(y), F_i^2(y), \cdots \to P_{\lambda_i}$. Therefore,

$$\begin{aligned} d(y, P_{\lambda_i}) &\leq d(y, F_i(y)) + d(F_i(y), F_i^2(y)) + \cdots \\ &\leq d(y, F_i(y)) + k \cdot d(y, F_i(y)) + \cdots \leq (1-k)^{-1}\, d(y, F_i(y)). \end{aligned}$$

In particular, $d(P_{\lambda_1}, P_{\lambda_0}) \leq (1-k)^{-1}\, d(F_1(P_{\lambda_0}), P_{\lambda_0}) = (1-k)^{-1}\, d(F_1(P_{\lambda_0}), F_0(P_{\lambda_0}))$. Since p_λ is a continuous function of λ, for any fixed y, $F_{(\lambda, p(\lambda))}(y)$ is continuous in λ. Therefore, $d(F_1(P_{\lambda_0}), F_0(P_{\lambda_0})) \to 0$ as $\lambda_1 \to \lambda_0$, and so P_λ is continuous. □

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Proof of Theorems 3.1 and 4.4. If $\tau_m \leq 1$, then $\max\{\tau_M(\tau_m)^{-i} : 1 \leq i \leq k\} = \tau_M(\tau_m)^{-k} < \alpha^\theta \leq \alpha$, and the condition $\tau_M/\tau_m < \alpha$ is superfluous. If $\tau_m \geq 1$, then $\max\{\tau_M(\tau_m)^{-i} : 1 \leq i\} = \tau_M/\tau_m$. In either case, we can conclude that we also have $\tau_M(\tau_m)^{-i} < \alpha$ for $1 \leq i \leq k-1$.

Since α is the inner boundary for the spectrum of A, by changing to an equivalent norm on $\mathbb{E}_1$ we can assume $m(A) := \|A^{-1}\|^{-1} \leq \alpha$ is as close to α as we wish. Likewise, by changing norms on $\mathbb{E}_2$ we can assume $\|N\| \cong \tau_M$, $m(N) \cong \tau_m$. Consequently, since the conditions on τ_m, τ_M, and α are strict inequalities, we can assume $m(N)$, $\|N\|$, and $m(A)$ satisfy the same conditions. Namely,

$$\|A^{-1}\| < 1,$$
$$\|N\| \cdot \|N^{-1}\|^i \cdot \|A^{-1}\| < 1, \quad \text{for } i = 1, \ldots, k-1$$

and

$$\|N\| \cdot \|N^{-1}\|^k \cdot \|A^{-1}\|^\theta < 1.$$

Define the norm on $\mathbb{E}$ as the product of these new norms, $\|(x, y)\| := \max\{\|x\|, \|y\|\}$.

Since $gr(\lambda)$ is tangent to $\mathbb{E}_1$ at zero, we can make the C^r change of variable $(x, y) \mapsto (x, y - \lambda(x))$ which takes $gr(\lambda)$ to $\mathbb{E}_1(1)$ and leaves $(Df)_0$ unchanged. Therefore, we can assume, without loss of generality, that $\lambda \equiv 0$ and $\mathbb{E}_1(1)$ is f-overflowing invariant. In this case, $f_2(x, 0) \equiv 0$.

We are trying to prove that there exists a unique equivalence class of foliations, where the equivalence relation is based on the k-jets of the holonomy maps between points in $gr(\lambda)$, with the property that if $\mathcal{F}$ is in the equivalence class, then $f(\mathcal{F})$ is also. By Lemma 4.1, the equivalence classes (local (k, r)-jets of a foliations along $gr(\lambda)$) are homeomorphic to the set of $\mathbf{g} = (g^{(0)}, \ldots, g^{(k)}) \in C^r(\mathbb{E}_1(1), \mathbb{E}_2) \times \cdots \times C^{r-k}(\mathbb{E}_1(1), L^k_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2))$ with $g^{(0)} = \lambda$, $g^{(i)}(0) = 0$ $(i = 1, \ldots, k)$, and $I + g_x^{(1)}$ non-singular for all $x \in \mathbb{E}_1(1)$. (Since $\lambda \equiv 0$, we can drop $g^{(0)}$ and reduce this to a k-tuple.) Therefore, we want to look at how f_* acts on this set and show that it has a unique fixed point. We make the following definitions:

$$\mathcal{C}_i := \{g^{(i)} \in C^{r-i}(\mathbb{E}_1(1), L^i_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2)) : g^{(i)}(0) = 0\} \quad (i = 1, \ldots, k),$$

and

$$\mathcal{G}_i := \mathcal{C}_1 \times \cdots \times \mathcal{C}_i = \mathcal{G}_{i-1} \times \mathcal{C}_i.$$

Let $\Gamma_k : \mathcal{G}_k \to \mathcal{G}_k$ be the action of f_* on $\mathcal{G}_k$. We claim that Γ_k is indeed well defined on all of $\mathcal{G}_k$, not just for those $(g^{(1)}, \ldots, g^{(k)})$ with $I + g_x^{(1)}$ non-singular for $x \in \mathbb{E}_1(1)$. Therefore, we need to show there exists a unique fixed point of Γ_k, and that for the fixed point $I + g_x^{(1)}$ is non-singular for all $x \in \mathbb{E}_1(1)$. In fact, we will show more. Denote the integer and fractional differentiability class norms on these spaces by $\|\mathbf{g}\|_{C^{[r]}} := \max_{1 \leq s \leq i}\{\|g^{(s)}\|_{C^{[r]-s}}\}$ and $\|\mathbf{g}\|_{C^r} := \max_{1 \leq s \leq i}\{\|g^{(s)}\|_{C^{r-s}}\}$ for $\mathbf{g} \in \mathcal{G}_i$. We will show the fixed point is globally attracting with respect to $\|\mathbf{g}\|_{C^r}$. The topology of the local jets of foliations is controlled by convergence with respect to $\|\mathbf{g}\|_{C^{[r]}}$. Since $\|\mathbf{g}\|_{C^{[r]}} \leq \|\mathbf{g}\|_{C^r}$, this will imply that the fixed point is also globally attracting with respect to $\|\mathbf{g}\|_{C^{[r]}}$.

We outlined the construction of Γ_k earlier. With Lemma 4.2 we can now fill in the details. Given $(0, g^{(1)}, \ldots g^{(k)})$, let g be the extension defined in Lemma 4.1 with

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$g(0, y) \equiv 0$ and $(D_2^i g)_{(x,0)} = g_x^{(i)}$. Let $G(x, y) := (x, y + g(x, y))$. If $I + g_x^{(1)}$ is non-singular, then G is an embedding, and $\mathcal{F} := G(\mathcal{H})$ is the foliation with standard presentation G. Since $G(x, 0) = (x, 0)$, $f \circ G$ is an embedding that maps $\mathbb{E}_1(1)$ over itself. Therefore, applying Lemma 4.2 to $f \circ G$ with $y_0 = 0$, the standard presentation for $f(\mathcal{F})$ is $\tilde{G} := (f \circ G) \circ F^{-1}$, where $F(x, y) = [(f_1 \circ G)(x, y), (f_2 \circ G)(h(y), y)]$ and h solves $(f_1 \circ G)(hy, y) \equiv 0$ with $h(0) = 0$. Now set $\tilde{g}(x, y) := (\pi_2 \circ \tilde{G})(x, y) - y$, so that $\tilde{G}(x, y) = (x, y + \tilde{g}(x, y))$, and define $\tilde{g}_x^{(i)} := (D_2^i \tilde{g})_{(x,0)}$.

If $I + g_x^{(1)}$ is singular for some x, then G and $f \circ G$ are not embeddings. However, for the above calculations to go through, Lemma 4.2 requires only $D(f \circ G)_0 = (Df)_0 (DG)_0$ to be non-singular. f is a diffeomorphism and below we show $(DG)_0 = I$. Therefore, the $\tilde{g}^{(i)}$ are well defined for any $(g^{(1)}, \ldots, g^{(k)}) \in \mathcal{G}_k$. By Lemma 4.2, $\tilde{g}$ is C^r with $\tilde{g}(0, y) \equiv 0$. Hence, $\tilde{g}_x^{(i)}$ is C^{r-i} and $\tilde{g}^{(i)}(0) = 0$, i.e. $\tilde{g}^{(i)} \in \mathcal{C}_i$. Since both F and $f \circ G$ map $\mathbb{R}^d \times \{0\}$ into itself, $\tilde{G}$ does also. Hence, $\tilde{g}(x, 0) =: \tilde{g}_x^{(0)} \equiv 0$. Consequently, $(0, g^{(1)}, \ldots, g^{(k)}) \mapsto (0, \tilde{g}^{(1)}, \ldots, \tilde{g}^{(k)})$, and Γ_k defined by $\Gamma_k(g^{(1)}, \ldots, g^{(k)}) = (\tilde{g}^{(1)}, \ldots, \tilde{g}^{(k)})$ is well defined and maps $\mathcal{G}_k$ into itself. We could do the same for any $i = 1, \ldots, k$, and so we have a family of maps $\Gamma_i : \mathcal{G}_i \to \mathcal{G}_i$.

This description of Γ_k is rather complicated. As a result, analysing Γ_k to find a fixed point will entail a certain amount of calculations. We segregate most of these tedious but straightforward calculations in the proof of Lemma 4.8, which asserts the general form of this map. We can then use this general form to show the existence of a fixed point without having to delve too much into the details. An added benefit of the lemma is that we can see directly that the formal calculations needed in a Banach space are well defined. Lemma 4.8 relies on the higher-order chain rule, which we state here for reference.

THEOREM 4.7. [**Pu**] (Higher order chain rule) *Let $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ be Banach spaces, $U \subset \mathbb{E}$, $V \subset \mathbb{F}$ open sets, and $f : U \to V$, $g : V \to \mathbb{G}$ maps of class C^k. Then $g \circ f$ is of class C^k, and for $x \in U$, and $v_1, \ldots, v_k \in \mathbb{E}$, we have*

$$D^k(g \circ f)_x(v_1, \ldots, v_k) = \sum_{s=1}^{k} \sum_{\beta \in \mathcal{B}_{s,k}} (D^s g)_{f(x)} (D^\beta f)_x(v_\beta)$$

where $\mathcal{B}_{s,k}$ is the set of all partitions of $\{1, \ldots, k\}$ into s disjoint, non-empty subsets, and for $\beta \in \mathcal{B}_{s,k}$ ($\beta = \{\beta_1, \ldots, \beta_s\}$ with $\beta_1 \sqcup \cdots \sqcup \beta_s = \{1, \ldots, k\}$), we define the following: $|\beta_i| =$ order of β_i, $v_{\beta_i} := \{v_j : j \in \beta_i\}$ (so $v_{\beta_1} \sqcup \cdots \sqcup v_{\beta_s} = \{v_1, \ldots, v_k\}$) and $(D^\beta f)_x(v_\beta) := [(D^{|\beta_1|} f)_x(v_{\beta_1}), \ldots, (D^{|\beta_s|} f)_x(v_{\beta_s})]$.

LEMMA 4.8. *Let $g^{(i)}, \tilde{g}^{(i)} (1 \le i \le r)$ be as defined above. Let $\varphi(x) := f_1(x, 0)$. Then*

$$\tilde{g}_x^{(i)} = (D_2 f_2)_{(\varphi^{-1}x, 0)} g^{(i)}_{\varphi^{-1}x} [\{N^{-1}\}^i] + \Re_i(x)$$

where $\Re_i$ is a polynomial in the following (only): the functions $g^{(s)}_{\varphi^{-1}x}$ with $s < i$, and their derivatives $(D^t g^{(s)})_{\varphi^{-1}x}$ with $s + t \le i$; the functions $(D^j f)_{(\varphi^{-1}x, 0)}$ $(j \le i)$ and $(Df)^{-1}_{(\varphi^{-1}x, 0)}$; together with constant operators obtained by evaluating any of the above at $x = 0$. Here $\{N^{-1}\}^i$ denotes i copies of N^{-1}.

Proof. First, since f is a diffeomorphism and $f[\mathbb{E}_1(1)] \supset \mathbb{E}_1(1)$, φ is an embedding and covers $\mathbb{E}_1(1)$. Therefore, $\varphi^{-1} : \mathbb{E}_1(1) \to \mathbb{E}_1(1)$ exists and is C^r. By construction,

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$g(x,0) = g^{(0)}(x) \equiv 0$. Hence, $G(x,0) = (x,0)$. Since $h(0) = 0$, $F(x,0) = [(f_1 \circ G)(x,0), (f_2 \circ G)(0,0)] = [f_1(x,0), 0] = [\varphi(x), 0]$. Therefore, $F^{-1}(x,0) = [\varphi^{-1}(x), 0]$ and $(G \circ F^{-1})(x,0) = [\varphi^{-1}(x), 0]$. A partial derivative is the same as the full derivative restricted to vectors from the appropriate subspace. Therefore, $\tilde{g}_x^{(i)} = D_2^i(f_2 \circ G \circ F^{-1} - y)_{(x,0)} = D^i(f_2 \circ G \circ F^{-1} - y)_{(x,0)}|_{\mathbb{E}_2^i}$. We handle the case $i = 1$ first. Since $F^{-1}(x,0) = [\varphi^{-1}(x), 0] = (G \circ F^{-1})(x,0)$, we have

$$\begin{aligned} D(f_2 \circ G \circ F^{-1})_{(x,0)} &= (Df_2)_{G \circ F^{-1}(x,0)}(DG)_{F^{-1}(x,0)}(DF)^{-1}_{F^{-1}(x,0)} \\ &= (Df_2)_{(\varphi^{-1}x,0)}(DG)_{(\varphi^{-1}x,0)}(DF)^{-1}_{(\varphi^{-1}x,0)}. \end{aligned}$$

Computing $(DG)_{(\varphi^{-1}x,0)}$:

$$(DG)_{(\varphi^{-1}x,0)} = \begin{bmatrix} I & 0 \\ (D_1 g)_{(\varphi^{-1}x,0)} & I + (D_2 g)_{(\varphi^{-1}x,0)} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I + g^{(1)}_{\varphi^{-1}x} \end{bmatrix},$$

from which it follows that $(DG)_0 = I_{\mathbb{E}}$. Computing the components of $(DF)_{(\varphi^{-1}x,0)}$ separately we have

$$\begin{aligned} (DF_1)_{(\varphi^{-1}x,0)} &= (Df_1)_{(\varphi^{-1}x,0)}(DG)_{(\varphi^{-1}x,0)} \\ &= \left[(D_1 f_1)_{(\varphi^{-1}x,0)}(D_2 f_1)_{(\varphi^{-1}x,0)}(I + g_{\varphi^{-1}x})\right], \end{aligned}$$

and

$$(DF_2)_{(\varphi^{-1}x,0)} = (Df_2)_0(DG)_0 \begin{bmatrix} 0 & (Dh)_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & N \end{bmatrix}.$$

Both $(D_1 f_1)_{(x,0)}$ (any $x \in \mathbb{E}_1(1)$) and N are non-singular by assumption. Therefore,

$$(DF)^{-1}_{(\varphi^{-1}x,0)} = \begin{bmatrix} (D_1 f_1)^{-1}_{(\varphi^{-1}x,0)} & * \\ 0 & N^{-1} \end{bmatrix}$$

where $* = -(D_1 f_1)^{-1}_{(\varphi^{-1}x,0)}(D_2 f_1)_{(\varphi^{-1}x,0)}[I + g^{(1)}_{\varphi^{-1}x}]N^{-1}$ is an affine function of $g^{(1)}_{\varphi^{-1}x}$. Note that the invertibility of (DF) does not depend upon $I + g^{(1)}_{\varphi^{-1}x}$ being non-singular.

Combining the three parts and using $f_2(x,0) \equiv 0$, so that $(D_1 f_2)_{(\varphi^{-1}x,0)} = 0$, we have:

$$\begin{aligned} D_2(f_2 \circ G \circ F^{-1})_{(x,0)} &= (Df_2)_{(\varphi^{-1}x,0)}(DG)_{(\varphi^{-1}x,0)}(DF)^{-1}_{(\varphi^{-1}x,0)}|_{\mathbb{E}_2} \\ &= \begin{bmatrix} 0 & (D_2 f_2)_{(\varphi^{-1}x,0)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I + g^{(1)}_{\varphi^{-1}x} \end{bmatrix} \begin{bmatrix} * \\ N^{-1} \end{bmatrix} \\ &= (D_2 f_2)_{(\varphi^{-1}x,0)}[I + g^{(1)}_{\varphi^{-1}x}]N^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{g}_x^{(1)} &= (D_2 f_2)_{(\varphi^{-1}x,0)}[I + g^{(1)}_{\varphi^{-1}x}]N^{-1} - I \\ &= (D_2 f_2)_{(\varphi^{-1}x,0)} g^{(1)}_{\varphi^{-1}x} N^{-1} + [(D_2 f_2)_{(\varphi^{-1}x,0)} N^{-1} - I], \end{aligned}$$

which proves the lemma for $i = 1$.

For $i > 1$ we want to show that $\tilde{g}_x^{(i)}$ as a whole is a polynomial in the specified functions together with $g^{(i)}_{\varphi^{-1}x}$, and that the only dependence on $g^{(i)}_{\varphi^{-1}x}$ is the term shown in the

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lemma. From the higher-order chain rule we can see immediately that $\tilde{g}_x^{(i)}$ is a polynomial in $(D^t f_2)_{(\varphi^{-1}x,0)}$, $(D^t G)_{(\varphi^{-1}x,0)}$ and $(D^t F^{-1})_{(x,0)}$ for $1 \le t \le i$.

For $j > 1$, $(D^j G)_{(\varphi^{-1}x,0)} = [0 \quad (D^j g)_{(\varphi^{-1}x,0)}]^\top$, and $(D^j g)_{(\varphi^{-1}x,0)}$ is the 'matrix' with entries $(D^t g^{(s)})_{\varphi^{-1}x}$ for $t + s = j$. Therefore, the assertion that $\tilde{g}_x^{(i)}$ is a polynomial in the specified functions is true provided this holds for each $D^t(F^{-1})_{(x,0)}$ with $t \le i$. For $t > 1$, we can expand $D^t(F \circ F^{-1})_{(x,0)} \equiv 0$ by the chain rule and solve for $D^t(F^{-1})_{(x,0)}$:

$$D^t(F^{-1})_{(x,0)}(v_1, \ldots, v_t) = \sum_{l=2}^{t} \sum_{\chi \in \mathcal{B}_{l,t}} -(DF)^{-1}_{(\varphi^{-1}x,0)}(D^l F)_{(\varphi^{-1}x,0)}(D^\chi F^{-1})_{(x,0)}(v_\chi).$$

Since $l \ge 2$, any $\chi \in \mathcal{B}_{l,t}$ is a partition of $\{1, \ldots, t\}$ into more than one non-empty subset, which implies all have order strictly less than t. Hence, the right-hand side involves only those $D^s(F^{-1})_{(x,0)}$ with $s < t$ and inductively defines $D^t(F^{-1})_{(x,0)}$. Since $(DF)^{-1}_{(\varphi^{-1}x,0)}$ is a polynomial in $(Df)_{(\varphi^{-1}x,0)}$, $(Df)^{-1}_{(\varphi^{-1}x,0)}$, and $g^{(1)}_{\varphi^{-1}x}$, if each $(D^l F)_{(\varphi^{-1}x,0)}$ $(l \le t)$ is a polynomial in the claimed functions, the same holds for $D^t(F^{-1})_{(x,0)}$:

$$(D^l F)_{(\varphi^{-1}x,0)} = \begin{bmatrix} D^l(f_1 \circ G)_{(\varphi^{-1}x,0)} \\ D^l(f_2 \circ G \circ (gr\, h))_0 \end{bmatrix}.$$

Expanding as above, we get that $(D^l F)_{(\varphi^{-1}x,0)}$ is a polynomial in the derivatives of f, $(D^j G)_{(\varphi^{-1}x,0)}$, $(D^j G)_0$ and $(D^j h)_0$, all up to order $l \le i$. Finally, h solves $(f_1 \circ G)(hy, y) \equiv 0$, so by the implicit function theorem, $(Dh)_y = [D_1(f_1 \circ G)_{(hy,y)}]^{-1}[D_2(f_1 \circ G)_{(hy,y)}]$, and $(Dh)_0 = A^{-1}(D_2 f_1)_0$. Applying the same trick as for $D^m(F^{-1})_{(x,0)}$, we can conclude $(D^j h)_0$ is a polynomial in A^{-1}, $(D^s f)_0$ and $(D^s G)_0$, $s = 1, \ldots, j$. Therefore, $\tilde{g}_x^{(i)}$ is indeed a polynomial in only the claimed function.

To find the dependence of $\tilde{g}_x^{(i)}$ on $g^{(i)}$, we observe that $g^{(i)}$ can only occur in terms involving $(D^i G)$. Consequently,

$$\begin{aligned}
&D^i(f_2 \circ G \circ F^{-1})_{(x,0)} \\
&\quad = (Df_2)_{(\varphi^{-1}x,0)}(D^i G)_{(\varphi^{-1}x,0)}[\{(DF)^{-1}_{(\varphi^{-1}x,0)}\}^i] \\
&\qquad + (Df_2)_{(\varphi^{-1}x,0)}(DG)_{(\varphi^{-1}x,0)} D^i(F^{-1})_{(x,0)} \\
&\qquad + \text{ terms not involving } g^{(i)} \\
&\quad = (Df_2)_{(\varphi^{-1}x,0)}(D^i G)_{(\varphi^{-1}x,0)}[\{(DF)^{-1}_{(\varphi^{-1}x,0)}\}^i] \\
&\qquad - (Df_2)_{(\varphi^{-1}x,0)}(DG)_{(\varphi^{-1}x,0)}(DF)^{-1}_{(\varphi^{-1}x,0)}(D^i F)_{(\varphi^{-1}x,0)}[\{(DF)^{-1}_{(\varphi^{-1}x,0)}\}^i] \\
&\qquad + \text{ terms not involving } g^{(i)}.
\end{aligned}$$

Further breaking down the first term we have:

$$\begin{aligned}
&(Df_2)_{(\varphi^{-1}x,0)}(D^i G)_{(\varphi^{-1}x,0)}[\{(DF)^{-1}_{(\varphi^{-1}x,0)}\}^i] \\
&\qquad = \begin{bmatrix} 0 & (D_2 f_2)_{(\varphi^{-1}x,0)} \end{bmatrix} \begin{bmatrix} 0 \\ (D^i g)_{(\varphi^{-1}x,0)} \end{bmatrix} [\{(DF)^{-1}_{(\varphi^{-1}x,0)}\}^i] \\
&\qquad = (D_2 f_2)_{(\varphi^{-1}x,0)}(D^i g)_{(\varphi^{-1}x,0)}[\{(DF)^{-1}_{(\varphi^{-1}x,0)}\}^i],
\end{aligned}$$

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$(D^i g)_{(\varphi^{-1}x,0)}$ has entries $(D^t g^{(s)})_{\varphi^{-1}x}$, $t+s=i$ and $(D^i g)_{(\varphi^{-1}x,0)}[\{(u,v)\}^i] = g^{(i)}_{\varphi^{-1}x}[\{v\}^i]$ plus terms not involving $g^{(i)}_{\varphi^{-1}x}$. Therefore,

$$(Df_2)_{(\varphi^{-1}x,0)}(D^i G)_{(\varphi^{-1}x,0)}[\{(DF)^{-1}_{(\varphi^{-1}x,0)}\}^i] = (D_2 f_2)_{(\varphi^{-1}x,0)} g^{(i)}_{\varphi^{-1}x}\left\{\begin{bmatrix}0 & N^{-1}\end{bmatrix}\right\}^i + \text{terms not involving } g^{(i)}.$$

For the second term,

$$\begin{aligned}
&(Df_2)_{(\varphi^{-1}x,0)}(DG)_{(\varphi^{-1}x,0)}(DF)^{-1}_{(\varphi^{-1}x,0)}(D^i F)_{(\varphi^{-1}x,0)} \\
&\quad= \begin{bmatrix}0 & (D_2 f_2)_{(\varphi^{-1}x,0)}\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & I+g^{(1)}_{\varphi^{-1}x}\end{bmatrix}\begin{bmatrix}(D_1 f_1)^{-1}_{(\varphi^{-1}x,0)} & *\\ 0 & N^{-1}\end{bmatrix}(D^i F)_{(\varphi^{-1}x,0)} \\
&\quad= \begin{bmatrix}0 & (D_2 f_2)_{(\varphi^{-1}x,0)}(I+g^{(1)}_{\varphi^{-1}x})N^{-1}\end{bmatrix}\begin{pmatrix} D^i(f_1\circ G)_{(\varphi^{-1}x,0)} \\ D^i(f_2\circ G\circ(gr\,h))_0\end{pmatrix} \\
&\quad= (D_2 f_2)_{(\varphi^{-1}x,0)}(I+g^{(1)}_{\varphi^{-1}x})N^{-1}D^i(f_2\circ G\circ(gr\,h))_0.
\end{aligned}$$

However, $D^i(f_2\circ G\circ(gr\,h))_0$ is a function of only $(D^t g^{(s)})_0$ with $t+s\le i$. In particular, the only occurrence of $g^{(i)}$ is as $g^{(i)}_0 = 0$. Therefore, the second term does not depend upon $g^{(i)}$. Consequently, for $i>1$,

$$\begin{aligned}
\tilde{g}^{(i)}_x &= D^i(f_2\circ G\circ F^{-1})_{(x,0)}|_{\mathbb{E}^i_2} \\
&= (D_2 f_2)_{(\varphi^{-1}x,0)} g^{(i)}_{\varphi^{-1}x}\left\{\begin{bmatrix}0 & N^{-1}\end{bmatrix}\right\}^i|_{\mathbb{E}^i_2} + \text{ terms not involving } g^{(i)} \\
&= (D_2 f_2)_{(\varphi^{-1}x,0)} g^{(i)}_{\varphi^{-1}x}[\{N^{-1}\}^i] + \text{ terms not involving } g^{(i)}
\end{aligned}$$

which proves the lemma for $i>1$. □

We can now proceed with the proof of Theorems 3.1 and 4.4. We want to show there is a unique fixed point of Γ_k. We start by making some initial preparations.

As a shorthand let $\upsilon=\theta$ when $i=k$, and $\upsilon=1$ otherwise. In either case,

$$\|N\|\cdot\|N^{-1}\|^i\cdot\|A^{-1}\|^\upsilon<1.$$

Choose $\varepsilon>0$ and $\kappa<1$ sufficiently small that for all $i=1,\dots,k$ we have

$$(\|N\|+\varepsilon)(\|N^{-1}\|)^i(\|A^{-1}\|+\varepsilon)^\upsilon<\kappa<1.$$

Let $a:=\|A^{-1}\|+\varepsilon$ and $b:=\|N\|+\varepsilon$, so $b\|N^{-1}\|^i a^\upsilon<\kappa<1$.

Write $(Df)_0=\left[\begin{smallmatrix}A & B\\ 0 & N\end{smallmatrix}\right]$. For any $\tau>0$, the linear change of variable $(x,y)\mapsto(\tau x,y)$ conjugates $(Df)_0$ and converts B to $(1/\tau)B$ while leaving A and N unchanged. Therefore, a preliminary change of variable with τ sufficiently large will force $\|B\|<\delta\le\varepsilon$ for any $\delta>0$.

Define $T=A\oplus N$ and write $f=T+\rho$. Then $\rho(0)=0$ and $\|(D\rho)_0\|=\|B\|<\delta$. Therefore, there exists a $d>0$ such that $\|\rho|_{\mathbb{E}(d)}\|_{C^1}<\delta$. Since $\mathbb{E}_1(1)$ is overflowing and expanding, there will exist n such that $\varphi^n[\mathbb{E}_1(d)]\supset\mathbb{E}_1(1)$. If $\mathbf{g}=(g^{(1)},\dots,g^{(k)})$ is defined on a set V, than $\Gamma_k(\mathbf{g})$ is defined on $\varphi(V)$. Therefore, it will suffice to prove the

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theorem on $\mathbb{E}(d)$, as any solution there can be propagated out to all of $\mathbb{E}_1(1)$ by applying Γ_k^n. By a constant rescaling we can therefore assume $\|\rho|_{\mathbb{E}(1)}\|_{C^1} < \delta$.

Since f is C^r, by restricting to a smaller neighbourhood of zero if necessary (and then again rescaling), we can assume $\|f|_{\mathbb{E}(1)}\|_{C^r}, \|(Df)^{-1}|_{\mathbb{E}(1)}\|_{C^0} < \infty$. (This is to side step the non-compactness of the unit ball in a Banach space.)

For any two points $x_1, x_2 \in \mathbb{E}_1(1)$:

$$\begin{aligned} f(x_1,0) - f(x_2,0) &= f_1(x_1,0) - f_1(x_2,0) \\ &= A[x_1 - x_2] + [\rho_1(x_1,0) - \rho_1(x_2,0)] \\ \|f(x_1,0) - f(x_2,0)\| &\geq m(A)\|x_1 - x_2\| - Lip(\rho)\|x_1 - x_2\| \\ &\geq [m(A) - \delta]\|x_1 - x_2\|. \end{aligned}$$

Therefore, f expands point in $\mathbb{E}_1(1)$ by at least $[m(A) - \delta]$, which implies φ^{-1} contracts $\mathbb{E}_1(1)$ by at least $[m(A) - \delta]^{-1}$. Choose δ small so that $Lip(\varphi^{-1}) \leq \|A^{-1}\| + \varepsilon = a < 1$.

We will prove the existence of the fixed point by induction. Therefore, fix $i \in \{1, \ldots, k\}$, and let $\gamma_i : \mathcal{G}_i \to \mathcal{C}_i$ be the last component of Γ_i, $\gamma_i(g^{(1)}, \ldots, g^{(i)}) := \tilde{g}^{(i)}$. By Lemma 4.8, γ_i is a polynomial in $(D^t g^{(s)})_{\varphi^{-1}x}$ for $s + t \leq i$. Right composition by a fixed function is a linear operation. Therefore, $g^{(s)} \mapsto (D^t g^{(s)}) \circ \varphi^{-1}$ is continuous with respect to both the integer and fractional class norms. Products and sums are continuous operations, so γ_i is continuous with respect to either norm. Therefore, Γ_i is a continuous function from $\mathcal{G}_i$ into itself with respect to either $\|\cdot\|_{C^{[r]}}$ or $\|\cdot\|_{C^r}$. Let $\mathbf{g}$ denote an element in $\mathcal{G}_{i-1}$, then since $\tilde{g}^{(j)}$ does not depend on $g^{(i)}$ for any $i > j$, we have

$$\Gamma_i(\mathbf{g}, g^{(i)}) = [\Gamma_{i-1}(\mathbf{g}), \gamma_i(\mathbf{g}, g^{(i)})] : \mathcal{G}_{i-1} \times \mathcal{C}_i \to \mathcal{G}_{i-1} \times \mathcal{C}_i.$$

Consequently, Γ_i has the correct form for Lemma 4.5. Assume we have already shown there exists a globally attracting fixed point of Γ_{i-1}. (For $i = 1$, this assumption is vacuous.) We want to show there is a unique globally attracting fixed point of Γ_i. This would follow from Lemma 4.5 if $\gamma_i(\mathbf{g}, \cdot)$ were a contraction for each $\mathbf{g} \in \mathcal{G}_{i-1}$. Unfortunately, it is not under the existing norm. Therefore, we will modify the norm on $\mathcal{C}_i$.

Let $l := [r] - i$, and $\omega := (r)$, so that elements of $\mathcal{C}_i$ are $C^{l+\omega}$. For $g^{(i)} \in \mathcal{C}_i$, define $|||g^{(i)}||| := \sup_{0<\|x\|\leq 1}[\|g_x^{(i)}\|/\|x\|^\upsilon]$. Clearly $|||g^{(i)}||| \geq \|g^{(i)}\|_{C^0}$. Since $\upsilon \leq r - i$, $g^{(i)}$ is at least C^υ, and since $g^{(i)}(0) = 0$, $|||g^{(i)}||| \leq H_\upsilon(g^{(i)}) < \infty$. Now let $\|g^{(i)}\|_* := \max\{|||g^{(i)}|||, \|(Dg^{(i)})\|_{C^0}, \ldots, \|(D^l g^{(i)})\|_{C^0}, H_\omega[(D^l g^{(i)})]\}$. (This is assuming $\omega \neq 0$. If $\omega = 0$, we drop the last term.) It is easy to check that both $|||\cdot|||$ and $\|\cdot\|_*$ are norms on $\mathcal{C}_i$. $\|g^{(i)}\|_*$ differs from the $C^{l+\omega}$-norm only in the first term. Therefore, $\|g^{(i)}\|_* \geq \|g^{(i)}\|_{C^{l+\omega}}$. On the other hand, if $l \geq 1$, then $|||g^{(i)}||| \leq Lip(g^{(i)}) = \|(Dg^{(i)})\|_{C^0} \leq \|g^{(i)}\|_{C^1}$ (since $\mathbb{E}_1(1)$ is convex). If $l = 0$, then $\upsilon \leq \omega$, so one either has $\omega \neq 0$ and $|||g^{(i)}||| \leq H_\omega(g^{(i)})$, or $\upsilon = \omega = 0$ and $|||g^{(i)}||| = \|g^{(i)}\|_{C^0}$. In all cases, $\|g^{(i)}\|_* \leq \|g^{(i)}\|_{C^{l+\omega}}$. Therefore, $\|\cdot\|_* = \|\cdot\|_{C^{l+\omega}}$, and this is just a different way of expressing the norm.

Define $\chi : \mathcal{C}_i \to \mathcal{C}_i$ by $\chi(g^{(i)})(x) = (D_2 f_2)_{(\varphi^{-1}x,0)} g^{(i)}_{\varphi^{-1}x}[\{N^{-1}\}^i]$. Note that χ is linear, and $\gamma_i(\mathbf{g}, g^{(i)}) = \chi(g^{(i)}) + \Re_i(\mathbf{g})$. Therefore, γ_i will be a uniform fibre contraction (i.e. satisfy the hypotheses of Lemma 4.5) if and only if the norm of $\chi \in L(\mathcal{C}_i, \mathcal{C}_i)$ is less than one.

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LEMMA 4.9. *There exists a norm $\|\cdot\|'$ on $\mathcal{C}_l$, equivalent to $\|\cdot\|_*$, for which $\|\chi\|' < 1$.*

Proof. If $\|g^{(i)}\|' = \max\{|||g^{(i)}|||, c_1\|(Dg^{(i)})\|_{C^0}, \dots, c_l\|(D^l g^{(i)})\|_{C^0}, c_\omega H_\omega[(D^l g^{(i)})]\}$, then $\|\cdot\|'$ will be a norm equivalent to $\|\cdot\|_*$ for any $c_1, \dots, c_l, c_\omega > 0$. We want to show that for some choice of the c_j this will make $\|\chi\|' < 1$. For $j = 0$,

$$\frac{\|(D_2 f_2)_{(\varphi^{-1}x,0)} g^{(i)}_{\varphi^{-1}x}[\{N^{-1}\}^i]\|}{\|x\|^\upsilon} \le \|(D_2 f_2)_{(\varphi^{-1}x,0)}\| \left(\frac{\|g^{(i)}_{\varphi^{-1}x}\|}{\|\varphi^{-1}(x)\|^\upsilon}\right)\left(\frac{\|\varphi^{-1}(x)\|}{\|x\|}\right)^\upsilon \|N^{-1}\|^i.$$

Since $\varphi^{-1}(\mathbb{E}_1(1)) \subset \mathbb{E}_1(1)$, $\sup_x\{\|g^{(i)}_{\varphi^{-1}x}\|/\|\varphi^{-1}x\|^\upsilon\} \le |||g^{(i)}|||$. $\|(D_2 f_2)_{(\varphi^{-1}x,0)}\| \le \|N\| + \varepsilon = b$, and since $\varphi^{-1}(0) = 0$, $\|\varphi^{-1}x\|/\|x\| \le Lip(\varphi^{-1}) \le a$. Therefore, $|||(D_2 f_2)_{(\varphi^{-1}x,0)} g^{(i)}_{\varphi^{-1}x}[\{N^{-1}\}^i]||| \le (b\|N^{-1}\|^i a^\upsilon)|||g^{(i)}||| \le \kappa|||g^{(i)}||| \le \kappa\|g^{(i)}\|'$ and $\kappa < 1$.

Now suppose for $j \ge 1$ we have $1 \ge c_1 \ge \cdots \ge c_{j-1} > 0$ such that for all $t = 1, \dots, j-1$, $c_t\|D^t\chi(g^{(i)})\|_{C^0} \le \kappa \max\{|||g^{(i)}|||, c_1\|(Dg^{(i)})\|_{C^0}, \dots, c_t\|(D^t g^{(i)})\|_{C^0}\} \le \kappa\|g^{(i)}\|'$. Expand $D^j\chi(g^{(i)})$:

$$D^j\chi(g^{(i)}) = (D_2 f_2)_{(\varphi^{-1}x,0)}(D^j g^{(i)})_{\varphi^{-1}x}[\{(D\varphi)^{-1}_{\varphi^{-1}x}\}^j, \{N^{-1}\}^i] + \mathcal{R}_j(x),$$

where $\mathcal{R}_j(x)$ is linear in $(g^{(i)}_{\varphi^{-1}x}, \dots, (D^{j-1}g^{(i)})_{\varphi^{-1}x})$. Therefore, we can choose $K > 0$ such that $\|\mathcal{R}_j\|_{C^0} \le K\max\{\|g^{(i)}\|_{C^0}, \dots, \|(D^{j-1}g^{(i)})\|_{C^0}\} = K\|g^{(i)}\|_{C^{j-1}}$. For the first term,

$$\|(D_2 f_2)_{(\varphi^{-1}x,0)}(D^j g^{(i)})_{\varphi^{-1}x}[\{(D\varphi)^{-1}_{\varphi^{-1}x}\}^j, \{N^{-1}\}^i]\|_{C^0} \le [b\|N^{-1}\|^i\alpha^j]\|(D^j g^{(i)})\|_{C^0}.$$

Since $a < 1$ and $j \ge \upsilon$, $b\|N^{-1}\|^i\alpha^j = \kappa\alpha^{j-\upsilon} < \kappa$. Choose c_j sufficiently small, $c_{j-1} \ge c_j > 0$, that $\kappa\alpha^{j-\upsilon} + K(c_{j-1}^{-1}c_j) \le \kappa$. We then have

$$\begin{aligned}
&c_j\|D^j\chi(g^{(i)})\|_{C^0} \\
&\le \kappa\alpha^{j-\upsilon}(c_j\|(D^j g^{(i)})\|_{C^0}) + K(c_{j-1}^{-1}c_j)(c_{j-1}\|g^{(i)}\|_{C^{j-1}}) \\
&\le (\kappa\alpha^{j-\upsilon} + K(c_{j-1}^{-1}c_j))\max\{c_{j-1}\|g^{(i)}\|_{C^0}, \dots, c_{j-1}\|(D^{j-1}g^{(i)})\|_{C^0}, c_j\|(D^j g^{(i)})\|_{C^0}\} \\
&\le \kappa\max\{|||g^{(i)}|||, c_1\|(Dg^{(i)})\|_{C^0}, \dots, c_j\|(D^j g^{(i)})\|_{C^0}\}.
\end{aligned}$$

We continue in this way to choose $c_1, \dots, c_l$. Finally, for c_ω, using again that $\mathcal{R}_l(x)$ is linear in $(g^{(i)}_{\varphi^{-1}x}, (Dg^{(i)})_{\varphi^{-1}x}, \dots, (D^{l-1}g^{(i)})_{\varphi^{-1}x})$, we can find K_1 such that $\|\mathcal{R}_l\|_{C^\omega} \le K_1\max\{\|g^{(i)}_{\varphi^{-1}x}\|_{C^\omega}, \dots, \|(D^l g^{(i)})_{\varphi^{-1}x}\|_{C^\omega}\}$. Since each of these is at least C^l, $H_\omega[(D^j g^{(i)})_{\varphi^{-1}x}] \le Lip[(D^j g^{(i)})_{\varphi^{-1}x}] \le Lip(\varphi^{-1})\,Lip[(D^j g^{(i)})] \le \|(D^{j+1}g^{(i)})_{\varphi^{-1}x}\|_{C^0}$, and $H_\omega[\mathcal{R}_l] \le \|\mathcal{R}_l\|_{C^\omega} \le K_1\|g^{(i)}\|_{C^l}$. Finally, using the

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multilinearity of the first term:

$$\begin{aligned}
&H_\omega[(D_2 f_2)_{(\varphi^{-1}x,0)}(D^l g^{(i)})_{\varphi^{-1}x}[\{(D\varphi)^{-1}_{\varphi^{-1}x}\}^l, \{N^{-1}\}^i]] \\
&\quad \le [\|(D_2 f_2)\|_{C^0}(\|(D\varphi)^{-1}\|_{C^0})^l \|N^{-1}\|^i] H_\omega[(D^l g^{(i)}) \circ \varphi^{-1}] \\
&\qquad + H_\omega[(D_2 f_2)_{(\varphi^{-1}x,0)}]\|(D^l g^{(i)})\|_{C^0}(\|(D\varphi)^{-1}\|_{C^0})^n \|N^{-1}\|^i \\
&\qquad + l\|(D_2 f_2)\|_{C^0}\|(D^l g^{(i)})\|_{C^0}(\|(D\varphi)^{-1}\|_{C^0})^{l-1}\|N^{-1}\|^i H_\omega[(D\varphi)^{-1}_{\varphi^{-1}x}] \\
&\quad \le (b\|N^{-1}\|^i a^{l+\omega}) H_\omega[(D^l g^{(i)})] + K_2\|(D^l g^{(i)})\|_{C^0}
\end{aligned}$$

for some $K_2 > 0$. Consequently, $H_\omega[D^l \chi(g^{(i)})] \le \kappa a^{l+\omega-\upsilon} H_\omega[(D^l g^{(i)})] + (K_1 + K_2)\|g^{(i)}\|_{C^l}$. Take c_ω with $c_l \ge c_\omega > 0$ and $\kappa a^{l+\omega-\upsilon} + (K_1 + K_2)c_l^{-1} c_\omega \le \kappa$, then $H_\omega[D^l \chi(g^{(i)})] \le \kappa \|g^{(i)}\|'$. Therefore, $\|\chi(g^{(i)})\|' \le \kappa \|g^{(i)}\|'$ and χ is a contraction under this norm.

In the above argument we have assumed $\omega \neq 0$. However, if $\omega = 0$ and the $H_\omega[\cdot]$ term is removed from both $\|\cdot\|'$ and $\|\cdot\|_*$, the argument works equally well. □

We can now finish the proof of the theorem. The spaces $\mathcal{G}_i$ and $\mathcal{C}_i (i = 1, \ldots, k)$ are Banach spaces under the respective C^r-norms. For $i = 1$, $\Gamma_1 : \mathcal{G}_1 \to \mathcal{G}_1$ ($\mathcal{G}_1 = \mathcal{C}_1$) is an affine contraction under $\|\cdot\|'$. Since this is equivalent to the original norm on $\mathcal{C}_1$, there exists a globally attracting fixed point $\hat{g}^{(1)} \in \mathcal{C}_1$. Now assume there exists a globally attracting fixed point $\hat{\mathbf{g}} = (\hat{g}^{(1)}, \ldots, \hat{g}^{(i-1)}) \in \mathcal{G}_{i-1}$ for Γ_{i-1} under $\|\cdot\|_{C^r}$. By the previous lemma, Γ_i is a fibre contraction if $\mathcal{C}_i$ has norm $\|\cdot\|'$. Therefore, by Lemma 4.5 we get a fixed point $(\hat{\mathbf{g}}, \hat{g}^{(i)}) \in \mathcal{G}_i$ that is globally attracting for Γ_i with respect to the norm $\|(\hat{\mathbf{g}}, \hat{g}^{(i)})\| = \max\{\|\hat{\mathbf{g}}\|_{C^r}, \|\hat{g}^{(i)}\|'\}$. As this is again equivalent to $\|\cdot\|_{C^r}$ on $\mathcal{G}_i$, the fixed point is globally attracting in the original norm. By induction, we get a globally attracting fixed point of Γ_k in $\mathcal{G}_k$. Since it is globally attracting, it is unique.

Let $\hat{\mathbf{g}} \in \mathcal{G}_k$ denote this final fixed point. We must still show $I + \hat{g}^{(1)}_x$ is non-singular for all x, so that $\hat{\mathbf{g}}$ corresponds to a local (k, r)-jet of a foliation. If $I + g^{(1)}_{\varphi^{-1}x}$ is non-singular, then $I + \Gamma_1(g^{(1)})(x) = (D_2 f_2)_{(\varphi^{-1}x,0)}[I + g^{(1)}_{\varphi^{-1}x}]N^{-1}$ is non-singular since $(D_2 f_2)_{(\varphi^{-1}x,0)}$ is invertible owing to the fact that $(Df)_{(\varphi^{-1}x,0)}$ is invertible and upper triangular. Therefore, if $I + g^{(1)}_x$ is non-singular for all $x \in V$, then $I + \Gamma_1(g^{(1)})(x)$ is non-singular for all $x \in \varphi(V)$. Since $\hat{g}^{(1)}$ is continuous and $\hat{g}^{(1)}(0) = 0$, there exists $d > 0$ such that $\|\hat{g}^{(1)}_x\| < 1$ on $\mathbb{E}_1(d)$, which implies $I + \hat{g}^{(1)}_x$ is non-singular on $\mathbb{E}_1(d)$. Therefore, since $\Gamma_1(\hat{g}^{(1)}) = \hat{g}^{(1)}$ and $\varphi^n[\mathbb{E}_1(d)] \supset \mathbb{E}_1(1)$ for some $n > 0$, $I + \hat{g}^{(1)}_x$ must be non-singular for all x. Consequently, $\hat{\mathbf{g}}$ represents a unique f overflowing invariant local (k, r)-jet of a foliation along $\mathbb{E}_1(1)$, which completes the existence and uniqueness part of the proof. We are left with proving continuity, and for this we are concerned only with finite dimensions.

Let $\bar{f} \in \mathrm{Diff}^r(\mathbb{E})$ satisfy the hypotheses of the theorem. We showed above that we could make a preliminary change of variable and norm so that if $A := (D_1 \bar{f}_1)_0$, $N := (D_2 \bar{f}_2)_0$, and $\rho := \bar{f} - A \oplus N$, then $\bar{f}(\mathbb{E}_1(1)) \supset \mathbb{E}_1(1)$; for some $d > 0$, $\|\rho|_{\mathbb{E}_1(d)}\|_{C^1} < \varepsilon$; for some n sufficiently large, $\mathrm{int}[\bar{f}^n(\mathbb{E}_1(d))] \supset \mathbb{E}_1(1)$; if $\bar{\varphi} := \bar{f}|_{\mathbb{E}_1(d)}$, then $Lip[\bar{\varphi}^{-1}] < \|A^{-1}\| + \varepsilon$; and ε was sufficiently small that $(\|N\| + \varepsilon)(\|N^{-1}\|)^i(\|A^{-1}\| + \varepsilon)^\upsilon < \kappa < 1$. Therefore, there exists a C^1 neighbourhood $\mathcal{N}$ of $\bar{f} \in \mathrm{Diff}^r(\mathbb{E})$ such that if $f \in \mathcal{N}$ and

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$f(gr\,\lambda) \supset gr\,\lambda$, then under the same change of variable and norm, f will satisfy these same conditions.

Consequently, for any such f there will exist a unique f-invariant local (k, r)-jet of a foliation along $gr(\lambda)$, ξ_f. We want to show the function $f \mapsto \xi_f$ is continuous, and this will follow from Corollary 4.6 provided the functions Γ_i are continuous in f. Unfortunately, this is not the case for the fractional differentiability classes. However, since $r > k + Lip$, we have $[r] \geq k + \theta$. Therefore, if we replace r by its integer part, the hypotheses of the theorem are still satisfied. Consequently, we can repeat the first part of the proof to again get that each Γ_i has a unique, globally attracting fixed point $(\hat{g}^{(1)}, \ldots, \hat{g}^{(i)}) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_i$. Now, however, $\mathcal{C}_i$ is a space of $C^{[r]-i}$ instead of C^{r-i} functions, and the convergence is in the $C^{[r]-i}$-topology. On the other hand, the new $\mathcal{C}_i$ contains the old $\mathcal{C}_i$ as a subset, and since the fixed point is unique, this is the same fixed point we had before. Moreover, the topology on $\mathcal{G}_k$ as a subset of the local jets of foliations is the $C^{[r]-i}$-topology on each $\mathcal{C}_i$. Therefore, to show the fixed point depends continuously on f, it will suffice to show Γ_i is continuous in f with respect to the integer differentiability classes. From here on we will take r to be an integer.

From Lemma 4.1, the extension operator $\mathbf{g} \mapsto g$ is a bounded linear operator from $\mathcal{G}_k$ into $C^r(\mathbb{R}^n, \mathbb{R}^{n-d})$. Therefore, $\mathbf{g} \mapsto G = \mathrm{id} + (0, g) \in C^r(\mathbb{R}^n, \mathbb{R}^n)$ is continuous. If $V \subset \mathbb{R}^n$ is any neighbourhood of B_1^d, the differential operator $D_2^i : \tilde{g} \mapsto (D_2^i \tilde{g})|_{B_1^d}$ from $C^r(V, \mathbb{R}^{n-d})$ to $C^{r-i}(B_1^d, L^i_{\mathrm{sym}}(\mathbb{R}^{n-d}, \mathbb{R}^{n-d}))$ is a bounded linear operator, and so $\tilde{g} \mapsto \tilde{\mathbf{g}} \in \mathcal{G}_k$ is continuous. Γ_k can be expressed as the composition of the three maps $(\mathbf{g}, f) \mapsto (G, f) \mapsto \pi_2 \circ f \circ G \circ F^{-1} = \tilde{g} \mapsto \tilde{\mathbf{g}}$. The first and last are continuous, so we want the middle map from $C^r(\mathbb{R}^n, \mathbb{R}^n) \times C^r(\mathbb{R}^n, \mathbb{R}^n)$ into $C^r(V, \mathbb{R}^{n-d})$ to be continuous. Composition $\circ : (G, f) \mapsto f \circ G$ is continuous at any (G, f) with f uniformly C^r, i.e. all derivatives of f are uniformly continuous. From this it follows that $(G, f) \mapsto f' := f \circ G|_U$ is continuous at all (G, f) if U is any compact set. In particular, let $U \subset \mathbb{R}^n$ be a neighbourhood of B_1^d. Finally, F^{-1} is obtained from f' by applying first the implicit and then the inverse function theorems, both of which depend continuously on the function. Consequently, $(G, f) \mapsto f' \mapsto \pi_2 \circ f' \circ F^{-1}$ is continuous, which proves each Γ_i is continuous in f. We can now apply Corollary 4.6 at each step and conclude there is a unique fixed point $\mathbf{g}_f$ of Γ_k depending continuously on f.

Finally, each of these solutions corresponds to a local jet along the smaller disk $\mathbb{E}_1(d)$. However, there is a single n for which $f^n(\mathbb{E}_1(d)) \supset \mathbb{E}_1(1)$ for all $f \in \mathcal{N}$. By the same argument $\mathbf{g}_f \mapsto G_f \mapsto f^n \circ G_f + gr(\lambda)$ depends continuously on f, and is a foliation chart for a foliation representing ξ_f, the unique f-invariant (k, r)-jet of a foliation along $gr(\lambda)$. Hence, by definition $f \mapsto \xi_f$ is continuous. □

Proof of Theorem 3.2. By choosing local coordinates about p we can assume $M = \mathbb{R}^n$ and $p = 0$. Set $T = (Df)_0$. Given $\varepsilon > 0$, there exists a bump function $\beta : [0, \infty) \to [0, 1]$, with $\beta \equiv 1$ near zero, such that $\chi(f) := \beta f + (1 - \beta)T$ satisfies $Lip(\chi(f) - T) \leq \varepsilon$. χ is a bounded linear operator, and hence continuous. Therefore, since $\chi(f) \equiv f$ near zero, it will suffice to show there exists a neighbourhood $\mathcal{N}$ of $\chi(f)$ for which the theorem holds. Consequently, without loss of generality, we can assume $Lip(f - T)$ is as small as we like.

From the stable manifold theorem [**HPS**, p. 53], there exists an $\varepsilon > 0$ such that if T

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satisfies either (1) or (2), and $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, with $Lip(f - T) \leq \varepsilon$, then f has a unique invariant manifold which is the graph of a C^r function $g : E_1 \to E_2$ with $Lip(g) \leq 1$. Moreover, this invariant manifold depends continuously on f in the $C^{[r]}$ sense among those functions fixing p, and is tangent to the generalized eigenspace of $(Df)_p$ corresponding to all eigenvalues $z \in sp[(Df)_p]$ with $|z| > \rho$ in case (1), and $|z| < \rho$ in case (2). In the strong unstable case this is the only invariant manifold tangent to this generalized eigenspace. In both cases, since $b_1 > 1$, the invariant manifold is expanding. Applying Theorem 3.1, we then get a unique local (k, r)-jet of a foliation along the given invariant manifold. Since the invariant manifold is unique in the strong unstable case, the local (k, r)-jet of a foliation is also unique.

If $\bar{f}$ satisfies the hypotheses of Theorem 3.2, and $Lip(\bar{f} - T) \leq \varepsilon/2$, there exists a C^1 neighbourhood $\mathcal{N}$ of $\bar{f}$ where $Lip(f - \bar{f}) \leq \varepsilon/2$ for $f \in \mathcal{N}$, and so $Lip(f - T) \leq \varepsilon$. Therefore, if $f \in \mathcal{N}_p$, there is a continuous function $f \mapsto g_f$ such that $gr(g_f)$ is the invariant manifold for f. Let $h_f(x, y) := (x, y - g_f(x))$ be the change of variable taking $gr(g_f)$ to E_1, and $f' := h_f \circ f \circ h_f^{-1}$. Then $f \mapsto f'$ is continuous and f' leaves E_1 invariant. Therefore, we can apply the previous theorem to conclude that for a possibly smaller $\mathcal{N}$ there exists a unique $\zeta \in \mathcal{F}_d^{(k,r)}(\mathbb{R}^n)$ along $E_1(1)$ which is f'-overflowing invariant and depends continuously on f', and so continuously on f. Define $\xi = (h_f^{-1})_*(\zeta)$. Then ξ depends continuously on f, is f-overflowing invariant, and is tangent to the appropriate subspace.

Finally, if zero is elementary, then there exists a neighbourhood $\mathcal{N}$ of $\bar{f}$ and U of zero such that if $f \in \mathcal{N}$, then f has a unique fixed point p_f in U, and $f \mapsto p_f$ is continuous. Therefore, a preliminary change of variable that translates $p_f \mapsto 0$ will produce a continuous function $\mathcal{N} \to \mathcal{N}_p$, and we can then apply the previous case. □

Proof of Theorem 3.3. Consider the compact sets $f^{-n}(gr\,\lambda_n)$, $n \geq 0$. Since $f(gr\,\lambda_n) \supset gr(\lambda_{n+1})$, these form a nested sequence with $f^{-n}(gr\,\lambda_n) \subset gr(\lambda_0)$ for all n. Since $gr(\lambda)$ is expanding and $\lambda_n \to \lambda$ in the C^0 topology, the diameters are going to zero. Hence, $\cap f^{-n}(gr\,\lambda_n)$ is a single point. Denote this point by p_0 and let $p_n = f^n(p_0)$. Since $p_n \in gr(\lambda_n)$ for all n, then $p_n \to 0$.

We assume we have made a preliminary change of variable taking $gr(\lambda)$ to B_1^d. As in the proof of Theorem 3.1, since $gr(\lambda) = B_1^d$ is expanding, it will suffice to work on a small ball about zero. By rescaling to bring this neighbourhood back to the unit ball, we can assume the functions λ_n are defined on a disk of radius greater than one. For simplicity, we will assume these are defined on all of $\mathbb{R}^n$. This will not affect the conclusions since we will only work in a neighbourhood of the origin.

Let $p_n =: (x_n, y_n)$, and define $h_n : \mathbb{R}^n \to \mathbb{R}^n$ by $h_n(x, y) = (x - x_n, y - \lambda_n(x))$. For any $(x, y) \in gr(\lambda_n)$, we have $(h_n)_2(x, y) = y - \lambda_n(x) = 0$. Therefore, h_n takes $gr(\lambda_n)$ to $\mathbb{R}^d$, with p_n being sent to the origin, and $h_n \to \mathrm{id}$ in the $C^{[r]}$ topology. Since $r > k + Lip$, we can, as in the proof of continuity in Theorem 3.1, replace r by $[r]$. Therefore, we can without loss of generality assume that r is an integer.

Define $f_n = h_{n+1} \circ f \circ h_n^{-1}$. $h_{n+1} \circ f \circ h_n^{-1}(0) = h_{n+1} \circ f(p_n) = h_{h+1}(p_{n+1}) = 0$, hence $f_n(0) = 0$. Likewise, f_n takes $\mathbb{R}^d$ to itself. Moreover, since $h_n \to \mathrm{id}$, we have $f_n \to f$. Since $D_1(\pi_1 \circ f)_0 = A$ and $m(A) \geq \alpha > 1$, and we assume we are working

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in a sufficiently small neighbourhood of zero (then rescaled), we can assume f expands points in B_1^d by $\approx\alpha$. Therefore, since $f_n \to f$, if n is sufficiently large, then $f_n(B_1^d) \supset B_1^d$. Taking n larger, f_n will satisfy all of the hypotheses of Theorem 3.1. For simplicity assume this holds for all n.

Let $\mathcal{G}_0 = \{p_n : n \geq 0\} \cup \{0\}$ with the topology induced from $\mathbb{R}^n$, and define $\Gamma_0 : \mathcal{G}_0 \to \mathcal{G}_0$ by $\Gamma_0(p_n) = p_{n+1}$ and $\Gamma_0(0) = 0$, i.e. $\Gamma_0 = f|_{\mathcal{G}_0}$. Γ_0 is clearly continuous with zero an attracting fixed point. Let the spaces $\mathcal{G}_i$ and $\mathcal{C}_i$ be as defined in the proof of Theorem 3.1, and $\Gamma_i(n)$ the function defined there, but with f_n replacing f. Define $\mathcal{G}_0' = \mathcal{G}_0$ and $\mathcal{G}_i' = \mathcal{G}_0 \times \mathcal{G}_i$. Define $\Gamma_i' : \mathcal{G}_i' \to \mathcal{G}_i'$ by $\Gamma_i'(p_n, \mathbf{g}) = (\Gamma_0(p_n), \Gamma_i(n)(\mathbf{g}))$ and $\Gamma_i'(0, \mathbf{g}) = (\Gamma_0(0), \Gamma_i(\mathbf{g}))$.

Since $\Gamma_0 : \mathcal{G}_0 \to \mathcal{G}_0$ has a globally attracting fixed point, we can use this as the starting space for working back up through the applications of Lemma 4.5. Since all the f_n satisfy the same estimates as f, for each fixed n, $\Gamma_i(n)$ is a fibre contraction. Hence, Γ_i' is a fibre contraction. Furthermore, we observed in the proof of Theorem 3.1 that each Γ_i depends continuously on f. Therefore, since $f_n \to f$, Γ_i' is continuous. Consequently, there exists a globally attracting fixed point of Γ_k', and this is over the fixed point for Γ_0. Hence, the fixed point is $(0, \hat{\mathbf{g}})$, where $\hat{\mathbf{g}} = (\hat{g}^{(1)}, \ldots, \hat{g}^{(k)})$ was the fixed point of Γ_k.

Now suppose ξ_0 is a local (k, r)-jet of a foliation along $gr(\lambda_0)$, then $h_{0*}(\xi_0)$ is a local jet along a disk containing B_1^d. Let ξ_0' be the restriction to B_1^d, and let ξ_0' be represented by $\mathbf{g}_0 = (g_0^{(1)}, \ldots, g_0^{(k)})$. We then have $(p_0, \mathbf{g}_0) \to (0, \hat{\mathbf{g}})$ under iteration by Γ_k'. In other words, $\Gamma_k(n-1) \circ \cdots \circ \Gamma_k(0)[\mathbf{g}_0] =: \mathbf{g}_n \to \hat{\mathbf{g}}$. Let G_n be the extension of $\mathbf{g}_n$ given by Lemma 4.1, and G the extension of $\hat{\mathbf{g}}$. $\Gamma_k(n)$ represents the action of $(f_n)_*$, which means that G_n is a standard presentation for a foliation in the same equivalence class as the foliation with foliation chart $f_{n-1} \circ G_{n-1}$. Continuing we get that G_n and $f_{n-1} \circ \cdots \circ f_0 \circ G_0 = h_n \circ f^n \circ h_0^{-1} \circ G_0$ define the same equivalence class. Therefore, if we set $H_n := h_n^{-1} \circ G_n$, then H_n and $f^n \circ H_0$ determine the same local (k, r)-jet of a foliation. On the other hand, G_0 is a foliation chart for a foliation in the equivalence class ξ_0', which means that H_0 determines the equivalence class $(h_0^{-1})_*(\xi_0') = \xi_0$. Therefore, ξ_n is determined by H_n. Now since $\mathbf{g}_n \to \hat{\mathbf{g}}$, by Lemma 4.1, $G_n \to G$. We also have $h_n \to \mathrm{id}$. Therefore, $H_n = h_n^{-1} \circ G_n \to G$, but by definition this means that $\xi_n \to \xi$. □

5. *Pseudo-hyperbolic sets*

This section is devoted to the proof of Theorem 3.5. The proof starts where the proof of the stable manifold theorem for pseudo-hyperbolic sets ends. Therefore, we begin by recalling the stage that is set in that proof. See [**HP, Sh**]. The idea is to reduce the case for general invariant sets to the fixed point case.

Let $f_\# : T_\Lambda M(\delta) \to T_\Lambda M$ be the result of lifting f via the exponential map to $T_\Lambda M$:

$$f_\#(v_x) := (\exp_{f(x)})^{-1} \circ f \circ \exp_x(v_x)$$

where $x \in \Lambda$ and $v_x \in T_x M$ with $\|v_x\| < \delta$. Since Λ is compact, δ can be chosen sufficiently small that $f_\#$ is well defined. If $f_{\#x} : T_x M \to T_{f(x)} M$ denotes the restriction of $f_\#$ to $T_x M$ for a given x, then $f_{\#x}$ is C^r and $(Df_{\#x})_0 = T_x f$. Given $\varepsilon > 0$, $f_\#$ can be extended to all of $T_\Lambda M$ so that $f_\# = Tf$ outside of a (compact) neighbourhood of the zero section, and $Lip(f_{\#x} - T_x f) \leq \varepsilon$ for all $x \in \Lambda$. The derivatives of $f_{\#x}$ as a

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function on T_xM, $(D^i f_{\#x})_{v_x}$, are uniformly continuous as functions of $(x, v_x) \in T_\Lambda M$, and if $r = l + \omega$, then $(D^l f_{\#x})_{v_x}$ is ω-Hölder with ω-Hölder constant independent of x.

Let $B(\Lambda, T_\Lambda M)$ be the Banach space of bounded functions from Λ into $T_\Lambda M$ with the sup norm, and $\mathbb{E} \subset B(\Lambda, T_\Lambda M)$ the subspace of bounded sections. Define $\hat{f} : \mathbb{E} \to \mathbb{E}$ by $\hat{f}(\sigma) = f_\# \circ \sigma \circ f^{-1}$. $\hat{f}$ is C^r with $(D\hat{f})_0(\sigma) = [(Tf) \circ f^{-1}](\sigma \circ f^{-1})$. That is, $(D\hat{f})_0(\sigma)$ is a bounded section defined by $(D\hat{f})_0(\sigma)(x) = (T_{f^{-1}(x)} f)(\sigma \circ f^{-1}(x))$.

Define $\mathbb{E}_i := \{\sigma \in \mathbb{E} : \text{image}(\sigma) \subset E_i\}, i = 1, 2$. We have $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$. Since E_1 and E_2 are Tf-invariant, $(D\hat{f})_0$ preserves this splitting. If $\hat{T} := (D\hat{f})_0$, and $\hat{A} := \hat{T}|_{\mathbb{E}_1}$, $\hat{N} := \hat{T}|_{\mathbb{E}_2}$, then $\hat{A}(\upsilon)(x) = A_{f^{-1}(x)}(\upsilon \circ f^{-1}(x))$ and $\hat{N}(\sigma)(x) = N_{f^{-1}(x)}(\sigma \circ f^{-1}(x))$ for $(\upsilon, \sigma) \in \mathbb{E}_1 \oplus \mathbb{E}_2$. Consequently, $m(\hat{A}) = \inf_{x\in\Lambda} m(A_x) > \alpha > \sup_{x\in\Lambda} \|N_x\| = \|\hat{N}\|$, and zero is a pseudohyperbolic fixed point for F. Moreover, $Lip(\hat{f} - \hat{T}) = \sup_{x\in\Lambda} Lip(f_{\#x} - T_x f) \le \varepsilon$, and since $f_\# - Tf$ has compact support, $\|\hat{f} - \hat{T}\|_{C^r} < \infty$.

At this point the proof of the stable manifold theorem concludes (from the fixed point case) that there exists a unique $\hat{g} : \mathbb{E}_1 \to \mathbb{E}_2$ such that $\hat{g}$ is C^r, and $gr(\hat{g})$ is $\hat{f}$ invariant and tangent to $\mathbb{E}_1$ at zero. Furthermore, it is shown that $\hat{g}$ has the form $\hat{g}(\upsilon) = g \circ \upsilon$ where $g : E_1 \to E_2$ covers id_Λ. $g_x : E_{1x} \to E_{2x}$ is C^r with derivatives that vary continuously with $(x, u_x) \in E_1$, and $H_\omega[(D^l g_x)]$ is independent of x. If $\rho > 0$ is small, then $\Delta_x = \exp_x[gr(g_x) \cap T_xM(\rho)]$ defines a $C^{[r]}$-continuous family of embedded disks, and $\Delta_x = W^u_\rho(x; \alpha)$. In preparation for our proof, we will use $\hat{g}^{(0)}$ and $g^{(0)}$ in place of $\hat{g}$ and g.

From here we can proceed to the proof of Theorem 3.5. We begin with a lemma. Let F_1 and F_2 be finite-dimensional vector bundles over Λ. Let $g : F_1 \to F_2$ be a map covering a homeomorphism $h : \Lambda \to \Lambda$, and that is bounded on bounded sets (continuous will do, but that will be part of the hypotheses of the lemma). Let $\mathbb{F}_i$ be the Banach space of bounded sections of F_i $(i = 1, 2)$, and define $\hat{g} : \mathbb{F}_1 \to \mathbb{F}_2$ by $\hat{g}(\upsilon) = g \circ \upsilon \circ h^{-1}$. Finally, let $\mathbb{F}^c_i \subset \mathbb{F}_i$ denote the subspace of continuous sections.

LEMMA 5.1. *$\hat{g}$ is $C^r (r = l + \omega)$ and takes continuous sections to continuous sections, $\hat{g}(\mathbb{F}^c_1) \subset \mathbb{F}^c_2$, if and only if g is continuous, and for each $x \in \Lambda$, $g_x : F_{1x} \to F_{2x}$ is C^r, for $i = 1, \ldots, l$, $(D^i g_x)_u$ is continuous in $(x, u) \in F_1$, and $H_\omega[(D^l g_x)]$ is independent of x. Moreover,*

$$[(D^i \hat{g})_\upsilon(\sigma_1, \ldots, \sigma_i)](x) = (D^i g_z)_{\upsilon(x)}(\sigma_1(z), \ldots, \sigma_i(z))$$

where $z = h^{-1}(x)$. In particular, $\|(D^i \hat{g})_\upsilon\| = \sup_{x\in\Lambda}\{\|(D^i g_x)_{\upsilon(x)}\|\}$.

Proof. ($\Rightarrow$) Assume $\hat{g}$ is C^r and takes continuous sections to continuous sections. We want to show $g_x := g|_{F_{1x}} : F_{1x} \to F_{2x}$ is C^r and that its derivatives vary continuously with $(x, u_x) \in F_1$. Fix $x_0 \in \Lambda$. Let $\pi : F_1 \to \Lambda$ be the bundle projection, and $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^d$ a vector bundle chart for a neighbourhood $U \subset \Lambda$ of x_0, so $\varphi_x : F_{1x} \to \mathbb{R}^d$ is an isometry for each $x \in U$. Let $F_{1U} := \pi^{-1}(U)$, and $\Sigma : F_{1U} \to \mathbb{F}_1$ be the map $\Sigma(u) = \upsilon_u$, where υ_u is the section defined by

$$\upsilon_u(x) = \begin{cases} \varphi^{-1}(x, \varphi_2(u)), & x \in U \\ 0, & x \notin U \end{cases}$$

and $\varphi_2 : \pi^{-1}(U) \to \mathbb{R}^d$ is the second term of φ. For each $u \in F_{1U}$, υ_u is continuous at all points $x \in U$. $\|\upsilon_u(x) - \upsilon_v(x)\| = \|\varphi_x^{-1}\| \cdot \|\varphi_2(u) - \varphi_2(v)\| = \|\varphi_2(u) - \varphi_2(v)\|$ if

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$x \in U$, and equal to zero otherwise. Hence, $\|\Sigma(u) - \Sigma(v)\| = \sup_{x\in\Lambda} \|v_u(x) - v_v(x)\| = \|\varphi_2(u) - \varphi_2(v)\|$. Consequently, since φ is continuous and linear on fibres, Σ is continuous and linear on fibres, and $\|\Sigma|_{F_{1x}}\| = 1$ for $x \in U$. Let $ev : \mathbb{F}_2 \times \Lambda \to F_2$ be the evaluation map, $(\sigma, x) \mapsto \sigma(x)$. For each fixed x, $ev_x := ev(\cdot, x) : \mathbb{F}_2 \to F_{2x}$ is linear with $\|ev_x\| = 1$. Moreover, ev is continuous at all (σ, x) for which σ is continuous at x.

It will clearly suffice to show g_x is C^r with continuous derivatives for $x \in U$. In that case, if $u \in F_{1U}$ and $x = \pi(u)$, then $v_u(x) = u$. Therefore, $g(u) = g \circ v_u(x) = \hat{g}(v_u)(h \circ \pi(u))$, and $g|_{F_{1U}} = ev \circ (\hat{g}, h) \circ (\Sigma, \pi)$. For fixed $x \in U$, $g_x = ev_{h(x)} \circ \hat{g} \circ \Sigma|_{F_{1x}}$. Since $ev_{h(x)}$ and $\Sigma|_{F_{1x}}$ are linear and $\hat{g}$ is C^r, g_x is C^r and $(D^i g_x)_u[u_1, \ldots, u_i] = (ev_{h(x)})(D^i \hat{g})_{\Sigma(u)}[\Sigma(u_1), \ldots, \Sigma(u_i)]$. Therefore, $(D^i g_x)_u = ev((D^i \hat{g})_{\Sigma(u)}[\Sigma^{(i)}], h \circ \pi(u))$. Note this is the same form as $i = 0$.

We know $\hat{g}$ takes $\mathbb{F}_1^c$ to $\mathbb{F}_2^c$. These are closed subspaces, so $\hat{g}$ restricted to $\mathbb{F}_1^c$ is C^r and $(D^i \hat{g})_v \in L^i(\mathbb{F}_1^c, \mathbb{F}_2^c)$ for $v \in \mathbb{F}_1^c$. Since $\hat{g}(v)(x)$ depends only on $v(x)$, this must be true locally as well, i.e. if $v, v_1, \ldots, v_i \in \mathbb{F}_1$ are continuous on a relatively open set $U \subset \Lambda$, then $(D^i \hat{g})_v(v_1, \ldots, v_i) \in \mathbb{F}_2$ is continuous on U. If $X \subset \mathbb{F}_2$ is the subspace of sections that are continuous on U, then $ev : X \times U \to F_{2U}$ is continuous. Since $(D^i \hat{g})_{\Sigma(u)}(\Sigma^{(i)})$ maps into X, $(D^i g_x)_u$ is the composition of continuous functions and therefore continuous. In particular, g itself is continuous. Finally, if $(D^l \hat{g})$ is ω-Hölder, then for each $x \in U$, and each $u, v \in F_{1x}$,

$$\begin{aligned}\|(D^l g_x)_u - (D^l g_x)_v\| &\le \|ev_{h(x)}\| \cdot \|(D^l \hat{g})_{\Sigma(u)} - (D^l \hat{g})_{\Sigma(v)}\| \cdot \|\Sigma|_{F_{1x}}\|^l \\ &\le H_\omega[(D^l \hat{g})]\|\Sigma|_{F_{1x}}\|^{l+\omega}\|u - v\|^\omega.\end{aligned}$$

Therefore, $H_\omega[(D^l g_x)] \le H_\omega[(D^l \hat{g})]$ if $x \in U$. However, Λ is covered by vector bundle charts of this type. Therefore, this holds for all $x \in \Lambda$, and $H_\omega[(D^l g_x)]$ is bounded independent of x.

($\Leftarrow$) Now assume g is continuous and g_x is C^r. We start by showing $\hat{g}$ is continuous. Let $v \in \mathbb{F}_1$ and set $R_0 := \|v\|_{\sup} + 1$. If $v' \in \mathbb{F}_1$ with $\|v - v'\|_{\sup} \le 1$, then for all $x \in \Lambda$, $v(x), v'(x) \in F_2(R_0) := \{v \in F_2 : \|v\| \le R_0\}$, which is compact. Therefore, g is uniformly continuous on $F_2(R_0)$. $\|\hat{g}(v) - \hat{g}(v')\|_{\sup} = \sup_{x\in\Lambda} \|g(v(x)) - g(v'(x))\|$. Since g is uniformly continuous, and $\|v(x) - v'(x)\| \le \|v - v'\|_{\sup}$, $\|\hat{g}(v) - \hat{g}(v')\|_{\sup} \to 0$ as $\|v - v'\|_{\sup} \to 0$, and $\hat{g}$ is continuous. If g_x is C^ω with $H_\omega[g_x] \le K$ for all x, then $\|g(v(x)) - g(v'(x))\| \le K\|v(x) - v'(x)\|^\omega$. In this case, $\|\hat{g}(v) - \hat{g}(v')\|_{\sup} \le K(\|v - v'\|_{\sup})^\omega$, and $\hat{g}$ is C^ω. Finally, if $v \in \mathbb{F}_1^c$, then v is continuous, so $g \circ v \circ h^{-1}$ is continuous, and $\hat{g}(v) \in \mathbb{F}_2^c$. This proves the second half of the lemma for $l = 0$.

Now suppose $l \ge 1$. Define $T : \mathbb{F}_1 \to L(\mathbb{F}_1, \mathbb{F}_2)$ by $T_v(\sigma)(x) = (Dg_z)_{v(z)}(\sigma(z))$, where $z = h^{-1}(x)$. $T_v(\sigma)$ is linear in σ, and $\|T_v(\sigma)\|_{\sup} \le \sup_{z\in\Lambda}\{\|(Dg_z)_{v(z)}\|\|\sigma(z)\|\} \le \sup_{x\in\Lambda}\{\|(Dg_x)_{v(x)}\|\}\|\sigma\|_{\sup}$. Again $v(x)$ is taking values in a compact set, and $(Dg_x)_v$ is continuous and hence bounded on compact sets. Therefore, $\sup_{x\in\Lambda}\{\|(Dg_x)_{v(x)}\|\}$ is finite, and so $T_v \in L(\mathbb{F}_1, \mathbb{F}_2)$. The same argument that was applied to $\hat{g}$ shows T is continuous, and if (Dg_x) is C^ω with $H_\omega[(Dg_x)] \le K$ for all x, then T is C^ω. We claim $T_v = (D\hat{g})_v$, and so in particular, $\hat{g}$ is C^1.

Since g_z is C^l, $\|g_z(u+v) - g_z(u) - (Dg_z)_u(v)\| \le (\sup_{0\le t\le 1}\|(Dg_z)_{u+tv} - (Dg_z)_u\|)\|v\|$. Therefore, $\|\hat{g}(v + \sigma) - \hat{g}(v) - T_v(\sigma)\|_{\sup}(\|\sigma\|_{\sup})^{-1} \le \sup_{0\le t\le 1}\|T_{v+t\sigma} - T_v\|$, and since T is continuous, this goes to zero as $\|\sigma\|_{\sup} \to 0$. Consequently, $T_v =$

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$(D\hat{g})_\upsilon$. In the same manner, we can define $T^i_\upsilon \in L^i(\mathbb{F}_1, \mathbb{F}_2)$ by $T^i_\upsilon(\sigma_1, \ldots, \sigma_i)(x) = (D^i g_z)_{\upsilon(z)}(\sigma_1(z), \ldots, \sigma_i(z))$ for $i = 2, \ldots, l$ and $z = h^{-1}(x)$. In each case, the continuity of $(D^i g_z)$ implies T^i is continuous, and C^ω if $(D^i g_z)$ is with $\sup_{z\in\Lambda} H_\omega[(D^i g_z)] < \infty$. If $i < l$, then $(D^i g_z)$ is C^1 with derivative $(D^{i+1} g_z)$, allowing us to again deduce as above $\|T^i_{\upsilon+\sigma} - T^i_\upsilon - T^{i+1}_\upsilon(\sigma)\|/\|\sigma\|_{\sup} \le \sup_{0\le t\le 1}\|T^{i+1}_{\upsilon+t\sigma} - T^{i+1}_\upsilon\|$. Since T^{i+1} is continuous, we then get $T^{i+1} = (DT^i)$, and by induction $T^i = (D^i\hat{g})$. Therefore, $\hat{g}$ is C^r and $(D^i\hat{g})_\upsilon(\sigma_1, \ldots, \sigma_i)$ has the desired form.

Finally, by the same argument as for $i = 1$, $\|(D^i\hat{g})_\upsilon\| \le \sup_{x\in\Lambda}\{\|(D^i g_x)_{\upsilon(x)}\|\}$. On the other hand, if we fix x, and $u_1, \ldots, u_i$ are any vectors in F_{1x}, then if σ_j is the section which is u_j at x and zero otherwise, then $(D^i\hat{g})_\upsilon(\sigma_1, \ldots, \sigma_i)(w) = (D^i g_x)_{\upsilon(x)}(u_1, \ldots, u_i)$ for $w = h(x)$ and zero otherwise. Therefore, $\|(D^i\hat{g})_\upsilon\| \ge \|(D^i g_x)_{\upsilon(x)}(u_1, \ldots, u_i)\|/(\|u_1\| \ldots \|u_i\|)$, and so $\|(D^i\hat{g})_\upsilon\| \ge \|(D^i g_x)_{\upsilon(x)}\|$. Since this holds for all x, $\|(D^i\hat{g})_\upsilon\| = \sup_{x\in\Lambda}\{\|(D^i g_x)_{\upsilon(x)}\|\}$. □

We want to find a continuous function $\xi : \Lambda \to \mathcal{F}^{(k,r)}_d(M)$ such that ξ_x is a local (k, r)-jet of a foliation along $\Delta_x = W^u_\rho(x; \alpha)$, and $f_*(\xi_x) \subset \xi_{f(x)}$. Suppose $\mathcal{F}_x$ is a foliation containing Δ_x as a disk plaque. If $\mathcal{F}'_x := \exp_x^{-1}(\mathcal{F}_x)$, then $\mathcal{F}'_x$ contains $gr(g^{(0)}_x)$ as a disk plaque. Suppose $\mathcal{F}_{f(x)}$ and $\mathcal{F}'_{f(x)}$ are defined analogously and containing $\Delta_{f(x)}$ and $gr(g^{(0)}_{f(x)})$ respectively. It is clear from the definition of $f_\#$ that $\mathcal{F}_{f(x)}$ and $f(\mathcal{F}_x)$ will define the same local jet of a foliation if and only if $\mathcal{F}'_{f(x)}$ and $f_\#(\mathcal{F}'_x)$ do. Therefore, it will be sufficient to find $\zeta_x \in \mathcal{F}^{(k,r)}_d(T_xM)$ for each x such that ζ_x is a local jet of a foliation along $gr(g^{(0)}_x)$ and $f_{\#*}(\zeta_x) \supset \zeta_{f(x)}$. However, if ζ_x is a local jet of a foliation along $gr(g^{(0)}_x)$, then it is represented by a $(k+1)$-tuple $\mathbf{g}_x = (g^{(0)}_x, \ldots, g^{(k)}_x)$, where $g^{(i)}_x \in C^{r-i}(E_{1x}(\rho), L^i_{\text{sym}}(E_{2x}, E_{2x}))$ and $g^{(i)}_x(0_x) = 0$. (In the fixed point case we used $g^{(i)}_x$ to denote the value of $g^{(i)}$ at x. We now use this to denote the base point. Its value at $u_x \in E_{1x}(\rho)$ will be $g^{(i)}(u_x)$.) Therefore, if $\mathcal{L}^i(E_2)$ denotes the vector bundle over Λ with fibres $L^i_{\text{sym}}(E_{2x}, E_{2x})$, then we want a $\mathbf{g} = (g^{(1)}, \ldots, g^{(k)})$, where $g^{(i)}$ is a map from $E_1(\rho)$ to $\mathcal{L}^i(E_2)$ covering id_Λ, such that for each x, $g^{(i)}_x$ is C^{r-i} and $f_{\#*}(\mathbf{g}_x) = \mathbf{g}_{f(x)}$. If $\xi : \Lambda \to \mathcal{F}^{(k,r)}_d(M)$ is to be continuous, then the derivatives of $g^{(i)}_x$ must be continuous in x, and if $\omega \neq 0$, we also ask that $H_\omega[(D^{r-i} g^{(i)}_x)]$ be independent of x.

Let $\mathbb{L}_i$ be the Banach space of bounded sections of $\mathcal{L}^i(E_2)$. That is, if $T \in \mathbb{L}_i$, then for any $x \in \Lambda$, $T(x) \in L^i_{\text{sym}}(E_{2x}, E_{2x})$. There is a natural isometry from $\mathbb{L}_i$ into (not onto) $L^i_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2)$. If $T \in \mathbb{L}_i$, let $T \in L^i_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2)$ be $T(\sigma_1, \ldots, \sigma_i)(x) = T(x)[\sigma_1(x), \ldots, \sigma_i(x)]$. Therefore, we can treat $\mathbb{L}_i$ as a subspace of $L^i_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2)$. Given $\mathbf{g} = (g^{(1)}, \ldots, g^{(k)})$ as above, for each $g^{(i)}$ define $\hat{g}^{(i)} : \mathbb{E}_1(1) \to \mathbb{L}_i$ by $\hat{g}^{(i)}(\upsilon) = g^{(i)} \circ \upsilon$. Since $g^{(i)}$ covers id_Λ, by the previous lemma $\hat{g}^{(i)}$ is C^{r-i}. Therefore, $\hat{g}^{(i)} \in C^{r-i}(\mathbb{E}_1(1), L^i_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2))$ with identifying $\mathbb{L}_i \hookrightarrow L^i_{\text{sym}}(\mathbb{E}_2, \mathbb{E}_2)$. $\hat{g}^{(i)}(0)(x) = g^{(i)}(0_x) = 0$ for all x. Hence, $\hat{g}^{(i)}(0) = 0$, and $\hat{g}^{(i)}$ is in the space $\mathcal{C}_i$ defined in the proof of Theorem 3.1. If $\hat{\mathbf{g}} := (\hat{g}^{(1)}, \ldots, \hat{g}^{(k)})$, then $\hat{\mathbf{g}} \in \mathcal{G}_k$.

Let $\mathcal{C}'_i$ be the subspace of $\mathcal{C}_i$ consisting of those $\hat{g}^{(i)}$ obtained in this way, i.e. $\hat{g}^{(i)}(\upsilon) = g^{(i)} \circ \upsilon$ for some $g^{(i)}$ as above. This is a closed subspace, and so $\mathcal{G}'_i := \mathcal{C}'_1 \times \cdots \times \mathcal{C}'_i \subset \mathcal{G}_i$ is also a closed subspace for each i. We want to show that the action of $f_\#$ on $\mathbf{g}$ is the same

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as Γ_k on $\hat{\mathbf{g}}$, in which case we will be reduced to finding a fixed point of Γ_k, and can revert to Theorem 3.1. We start by determining the action of $f_\#$.

Given $(g^{(0)}, \dots, g^{(k)})$, for each x we can use Lemma 4.1 to get $g_x : E_{1x} \times E_{2x} \to E_{2x}$ that is a C^r extension of $(g_x^{(0)}, \dots, g_x^{(k)})$. We claim this can be done so that the derivatives of g_x depend continuously on x. Given $x \in \Lambda$, we can find an open set $U \subset \Lambda$ with $x \in U$, and a vector bundle chart $\varphi : E_{1U} \oplus E_{2U} \to U \times \mathbb{R}^d \times \mathbb{R}^{n-d}$. This gives us a consistent choice of a coordinate system on $T_xM = E_{1x} \times E_{2x}$ for $x \in U$. Relative to these coordinates we can apply the extension operator of Lemma 4.1 to get g_x defined for each $x \in U$. Since the extension operator is a bounded linear operator between the two function spaces, if we first use the C^{l-i} topology ($l = [r]$) for each $g^{(i)}$, then since each $g_x^{(i)}$ is continuous in this topology, g_x will depend continuously on $x \in U$ in the C^l topology. Moreover, $g_x^{(i)}$ is bounded in the C^{r-i} topology, so g_x is bounded on U in the C^r topology, i.e. $H_\omega[(D^l g_x)]$ is independent of x. Now choose a partition of unity subordinate to a cover of Λ by vector bundle charts. To each chart there will exist a locally defined g_x. Using the partition of unity to add these together we get a function $g : E_1 \oplus E_2 \to E_2$ such that for each $x \in \Lambda$, g_x is C^r with derivatives that depend continuously on x, and $H_\omega[(D^l g_x)]$ is independent of x. Since the partition of unity is on Λ, for any fixed x, g_x is a *constant* linear combination of a finite number of different extensions of the same $(k+1)$-tuple $(g_x^{(0)}, \dots, g_x^{(k)})$. Therefore, g_x is also an extension, which proves the claim.

Define $G : E_1 \oplus E_2 \to E_1 \oplus E_2$ by $G := \mathrm{id} + (0, g)$. G has the property that for each $x \in \Lambda$, G_x is a standard presentation for a foliation on T_xM that is in the equivalence class determined by $(g_x^{(0)}, \dots, g_x^{(k)})$. The image of this foliation under $f_\#$ is a foliation on $T_{f(x)}M$ with chart $(f_\# \circ G)_x : T_xM \to T_{f(x)}M$. If we are to have an invariant family of local (k, r)-jets of foliations, the new foliation on T_xM with chart $(f_\# \circ G)_{f^{-1}(x)}$ must be in the same equivalence class as the original, i.e. determined by $\mathbf{g}_x$. To check this we must convert $(f_\# \circ G)_{f^{-1}(x)}$ to a standard presentation. Except that $(f_\# \circ G)_{f^{-1}(x)}$ maps between different tangent planes, this is the same as Lemma 4.2. The same argument yields that the standard presentation will be $(f_\# \circ G)_{f^{-1}(x)} \circ (F_{f^{-1}(x)})^{-1}$, where in this case, $F : E_1 \oplus E_2 \to E_1 \oplus E_2$ is the map covering f defined for each $x \in \Lambda$ by $F_x = [\pi_1 \circ (f_\# \circ G)_x, \pi_2 \circ (f_\# \circ G)_x \circ gr(h_x)]$, and $h : E_2 \to E_1$ covering id_Λ is defined by h_x solving $\pi_1 \circ (f_\# \circ G)_x(h_x(v), v) \equiv 0$, $h_x(0) = 0$. This can be written in terms of the maps on the full bundles as $F = [\pi_1 \circ f_\# \circ G, \pi_2 \circ f_\# \circ G \circ gr(h)]$, and $\pi_1 \circ f_\# \circ G \circ gr(h) \equiv 0$ with $h(0_x) = 0_x$ for all $x \in \Lambda$. (Here $\pi_i : E_1 \oplus E_2 \to E_i$ is the bundle map that for each x projects T_xM onto E_{ix} $(i = 1, 2)$, and $gr(h)$ is the map $gr(h)(v_x) := (h(v_x), v_x)$, which is abusing notation as this is not truly a graph!)

Since F_x is invertible for each x, and f is a homeomorphism on Λ, F is invertible with $(F^{-1})_x = (F_{f^{-1}(x)})^{-1}$, and F^{-1} covers f^{-1}. Therefore, $G' := f_\# \circ G \circ F^{-1}$ covers id_Λ and has the property that for each x, G'_x is a standard presentation for the foliation with chart $(f_\# \circ G)_{f^{-1}(x)}$. Consequently, if $g' : E_1 \oplus E_2 \to E_2$ is defined by $G' =: \mathrm{id} + (0, g')$, then for the family $\{\mathbf{g}_x : x \in \Lambda\}$ to be $f_\#$ invariant, we must have $(D_2^i g'_x)_{(u_x,0)} = g_x^{(i)}(u_x)$ for all x, u_x.

Now define $\hat{g} : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{E}_2$ by $\hat{g}(\upsilon) = g \circ \upsilon$. By Lemma 5.1 $\hat{g}$ is C^r. Moreover, $[(D_2^i \hat{g})_{(\upsilon,0)}(\sigma_1, \dots, \sigma_i)](x) = (D_2^i g_x)_{(\upsilon(x),0)}(\sigma_1(x), \dots, \sigma_i(x))$. $\upsilon \in \mathbb{E}_1(1)$ if and only if for all $x, \upsilon(x) \in E_{1x}(1)$. Therefore, $(D_2^i \hat{g})_{(\upsilon,0)} = g^{(i)} \circ \upsilon = \hat{g}^{(i)}(\upsilon)$, and so

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$(D_2^i \hat{g})|_{\mathbb{E}_1(1)} = \hat{g}^{(i)}$. Hence, $\hat{g}$ is an extension of $\hat{\mathbf{g}} = (\hat{g}^{(1)}, \dots, \hat{g}^{(k)})$. Consequently, each $\hat{\mathbf{g}} \in \mathcal{G}'_k$ has a C^r extension in spite of the fact that $\mathbb{E}$ is a Banach space, and we do not need to rely on formal calculations for Γ_k.

Define $\hat{G} : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{E}_1 \times \mathbb{E}_2$ by $\hat{G} := \mathrm{id} + (0\, \hat{g})$. We then have $\hat{G}(\upsilon) = G \circ \upsilon$, and $(\hat{f} \circ \hat{G})(\upsilon) = (f_\# \circ G) \circ \upsilon$. Γ_k is obtained by applying Lemma 4.2 to $\hat{f} \circ \hat{G}$. Define $\hat{h} : \mathbb{E}_2 \to \mathbb{E}_1$ by $\hat{h}(\upsilon) = h \circ \upsilon$. We observe that $\pi_1 \circ \hat{f} \circ \hat{G} \circ gr(\hat{h})(\upsilon)(x) = \pi_1 \circ f_{\#x} \circ G_x \circ gr(h_x)(\upsilon(x)) = 0$, and $\hat{h}(0)(x) = h_x(0_x) = 0$. Therefore, $\hat{h}$ is in fact the unique solution obtained from the implicit function theorem. Similarly, if we define $\hat{F} := [\pi_1 \circ \hat{f} \circ \hat{G}, \pi_2 \circ \hat{f} \circ \hat{G} \circ gr(\hat{h})]$, then $\hat{F}(\upsilon) = F \circ \upsilon$, and one can verify directly that $\hat{F}^{-1}(\upsilon) = F^{-1} \circ \upsilon$. Consequently, if $\hat{G}' := \hat{f} \circ \hat{G} \circ \hat{F}^{-1}$, then $\hat{G}'(\upsilon) = \hat{f} \circ \hat{G} \circ \hat{F}^{-1}(\upsilon) = f_\# \circ G \circ F^{-1} \circ \upsilon = G' \circ \upsilon$. Likewise, if we define $\hat{g}'$ by $\hat{G}' =: \mathrm{id} + (0, \hat{g}')$, then $\hat{g}'(\upsilon) = g' \circ \upsilon$. Since $\hat{g}'$ was obtained from Lemma 4.2, it is known to be C^r. From this it follows by Lemma 5.1 that g', which was constructed for each x independently, has the property that not only is each g'_x C^r, but its derivatives are continuous in x and have Hölder constants that are bounded in x. Moreover, $(D_2^i \hat{g}')_{(\upsilon,0)} = (D_2^i g'_x) \circ (\upsilon, 0)$.

From this we can conclude two things. First, the solution we are looking for does indeed correspond to a fixed point of Γ_k that is in $\mathcal{G}'_k$. Second, $\mathcal{G}'_k$ is invariant under Γ_k. We can now revert to Theorem 3.1. We already know $0 \in \mathbb{E}$ is α-pseudohyperbolic for $\hat{f}$. Recall $\hat{A} := (D_1 \hat{f}_1)_0$ and $\hat{N} := (D_2 \hat{f}_2)_0$. We cannot apply Theorem 3.1 directly, because we have not assumed that the conditions on the spectra of $\hat{A}$ and $\hat{N}$ are satisfied. However, these were only used in the proof of Theorem 3.1 to get a norm for which $\|\hat{N}\| \cdot \|\hat{N}^{-1}\|^i \cdot \|\hat{A}^{-1}\| < 1$ $(i = 1, \dots, k)$, and $\|\hat{N}\| \cdot \|\hat{N}^{-1}\|^k \cdot \|\hat{A}^{-1}\|^\theta < 1$, which in turn were only used in the proof of Lemma 4.9. This is similar to what is assumed for Theorem 3.5. Here we have $\|N_x\| \cdot \|N_x^{-1}\|^k \cdot \|A_x^{-1}\|^\theta < 1$. There are two cases. Either $\|N_x^{-1}\| > 1$, in which case $\|N_x\| \cdot \|N_x^{-1}\|^i \cdot \|A_x^{-1}\| \le \|N_x\| \cdot \|N_x^{-1}\|^k \cdot \|A_x^{-1}\|^\theta$ for all i, or $\|N_x^{-1}\| \le 1$, in which case $\|N_x\| \cdot \|N_x^{-1}\|^i \cdot \|A_x^{-1}\| \le \|N_x\| \cdot \|A_x^{-1}\|$. By assumption $\|N_x\| < \alpha < \|A_x^{-1}\|^{-1}$. Therefore, in either case we have $\|N_x\| \cdot \|N_x^{-1}\|^i \cdot \|A_x^{-1}\| < 1$ $(i = 1, \dots, k)$. However, these conditions are assumed only point by point. From Lemma 5.1 $\|\hat{N}\| \cdot \|\hat{N}^{-1}\|^i \cdot \|\hat{A}^{-1}\| = (\sup_{x\in\Lambda} \|N_x\|)(\sup_{x\in\Lambda} \|N_x^{-1}\|)^i (\sup_{x\in\Lambda} \|A_x^{-1}\|)$. Therefore, the hypothesis $\|\hat{N}\| \cdot \|\hat{N}^{-1}\|^i \cdot \|\hat{A}^{-1}\| < 1$ used in Theorem 3.1 is much stronger than we have assumed.

We claim that with a slight modification, Lemma 4.9 will still be true under the weaker assumptions of Theorem 3.5. The modification is in the definition of the norm $|||\cdot|||$. In the notation of this proof, this was defined as $|||\hat{g}^{(i)}||| := \sup_{0<\|\upsilon\|\le 1} [\|\hat{g}^{(i)}(\upsilon)\|/\|\upsilon\|^\tau]$, where $\tau = \theta$ when $i = k$, and $\tau = 1$ otherwise. υ is a bounded section, so $\|\upsilon\| := \sup_{x\in\Lambda} \|\upsilon(x)\|$, and if $\hat{g}^{(i)} \in \mathcal{C}'_i$, then $\|\hat{g}^{(i)}(\upsilon)\| = \sup_{x\in\Lambda} \|g^{(i)}(\upsilon(x))\|$. Modify $|||\cdot|||$ by pulling the supremum over x out of the ratio: $|||\hat{g}^{(i)}||| := \sup_{0<\|\upsilon\|\le 1} \sup_{x\in\Lambda} [\|g^{(i)}(\upsilon(x))\|/\|\upsilon(x)\|^\tau]$. Since υ need not be continuous, this is equivalent to $|||\hat{g}^{(i)}||| = \sup_{x\in\Lambda} \sup_{u_x \in E_{1x}(1)} [\|g^{(i)}(u_x)\|/\|u_x\|^\tau]$. Similarly, $\|\hat{g}^{(i)}\|_{C^0} = \sup_{x\in\Lambda} \sup_{u_x\in E_{1x}(1)} \|g^{(i)}(u_x)\|$. Since $\|u_x\| \le 1$, it is clear that $\|\hat{g}^{(i)}\|_{C^0} \le |||\hat{g}^{(i)}|||$. On the other hand, we have $\sup_{u_x\in E_{1x}(1)} [\|g^{(i)}(u_x)\|/\|u_x\|^\tau] \le H_\tau[g_x^{(i)}] \le H_\tau[\hat{g}^{(i)}]$. Therefore, $|||\hat{g}^{(i)}||| \le H_\tau[\hat{g}^{(i)}]$. Consequently, if $\|\hat{g}^{(i)}\|_* := \max\{|||\hat{g}^{(i)}|||, \|(D\hat{g}^{(i)})\|_{C^0}, \dots, \|(D^l \hat{g}^{(i)})\|_{C^0}, H_\omega[(D^l \hat{g}^{(i)}]\}$, as before, then not only do

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we again have $\|\cdot\|_* = \|\cdot\|_{C^{r-i}}$ $(r - i = l + \omega)$, but we also have $\|\hat{g}^{(i)}\|_* = \sup_{x\in\Lambda} \max\{|||g_x^{(i)}|||, \|(Dg_x^{(i)})\|_{C^0}, \ldots, \|(D^l g_x^{(i)})\|_{C^0}, H_\omega[(D^l g_x^{(i)})]\} = \sup_{x\in\Lambda}\|g_x^{(i)}\|_*$. Therefore, all calculations in the proof of Lemma 4.9 reduce to pointwise calculations, and the pointwise assumptions are sufficient.

Consequently, Γ_i restricted to $\mathcal{G}_i' = \mathcal{G}_{i-1}' \times \mathcal{C}_i'$ is a fibre contraction for each i, from which we conclude, as in the proof of Theorem 3.1, that there exists a unique fixed point of Γ_k in $\mathcal{G}_k'$. This then provides the function $\xi : \Lambda \to \mathcal{F}_d^{(k,r)}(M)$ that we sought, which completes the proof. □

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