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ARTICLE

Efficient homogeneous partitions and slicewise VC-dimension

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Abstract

Previous work of the author and Wolf, and independently Chernikov and Towsner, showed that 3-uniform hypergraphs of small slicewise VC-dimension admit homogeneous partitions. The results of Chernikov and Towsner did not produce explicit bounds, while the work of the author and Wolf relied on a strong version of hypergraph regularity, and consequently produced a Wowzer type bound on the size of the partition. This paper gives a new proof of this result, yielding ϵ-homogeneous partitions of size at most $2^{2^{\epsilon^{-K}}}$, where K is a constant depending on the slicewise VC-dimension. This is the first in a series of papers about growth of regular partitions of 3-uniform hypergraphs. Our main result is a crucial ingredient in the second and third papers in the series, which investigate the growth of two types of regular partitions in hereditary properties of 3-uniform hypergraphs, namely the weak type first defined by Chung and Haviland–Thomason, and the stronger type first developed by Gowers, and independently, Frankl and Rödl.

Keywords: hypergraph regularity; VC-dimension

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1. Introduction

The notion of VC-dimension (named for Vapnik and Chervonenkis) is an important measure of complexity in computer science, graph theory, and logic. The *VC-dimension* of a graph $G = (V, E)$, denoted $\mathrm{VC}(G)$, is the maximum k for which there exist vertices $\{a_1, \dots, a_k\} \cup \{b_S : S \subseteq [k]\}$ so that $a_i b_S \in E$ if and only if $i \in S$. In other words, $\mathrm{VC}(G)$ is the VC-dimension of the set system $(V, \mathcal{S})$, where

$$\mathcal{S} = \{N_G(x) : x \in V(G)\}.$$

Graphs with small VC-dimension satisfy "better" structure theorems than can be proved in the setting of arbitrary graphs. Examples of this type of result include theorems related to property testing [1, 2], bounds in Ramsey's Theorem [7, 9, 20], removal lemmas [11], and regularity lemmas [2, 4, 9, 15].

Szemerédi's Regularity Lemma is a general tool in graph theory that says any graph can be partitioned into a small number of parts, so that most pairs of parts behave quasirandomly (these are called the *regular pairs*). There is a parameter ϵ that measures the quasirandomness of the regular pairs, and a bound M, depending on ϵ, on the number of parts needed. It was shown by Gowers [13] that in general, the dependence of M on ϵ is at least a tower of exponentials of height

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polynomial in ϵ^{-1}. Fox and Lovász later obtained tight bounds for M as a tower of exponentials of height about ϵ^{-2} (see [8]).

In work related to property testing, Alon, Fischer, and Newman showed that in bipartite graphs of small VC-dimension, an improved regularity lemma holds, where the regular pairs all have density close to 0 or 1, and where the number of parts is only polynomial in ϵ^{-1} [2]. This result was extended to all graphs with small VC-dimension by Lovász–Szegedy [15]. A quantitative improvement was given by Fox–Pach–Suk in [9], which we will state below after some definitions. First, given a graph $G = (V, E)$ and sets $X, Y \subseteq V$, we say the pair (X, Y) is *ϵ-homogeneous with respect to G* if

$$d_G(X, Y) := \frac{|\{(x, y) \in X \times Y : \{x, y\} \in E\}|}{|X||Y|} \in [0, \epsilon) \cup (1 - \epsilon, 1].$$

We then say a partition $\mathcal{P}$ of V is *ϵ-homogeneous with respect to G* if at least $(1 - \epsilon)|V|^2$ pairs from V^2 come from an ϵ-homogeneous pair from $\mathcal{P}^2$ (see Definition 2.9).

Theorem 1.1 (Fox, Pach, Suk [9]). *For all $k \geq 1$ there is $c > 0$ so that the following holds. For all $\epsilon \in (0, 1/4)$, any graph with VC-dimension at most k has an ϵ-homogeneous equipartition of size t for some $8\epsilon^{-1} \leq t \leq c\epsilon^{-2k-1}$.*

In fact, Fox, Pach and Suk proved a much more general theorem which applies to k-uniform hypergraphs. For a k-uniform hypergraph $H = (V, E)$, they define the *VC-dimension of H*, denoted $\mathrm{VC}(H)$, to be the VC-dimension of the set system $(V, \mathcal{S})$, where

$$\mathcal{S} = \{N_H(v_1, \ldots, v_{k-1}) : v_1, \ldots, v_{k-1} \in V\}.$$

One can define homogeneous partitions for k-uniform hypergraphs in much the same way as above. In particular, if $G = (V, E)$ is a k-uniform hypergraph and $\mathcal{P}$ is a partition of V, then a k-tuple $(X_1, \ldots, X_k) \in \mathcal{P}^k$ is *ϵ-homogeneous with respect to G* if

$$d_G(X_1, \ldots, X_k) := \frac{|\{(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k : \{x_1, \ldots, x_k\} \in E\}|}{|X_1| \cdots |X_k|} \in [0, \epsilon) \cup (1 - \epsilon, 1].$$

We say then $\mathcal{P}$ is *ϵ-homogeneous with respect to G* if at least $(1 - \epsilon)|V|^k$ of the tuples from V^k come from an ϵ-homogeneous tuple from $\mathcal{P}^k$. Fox, Pach, and Suk proved the following extension of Theorem 1.1 in terms of the *dual VC-dimension* of a hypergraph.[1]

Theorem 1.2. *For all $d, k \geq 1$, there is a constant c so that the following holds. Suppose $H = (V, E)$ is an k-uniform hypergraph with dual VC-dimension at most d. Then it has an ϵ-homogeneous equipartition of size $\epsilon^{-1} \leq t \leq c\epsilon^{-2d-1}$.*

Similar results with a weaker bound were independently proved by Chernikov and Starchenko in [4]. The work in [4] is part a line of inquiry within model theoretic combinatorics seeking to understand analogues of VC-dimension in the setting of k-uniform hypergraphs for $k > 2$ (see e.g. [3, 5, 6, 14, 17–19]). Two papers in this vein [5, 19] show that there is a larger class of 3-uniform hypergraphs which admit homogeneous partitions, beyond just those of bounded VC-dimension. Stating these results requires a different analogue of VC-dimension for 3-uniform hypergraphs, which involves the following notion of "slice graphs." Given a 3-uniform hypergraph $H = (V, F)$ and $x \in V$, the *slice graph of H at x* is the graph

$$H_x = \left(V, \left\{yz \in \binom{V}{2} : xyz \in F\right\}\right).$$

We then define the *slicewise VC-dimension*[2] *of H* to be $\mathrm{SVC}(H) := \max\{\mathrm{VC}(H_x) : x \in V\}$.

[1] For the purposes of this introduction, it suffices to know the dual VC-dimension is bounded in terms of the VC-dimension. We refer the reader to [8] for more details.

[2] In [19] this is referred to as *weak VC-dimension*. Here we have adopted the more descriptive terminology of [5].

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It is an exercise to check that for any 3-uniform hypergraph H, $\mathrm{SVC}(H) \leq \mathrm{VC}(H)$. Work of Wolf and the author showed the existence of homogeneous partitions can be extended to the more general assumption of having bounded slicewise VC-dimension [19]. Non-quantitative versions of this result (as well as analogues for k-uniform hypergraphs) were independently proved by Towsner and Chernikov in [5].

Theorem 1.3 (Chernikov–Towsner [5], Terry–Wolf [19]). *For all $k \geq 1$ and $\epsilon > 0$, there is an integer $M(\epsilon)$ so that any 3-graph with slicewise VC-dimension less than k admits an ϵ-homogeneous partition with at most $M(\epsilon)$ parts.*

While the proof of the author and Wolf of Theorem 1.3 is quantitative, it relies on a strong version of the hypergraph regularity lemma, and thus yields a Wowzer style upper bound on the partition itself. The main result of the current paper is the following quantitative improvement.

Theorem 1.4. *For all $k \geq 1$ there are $K, \epsilon^* > 0$ so that the following holds. Suppose $0 < \epsilon < \epsilon^*$ and H is a sufficiently large 3-uniform hypergraph with slicewise VC-dimension less than k. Then there exists an ϵ-homogeneous partition for H of size at most $2^{2^{\epsilon^{-K}}}$.*

Theorem 1.4 answers a question posed in [19]. This theorem is also crucial for Part 2 of this series, which considers the possible growth rates of weak regular partitions in hereditary properties of 3-uniform hypergraphs.

The biggest open problem that remains from this paper and its sequels is whether the bound in Theorem 1.4 can be improved. It is shown in Part 2 that there exist arbitrarily large 3-uniform hypergraphs with slicewise VC-dimension 2, and which require at least $2^{\epsilon^{-C}}$ parts in any ϵ-homogeneous partition. Thus, if the bound in Theorem 1.4 can be improved, it will not be beyond a single exponential in ϵ^{-1}. The author conjectures the form of the bound in Theorem 1.4 is tight. We refer the reader to Part 2 for more discussion of this question.[3]

1.1 Notation

For a set V and an integer $k \geq 1$, define $\binom{V}{k} = \{X \subseteq V : |X| = k\}$. A *$k$-uniform hypergraph* is a pair (V, E), where $E \subseteq \binom{V}{k}$. Given a k-uniform hypergraph G, $V(G)$ will denote its vertex set of V, and $E(G)$ will denote its edge set of G. A *graph* is a 2-uniform hypergraphs. For $k \geq 3$, a *k-graph* is a k-uniform hypergraph. Given a k-graph (V, E), an induced sub-k-graph has the form (V', E'), where $V' \subseteq V$ and $E' = E \cap \binom{V'}{k}$. We let $G[V']$ be the induced sub-k-graph of G with vertex set V'.

We will use the notation xy for a two element set $\{x, y\}$, and xyz for a three element set $\{x, y, z\}$. For sets X, Y, Z, define

$$K_2[X, Y] = \{xy : x \in X, y \in Y, x \neq y\} \text{ and}$$

$$K_3[X, Y, Z] = \{xyz : x \in X, y \in Y, z \in Z, x \neq y, y \neq z, x \neq z\}.$$

Suppose $G = (V, E)$ is a graph. We let $\overline{E} = \{(x, y) \in V^2 : xy \in E\}$. Given sets $X, Y \subseteq V$, the *density of (X, Y) in G* is

$$d_G(X, Y) := \frac{|\overline{E} \cap (X \times Y)|}{|X||Y|}.$$

For any $y \in V$, the *neighbourhood of y in G* is $N_G(y) = \{x \in X : xy \in E\}$. Given a subset $E' \subseteq \binom{V}{2}$, we will also write

$$N_{E'}(y) = \{x \in X : xy \in E\}. \tag{1}$$

[3] This question has recently been resolved by Gishboliner, Shapira, and Widgerson, who showed a single exponential suffices [12].

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Suppose now $H = (V, E)$ is a 3-uniform hypergraph. We let

$$\overline{E} = \{(x, y, z) \in V^3 : xyz \in E\}.$$

For sets $X, Y, Z \subseteq V$, define

$$d_H(X, Y, Z) := \frac{|\overline{E} \cap (X \times Y \times Z)|}{|X||Y||Z|}.$$

For any $x, y \in V$, we let

$$N_H(x) = \left\{uv \in \binom{V}{2} : xuv \in E\right\} \text{ and } N_H(xy) = \left\{z \in V : xyz \in E\right\}.$$

For an edge set $E \subseteq \binom{V}{k}$, we will write $E^1 = E$ and $E^0 = \binom{V}{k} \setminus E$. More generally, given any set $E' \subseteq \binom{V}{3}$, we will also write

$$N_{E'}(x) = \left\{yz \in \binom{V}{2} : xyz \in E'\right\} \text{ and } N_{E'}(xy) = \left\{z \in V : xyz \in E'\right\}.$$

Given integers $1 \leq k \leq \ell$, We say a k-graph $G = (V, E)$ is *ℓ-partite* if there is a partition $V = V_1 \cup \cdots \cup V_\ell$ so that for every $e \in E$ and $1 \leq i \leq \ell$, we have $|e \cap V_i| \leq 1$. We will write $G = (V_1 \cup \cdots \cup V_\ell, E)$ to denote that G is ℓ-partite with the displayed partition of vertices.

For all integers $n \geq 1$, $[n] = \{1, \ldots, n\}$. An *equipartition* of a set V is a partition $V = V_1 \cup \cdots \cup V_t$ satisfying $||V_i| - |V_j|| \leq 1$ for all $1 \leq i, j \leq t$.

Given a set V, a partition $\mathcal{P}$ of V and a subset $X \subseteq V$, let $\mathcal{P}|_X := \{X \cap S : S \in \mathcal{P}\} \setminus \{\emptyset\}$. Note that in this case, $\mathcal{P}|_X$ is a partition of X.

Given two functions $f, g : \mathbb{R} \to \mathbb{R}$ and an integer $k \geq 1$, we use the notation $f(x) = O_k(g(x))$ to mean there is some constant C depending on k so that all $x, f(x) \leq Cg(x)$.

2. Preliminaries

This section contains some required preliminaries. These will include lemmas about averaging and symmetry in 3-graphs (Subsection 2.1), definitions and results about VC-dimension and homogeneity (Subsection 2.2), and finally, tripartite decompositions and related notions (Subsection 2.3).

2.1 Averaging and symmetry

This section contains two crucial lemmas. The first is a simple averaging lemma for "dense" subsets.

Lemma 2.1. *Let $a, b, \epsilon \in (0, 1)$ satisfy $ab = \epsilon$. Suppose $A \subseteq X$ and $|A| \geq (1 - \epsilon)|X|$. For any partition $\mathcal{P}$ of X, if we let $\Sigma = \{Y \in \mathcal{P} : |A \cap Y| \geq (1 - a)|Y|\}$, then $|\bigcup_{Y \in \Sigma} Y| \geq (1 - b)|X|$.*

Proof. Suppose towards a contradiction that $|\bigcup_{Y \in \Sigma} Y| < (1 - b)|X|$. Then

$$|X \setminus A| \geq \sum_{Y \in \mathcal{P} \setminus \Sigma} |Y \setminus A| > \sum_{Y \in \mathcal{P} \setminus \Sigma} a|Y| = a \sum_{Y \in \mathcal{P} \setminus \Sigma} |Y| \geq ab|X| = \epsilon|X|,$$

a contradiction. □

We will also use the analogue of Lemma 2.1 for very sparse sets. It is obtained immediately by applying Lemma 2.1 to complements.

Lemma 2.2. *Let $a, b, \epsilon \in (0, 1)$ satisfy $ab = \epsilon$. Suppose $A \subseteq X$ and $|A| \leq \epsilon|X|$. For any partition $\mathcal{P}$ of X, if we let $\Sigma = \{Y \in \mathcal{P} : |A \cap Y| > a|Y|\}$, then $|\bigcup_{Y \in \Sigma} Y| \leq b|X|$.*

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The second important lemma is a "symmetry lemma" for 3-partite 3-graphs We begin by stating Lemma 2.4 below, which is a simpler symmetry lemma in the setting of graphs. This will involve an adaptation of the notion of "good sets", first defined in [16].

Definition 2.3. Suppose $G = (A \cup B, E)$ is a bipartite graph and $\epsilon > 0$. We say a subset $X \subseteq B$ is almost ϵ-good with respect to G if for at least $(1-\epsilon)|A|$ many $a \in A$,

$$\frac{|N_G(a) \cap X|}{|X|} \in [0, \epsilon) \cup (1-\epsilon, 1].$$

The symmetry lemma for graphs, Lemma 2.4 below, says that if $G = (A \cup B, E)$ is a bipartite graph, and both A and B are almost ϵ-good with respect to G, then $d_G(A, B)$ is close to 0 or 1. A proof appears in [19] (see Lemma 4.9 there).

Lemma 2.4 (Symmetry Lemma for Graphs). *For all $0 < \epsilon < 1/4$ there is n such that the following holds. Suppose $G = (U \cup W, E)$ is a bipartite graph, $|U|, |W| \geq n$, and both U and W are almost ϵ-good sets in G. Then $d_G(U, W) \in [0, 2\epsilon^{1/2}) \cup (1 - 2\epsilon^{1/2}, 1]$.*

Our goal is to prove a ternary analogoue of Lemma 2.4. We begin with an analogue of Definition 2.3.

Definition 2.5. Suppose $H = (V_1 \cup V_2 \cup V_3, E)$ is a 3-partite 3-graph and $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. A set $X \subseteq V_{i_1}$ is almost ϵ-good with respect to H if at least $(1-\epsilon)|V_{i_2}||V_{i_3}|$ many $(y, z) \in V_{i_2} \times V_{i_3}$ satisfy

$$\frac{|N_H(yz) \cap X|}{|X|} \in [0, \epsilon) \cup (1-\epsilon, 1].$$

We now state and prove our symmetry lemma for 3-graphs, which says that a tripartite 3-graph in which all three parts are almost ϵ-good has density close to 0 or 1.

Lemma 2.6 (Symmetry Lemma for 3-graphs). *Let $0 < \epsilon < 1/9$, suppose $H = (V_1 \cup V_2 \cup V_3, E)$ is a tripartite 3-graph, and assume V_1, V_2, V_3 are each almost ϵ-good with respect to H. Then $d_H(V_1, V_2, V_3) \in [0, 4\epsilon^{1/16}) \cup (1 - 4\epsilon^{1/16}, 1]$.*

Proof. Given i, j, k satisfying $\{i, j, k\} = \{1, 2, 3\}$, define

$$E_{ij} = \{xy \in K_2[V_i, V_j] : |N_H(xy)| \leq \epsilon|V_k| \text{ or } |N_H(xy)| \geq (1-\epsilon)|V_k|\}.$$

Our assumption implies that for each $ij \in \binom{[3]}{2}$, $|E_{ij}| \geq (1-\epsilon)|V_i||V_j|$. Consider now the bipartite graph Γ with vertex set $V_1 \cup K_2[V_2, V_3]$ and edge set

$$E(\Gamma) = \{(x, yz) \in V_1 \cup K_2[V_2, V_3] : xyz \in H\}.$$

It suffices to show $d_\Gamma(V_1, K_2[V_2, V_3]) \in [0, 4\epsilon^{1/16}) \cup (1 - 4\epsilon^{1/16}, 1]$. We will show this by applying Lemma 2.4 to Γ. To do this, we need to verify the relevant hypotheses.

Since $|E_{23}| \geq (1-\epsilon)|K_2[V_2, V_3]|$, we already know V_1 is almost ϵ-good with respect to Γ. Our next goal is to show $K_2[V_2, V_3]$ is almost $2\epsilon^{1/4}$-good with respect to Γ. We begin by defining the following subsets of E_{12}.

$$E_{12}^1 = \{xy \in K_2[V_1, V_2] : |N_H(xy)| \geq (1-\epsilon)|V_3|\} \text{ and}$$

$$E_{12}^0 = \{xy \in K_2[V_1, V_2] : |N_H(xy)| \leq \epsilon|V_3|\}.$$

Note $E_{12} = E_{12}^1 \sqcup E_{12}^0$. Consider the following subset of V_1 (recall notation equation (1)).

$$V_1' = \{x \in V_1 : |N_{E_{12}}(x)| \geq (1-\epsilon^{1/2})|V_2| \text{ and } |N_{E_{13}}(x)| \geq (1-\epsilon^{1/2})|V_3|\}.$$

Since $|E_{12}| \geq (1-\epsilon)|V_1||V_2|$ and $|E_{13}| \geq (1-\epsilon)|V_1||V_3|$, we conclude by two applications of Lemma 2.1 that $|V_1'| \geq (1 - 2\epsilon^{1/2})|V_1|$.

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Claim 2.7. *For each $x \in V_1'$, either $|N_{E_{12}^1}(x)| \geq (1-\epsilon^{1/4})|V_2|$ or $|N_{E_{12}^0}(x)| \geq (1-\epsilon^{1/4})|V_2|$.*

Proof. Suppose towards a contradiction there is some $x \in V_1'$ so that

$$\max\{|N_{E_{12}^1}(x)|, |N_{E_{12}^0}(x)|\} < (1-\epsilon^{1/4})|V_2|.$$

Combining this with the fact that $x \in V_1'$, and since $N_{E_{12}}(x) = N_{E_{12}^0}(x) \cup N_{E_{12}^1}(x)$ by definition, we observe that

$$\min\{|N_{E_{12}^1}(x)|, |N_{E_{12}^0}(x)|\} \geq (\epsilon^{1/4} - \epsilon^{1/2})|V_2|.$$

To ease notation, set

$$V_2^1 = N_{E_{12}^1}(x) \text{ and } V_2^0 = N_{E_{12}^0}(x).$$

Then $V_2^1 \cup V_2^0 \subseteq V_2$ and $\min\{|V_2^1|, |V_2^0|\} \geq (\epsilon^{1/4} - \epsilon^{1/2})|V_2|$. Let H_x be the slice graph of H at x, i.e. the graph with vertex set $V(H)$ and edge set $\{yz \in \binom{V}{2} : xyz \in E(H)\}$. By definition of V_2^1 and V_2^0, the following hold.

$$|E(H_x) \cap K_2[V_2^1, V_3]| = \sum_{y \in V_2^1} |N_H(xy)| \geq (1-\epsilon)|V_2^1||V_3| \text{ and} \tag{2}$$

$$|K_2[V_2^0, V_3] \setminus E(H_x)| = \sum_{y \in V_2^0} |V_3 \setminus N_H(xy)| \geq (1-\epsilon)|V_2^0||V_3|. \tag{3}$$

Define

$$V_3^1 = \{z \in V_3 : |N_H(xz) \cap V_2^1| \geq (1-\epsilon^{1/2})|V_2^1|\} \text{ and}$$
$$V_3^0 = \{z \in V_3 : |V_2^0 \setminus N_H(xz)| \geq (1-\epsilon^{1/2})|V_2^0|\}.$$

By equations (2) and (3), and Lemma 2.1, we have $|V_3^1| \geq (1-\epsilon^{1/2})|V_3|$ and $|V_3^0| \geq (1-\epsilon^{1/2})|V_3|$. Consequently, using again that $x \in V_1'$, we have

$$|V_3^1 \cap V_3^0 \cap N_{E_{13}}(x)| \geq (1-3\epsilon^{1/2})|V_3| > 0,$$

where the last inequality is because $\epsilon < 1/9$. From this, we conclude there exists some $z \in V_3^1 \cap V_3^0 \cap N_{E_{13}}(x)$. Since $z \in V_3^1$,

$$|N_H(xz)| \geq (1-\epsilon^{1/2})|V_2^1| \geq (1-\epsilon^{1/2})(\epsilon^{1/4} - \epsilon^{1/2})|V_2| > \epsilon|V_2|,$$

where the last inequality uses that ϵ is sufficiently small. On the other hand, since $z \in V_3^0$,

$$|V_2 \setminus N_H(xz)| \geq (1-\epsilon^{1/2})|V_2^0| \geq (1-\epsilon^{1/2})(\epsilon^{1/4} - \epsilon^{1/2})|V_2| > \epsilon|V_2|.$$

But this contradicts that $xz \in E_{13}$. This finishes the proof of Claim 2.7. □

By Claim 2.7, we have that for every $x \in V_1'$, either $|N_{E_{12}^0}(x)| \geq (1-\epsilon^{1/4})|V_2|$, or $|N_{E_{12}^1}(x)| \geq (1-\epsilon^{1/4})|V_2|$. Given $x \in V_1'$, if $|N_{E_{12}^1}(x)| \geq (1-\epsilon^{1/4})|V_2|$, then

$$|N_H(x) \cap K_2[V_2, V_3]| \geq \sum_{y \in N_{E_{12}^1}(x)} |N_H(xy)| \geq |N_{E_{12}^1}(x)|(1-\epsilon)|V_3| \geq (1-\epsilon^{1/4})(1-\epsilon)|V_2||V_3|$$
$$> (1-2\epsilon^{1/4})|V_2||V_3|.$$

On the other hand, if $x \in V_1'$ and $|N_{E_{12}^0}(x)| \geq \epsilon^{1/4}|V_2|$, then a similar argument shows

$$|K_2[V_2, V_3] \setminus N_H(x)| > (1-2\epsilon^{1/4})|V_2||V_3|.$$

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We have shown that $|V_1'| \geq (1-2\epsilon^{1/2})$, and for all $x \in V_1'$, either $|N_\Gamma(x)| > (1-2\epsilon^{1/4})|V_2||V_3|$ or $|N_\Gamma(x)| < 2\epsilon^{1/4}|V_2||V_3|$. In other words, $K_2[V_2, V_3]$ is almost $2\epsilon^{1/4}$-good in Γ. We can thus conclude, by Lemma 2.4, that $d_\Gamma(V_1, K_2[V_2, V_3]) \in [0, 4\epsilon^{1/16}) \cup (1-4\epsilon^{1/16}, 1]$. This finishes the proof. □

2.2 VC-dimension and homogeneous partitions of graphs

In this section we discuss VC-dimension and homogeneous partitions for graphs. We begin with the definition of VC-dimension of a graph.

Definition 2.8. The VC-dimension of a graph $G = (V, E)$, denote $\mathrm{VC}(G)$, is the largest k so that there exist vertices $\{a_i : i \in [k]\} \cup \{b_S : S \subseteq [k]\} \subseteq V$, so that for all $i \in [k]$ and $S \subseteq [k]$, $a_i b_S \in E$ if and only if $i \in S$.

We now define homogeneous partitions for graphs.

Definition 2.9. Suppose $G = (V, E)$ is an graph.

(1) For subsets $X, Y \subseteq V$, the pair (X, Y) is ϵ-homogeneous with respect to G if $d_G(X, Y) \in [0, \epsilon) \cup (1-\epsilon, 1]$.

(2) We say a partition $\mathcal{P}$ of V is ϵ-homogeneous with respect to G if for at least $(1-\epsilon)|V|^2$ of the tuples $(x, y) \in V^2$, the unique pair $(X, Y) \in \mathcal{P}^2$ containing (x, y) is ϵ-homogeneous.

As mentioned in the introduction, work of Alon–Fischer–Newman, Lovász–Szegedy, and Fox–Pach–Suk have shown that graphs of small VC-dimension admit homogeneous partitions with efficient bounds (see Theorem 1.1 in the introduction). We will require a slightly more complicated version of Theorem 1.1, which allows us to find homogeneous equipartitions of bipartite graphs which also refine a distinguished bipartition. The proof follows from Theorem 1.1 via standard arguments (which we include in the appendix for completeness).

Theorem 2.10. *For all $k \geq 1$ there exist constants $C = C(k)$ and $D(k) = 4k+4$ so that for all sufficiently small $\epsilon > 0$ the following holds. Suppose $H = (A \cup B, E)$ is a bipartite graph with VC-dimension less than k and $|A| = |B|$. Then there exists an equipartition $\mathcal{P}$ of $A \cup B$ refining $\{A, B\}$ which is ϵ-homogeneous for H and so that $\epsilon^{-1} \leq |\mathcal{P}| \leq C\epsilon^{-D}$.*

2.3 Slicewise VC-dimension and homogeneous partitions of 3-graphs

We begin with the definition of a "slice graph" associated to a 3-graph.

Definition 2.11. Suppose H is a 3-graph. For each $x \in V(H)$, the slice graph of H at x, denoted H_x, is the graph with vertex set $V(H_x) = V(H)$ and edge set

$$E(H_x) = \left\{ yz \in \binom{V(H)}{2} : xyz \in E(H) \right\}.$$

We now define the slicewise VC-dimension of a 3-graph to be the maximum VC-dimension of its associated slice graphs.

Definition 2.12. Suppose $H = (V, E)$ is a 3-graph. The slicewise VC-dimension of H is

$$\mathrm{SVC}(H) := \max\{\mathrm{VC}(H_x) : x \in V\}.$$

We now define homogeneous partitions for 3-graphs in analogy to Definition 2.9.

Definition 2.13. Suppose $H = (V, E)$ is a 3-graph.

(1) Given subsets $X, Y, Z \subseteq V$, we say the triple (X, Y, Z) is ϵ-homogeneous with respect to H if $d_H(X, Y, Z) \in [0, \epsilon) \cup (1-\epsilon, 1]$.

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(2) We say a partition $\mathcal{P}$ of V is ϵ-homogeneous with respect to H if

$$\left| \bigcup_{(X,Y,Z)\in\mathcal{P}'} K_3[X,Y,Z] \right| \geq (1-\epsilon)|V|^3,$$

where $\mathcal{P}'$ is the set triples from $\mathcal{P}^3$ which are ϵ-homogeneous with respect to H.

It is known that 3-graphs of small slicewise VC-dimension admit homogeneous partitions (see Theorem 1.3 in the introduction). The goal of this paper is to prove they admit homogeneous partitions with a relatively small number of parts. A crucial tool in this proof will be the notion of an almost ϵ-good partition, which we now define (recall Definition 2.5 as well).

Definition 2.14. Suppose $H = (V_1 \cup V_2 \cup V_3, E)$ is a 3-partite 3-graph and $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. A partition $\mathcal{P}$ of V_{i_1} is almost ϵ-good with respect to H if

$$\left| \bigcup_{X\in\mathcal{P}'} X \right| \geq (1-\epsilon)|V_{i_1}|^3,$$

where $\mathcal{P}' = \{X \in \mathcal{P} : X \text{ is almost } \epsilon - \text{good with respect to } H\}$.

We now extend Definition 2.14 to partitions of the whole vertex set of a tripartite 3-graph.

Definition 2.15. Suppose $H = (V_1 \cup V_2 \cup V_3, E)$ is a 3-partite 3-graph. We say a partition $\mathcal{P}$ of $V(H)$ refining $\{V_1, V_2, V_3\}$ is almost ϵ-good with respect to H if $\mathcal{P}|_{V_1}$, $\mathcal{P}|_{V_2}$ and $\mathcal{P}|_{V_3}$ are almost ϵ-good with respect to H.

We now show that an almost ϵ-good partition of a tripartite 3-graph is automatically homogeneous. This relies crucially on our symmetry lemma, Lemma 2.6.

Proposition 2.16. *Let $0 < \epsilon < 3^{-9}$. Suppose $H = (V_1 \cup V_2 \cup V_3, E)$ is a tripartite 3-graph, and $\mathcal{P}$ is a partition of $V(H)$ refining $\{V_1, V_2, V_3\}$ which is an almost ϵ-good partition with respect to H. Then $\mathcal{P}$ is $5\epsilon^{1/64}$-homogeneous with respect to H.*

Proof. To ease notation, let $\mathcal{P}_1 = \mathcal{P}|_{V_1}$, $\mathcal{P}_2 = \mathcal{P}|_{V_2}$, and $\mathcal{P}_3 = \mathcal{P}|_{V_3}$. Define

$$\Gamma_1 = \bigcup_{Y\in\mathcal{P}_1} \left\{ xyz \in K_2[V_1,V_2,V_3] : \frac{|N_H(yz)\cap Y|}{|Y|} \in [0,\epsilon)\cup(1-\epsilon,1] \right\},$$

$$\Gamma_2 = \bigcup_{Y\in\mathcal{P}_2} \left\{ xyz \in K_2[V_1,V_2,V_3] : \frac{|N_H(xz)\cap Y|}{|Y|} \in [0,\epsilon)\cup(1-\epsilon,1] \right\}, \text{ and}$$

$$\Gamma_3 = \bigcup_{Y\in\mathcal{P}_3} \left\{ xyz \in K_2[V_1,V_2,V_3] : \frac{|N_H(xy)\cap Y|}{|Y|} \in [0,\epsilon)\cup(1-\epsilon,1] \right\}.$$

By assumption, each of Γ_1, Γ_2, and Γ_3 have size at least $(1-\epsilon)|V_1||V_2||V_3|$. Thus if we set

$$\Gamma := \Gamma_1 \cap \Gamma_2 \cap \Gamma_3,$$

then $|\Gamma| \geq (1-3\epsilon)|V_1||V_2||V_3|$. Let

$$\Sigma = \{(X,Y,Z) \in \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3 : |K_3[X,Y,Z]\cap\Gamma| \geq (1-\sqrt{3\epsilon})|X||Y||Z|\}.$$

By the lower bound above for $|\Gamma|$ and Lemma 2.1,

$$\left| \bigcup_{(X,Y,Z)\in\Sigma} K_3[X,Y,Z] \right| \geq (1-\sqrt{3\epsilon})|V_1||V_2||V_3|.$$

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Thus it suffices to show that each $(X, Y, Z) \in \Sigma$ is $5\epsilon^{1/64}$-homogeneous with respect to H. To this end, fix a triple $(X, Y, Z) \in \Sigma$. Let

$$E_{XY} = \{xy \in K_2[X, Y] : |N_\Gamma(xy) \cap Z| \geq (1 - (3\epsilon)^{1/4})|Z|\}.$$

Since $(X, Y, Z) \in \Sigma$, Lemma 2.1 implies $|E_{XY}| \geq (1 - (3\epsilon)^{1/4})|X||Y|$. For all $xy \in E_{XY}$, since $N_\Gamma(xy) \cap Z \neq \emptyset$, we have by definition of Γ that $|N_H(xy) \cap Z|/|Z| \in [0, \epsilon) \cup (1 - \epsilon, 1]$. Combining with the lower bound on E_{XY}, we have shown that Z is almost $(3\epsilon)^{1/4}$-good with respect to $H[X, Y, Z]$. A symmetric argument shows both X and Y are also almost $(3\epsilon)^{1/4}$-good with respect to $H[X, Y, Z]$. By Lemma 2.6, and since $4((3\epsilon)^{1/4})^{1/16} \leq 5\epsilon^{1/64}$, we have that (X, Y, Z) is $5\epsilon^{1/64}$-homogeneous with respect to H, as desired. □

2.4 Tripartite decompositions

In this section we define tripartite decompositions. These will play an important role in the proof of Theorem 4.1. They are essentially tripartite versions of more general decompositions typically used in a strong form of regularity for 3-graphs. Our notation is largely adapted from [10].

We begin by introducing the notion of a triad, which for us will be simply any tripartite graph.

Definition 2.17. A triad is a tripartite graph.

Given a triad $G = (X \cup Y \cup Z, E)$, we let $K_3^{(2)}(G)$ denote the set of triangles from G, i.e. we define

$$K_3^{(2)}(G) = \{xyz \in K_3[X, Y, Z] : xy, yz, xz \in E\}.$$

The notion of density of a triad, relative to a 3-graph, is crucial for our proofs.

Definition 2.18. Suppose H is a 3-graph, X, Y, Z are disjoint subsets of $V(H)$, and $G = (X \cup Y \cup Z, E)$ is a triad. Define

$$d_H(G) := \frac{|E(H) \cap K_3^{(2)}(G)|}{|K_3^{(2)}(G)|}.$$

We now define homogeneous triads relative to a 3-graph.

Definition 2.19. Suppose H is a 3-graph, X, Y, Z are disjoint subsets of $V(H)$, and $G = (X \cup Y \cup Z, E)$ is a triad. We say G is ϵ-homogeneous with respect to H if

$$d_H(G) \in [0, \epsilon) \cup (1 - \epsilon, 1].$$

We now define the notion of a tripartite decomposition.

Definition 2.20. Suppose $V = A \sqcup B \sqcup C$. A tripartite (A, B, C)-decomposition of V is a tuple $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ satisfying the following.

- $\mathcal{P}_1 = (\mathcal{P}_A, \mathcal{P}_B, \mathcal{P}_C)$, where $\mathcal{P}_A, \mathcal{P}_B, \mathcal{P}_C$ are partitions of of A, B, and C respectively,
- $\mathcal{P}_2 = \{\mathcal{P}_{XY} : (X, Y) \in (\mathcal{P}_A \times \mathcal{P}_B) \cup (\mathcal{P}_A \times \mathcal{P}_C) \cup (\mathcal{P}_B \times \mathcal{P}_C)\}$, where for each $(X, Y) \in (\mathcal{P}_A \times \mathcal{P}_B) \cup (\mathcal{P}_A \times \mathcal{P}_C) \cup (\mathcal{P}_B \times \mathcal{P}_C)$, $\mathcal{P}_{XY}$ is a partition of $K_2[X, Y]$.

In Definition 2.20, we will sometimes call $\mathcal{P}$ simply a *tripartite decomposition of* V when the relevant partition $A \sqcup B \sqcup C$ is clear from context. Tripartite decompositions come equipped with an associated collection of triads, which we now define.

Definition 2.21. Suppose $V = A \sqcup B \sqcup C$ and $\mathcal{P}$ is a tripartite (A, B, C)-decomposition of V.

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- A triad of $\mathcal{P}$ is a tripartite graph of the form
$$(X \cup Y \cup Z, P \cup Q \cup R),$$
where $(X, Y, Z) \in \mathcal{P}_A \times \mathcal{P}_B \times \mathcal{P}_C$, and $P \in \mathcal{P}_{XY}$, $Q \in \mathcal{P}_{XZ}$, and $R \in \mathcal{P}_{YZ}$.
- $\mathrm{Tri}(\mathcal{P})$ denotes the set of triads of $\mathcal{P}$.

Using the notation of Definition 2.21, we see that a tripartite decomposition $\mathcal{P}$ induces a partition
$$K_3[A, B, C] = \bigcup_{G \in \mathrm{Tri}(\mathcal{P})} K_3^{(2)}(G).$$
We now define homogeneous tripartite decompositions.

Definition 2.22. Suppose $V = A \sqcup B \sqcup C$, $H = (A \cup B \cup C, E)$ is a tripartite 3-graph, and $\mathcal{P}$ is a tripartite (A, B, C)-decomposition of V. We say $\mathcal{P}$ is ϵ-homogeneous with respect to H if
$$\sum_{G \in \Sigma} |K_3^{(2)}(G)| \geq (1 - \epsilon)|A||B||C|,$$
where $\Sigma = \{G \in \mathrm{Tri}(\mathcal{P}) : G \text{ is } \epsilon - \text{homogeneous with respect to } H\}$.

We note that there are major differences between a homogeneous tripartite decomposition in the sense of Definition 2.22 and a homogeneous partition in the sense of Definition 2.13. In particular, Definition 2.13 deals only with triples of sets from a vertex partition, while Definition 2.22 deals with tripartite graphs built from partitions of both vertices and pairs of vertices.

3. A structure theorem for edge-coloured bipartite graphs

This section contains a required result about bipartite edge-coloured graphs. As motivation, we begin by discussing one way in which these objects will show up in our proof of Theorem 4.1.

At some point in the proof of Theorem 1.4, we will apply the Theorem 2.10 to a bipartite graph $G = (A \cup B, E)$ with VC-dimension less than k. This will yield an ϵ-homogeneous equipartition of the form $A_1 \cup \cdots \cup A_t \cup B_1 \cup \cdots \cup B_t$. Such a decomposition naturally gives rise to a "reduced graph" with vertex set $\{a_1, \ldots, a_t, b_1, \ldots, b_t\}$, and with three types of edges: the pairs $a_i b_j$ where $d_G(A_i, B_j) \geq 1 - \epsilon$, the pairs $a_i b_j$ where $d_G(A_i, B_j) \leq \epsilon$, and the pairs $a_i b_j$ where (A_i, B_j) is not ϵ-homogeneous with respect to G. Making this precise is part of the motivation for the following definition.

Definition 3.1. A bipartite edge-coloured graph is a tuple $G = (A \cup B, E_0, E_1, \ldots, E_r)$, where $r \geq 1$ and $K_2[A, B] = E_0 \sqcup \cdots \sqcup E_r$. We refer to elements of $A \cup B$ as vertices, and the sets $E_0, \cdots, E_r$ as edge colours.

Given a bipartite edge-coloured graph $G = (A \cup B, E_0, E_1, \ldots, E_r)$, we will use similar notation as that laid out for graphs in Subsection 1.1. In particular, for $0 \leq u \leq r$ and $x \in A \cup B$, the *E_u-neighbourhood of x* is the set
$$N_{E_u}(x) := \{y \in A \cup B : xy \in E_u\}.$$
Given subsets $X \subseteq A$ and $Y \subseteq B$, we let $E_u[X, Y] = E_u \cap K_2[X, Y]$. We then let $G[X, Y]$ be the bipartite edge-coloured graph with vertex set $X \cup Y$, and edge colours $E_u[X, Y]$ for $0 \leq u \leq r$.

As discussed above, one way bipartite edge-coloured graphs will arise in our proof is as reduced graphs associated to homogeneous partitions of bipartite graphs of small VC-dimension. In this case, there will be 3 edge colorus: E_0 (the "sparse" colour), E_1 (the "dense" colour), and E_2 (the "error" colour). In this context, the resulting edge-coloured graph will omit certain special bipartite configurations. The bipartite configuration we are interested in is the following.

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Definition 3.2. Given $k \geq 1$, define $U(k)$ to be the bipartite graph

$$(\{a_i : i \in [k]\} \cup \{b_S : S \subseteq [k]\}, \{a_i b_S : i \in S\}).$$

We now give a definition of what it means to find a copy of $U(k)$ inside an the kinds of edge-coloured graphs we are interested in.

Definition 3.3. Let $G = (A \cup B, E_0, \ldots, E_r)$ be a bipartite edge-coloured graph, and let $0 \leq u \neq v \leq r$. An E_u/E_v-copy of $U(k)$ with right side in B consists of vertices $\{x_1, \ldots, x_k\} \subseteq A$ and $\{y_S \subseteq [k]\} \subseteq B$ so that for each $1 \leq i \leq k$ and $S \subseteq [k]$, if $i \in S$ then $x_i y_S \in E_u$ and if $i \notin S$, then $x_i y_S \in E_v$.

A E_u/E_v-copy of $U(k)$ in G is a copy of $U(k)$ in G with right side in A or with right side in B.

If $G = (A \cup B, E_0, E_1, E_2)$ is a bipartite edge-coloured graph containing no E_0/E_1-copies of $U(k)$ and with E_2 is small in size, then one can prove a structure theorem for G (such hypotheses will hold when G arises in the way discussed above). Our next goal is to state such a structure theorem. For this we will need the following definition.

Definition 3.4. Let $\epsilon > 0$ and suppose $G = (A \cup B, E_0, E_1, E_2)$ is a bipartite edge-coloured graph. Suppose $\mathcal{P} = (\mathcal{P}_A, \mathcal{P}_B)$, where $\mathcal{P}_A$ is a partition of A, and $\mathcal{P}_B$ is a partition of B. We say $\mathcal{P}$ is ϵ-homogeneous for G with respect to $\{E_0, E_1\}$ if

$$\sum_{(X,Y) \in \Sigma_0 \cup \Sigma_1} |K_2[X, Y]| \geq (1 - \epsilon)|A||B|,$$

where for each $\alpha \in \{0, 1\}$,

$$\Sigma_\alpha = \{(X, Y) \in \Sigma_\alpha : |E_\alpha \cap K_2[X, Y]| \geq (1 - \epsilon)|X||Y|\}.$$

The size of $\mathcal{P}$ is $\max\{|\mathcal{P}_A|, |\mathcal{P}_B|\}$.

We now state the structure theorem for edge-coloured graphs which will need in our proof of Theorem 1.4.

Proposition 3.5. *There is $\delta_0 > 0$ so that for all $k \geq 1$, there are $C_1 = C_1(k) > 0$ and $C_2 = C_2(k) \geq 1$ such that the following holds. Suppose $0 < \delta < \delta_0$ and $0 < \epsilon \leq C_1 \delta^{4k+4}$. Suppose $G = (A \cup B, E_0, E_1, E_2)$ is a bipartite edge-coloured graph where $|E_2| \leq \epsilon |A||B|$, and such that there is no E_0/E_1-copy of $U(k)$ in G.*

Then there are partitions $\mathcal{P}_A$ of A and $\mathcal{P}_B$ of B so that $\mathcal{P} = (\mathcal{P}_A, \mathcal{P}_B)$ is $\delta^{1/16}$-homogeneous for G with respect to $\{E_0, E_1\}$ and so that $\mathcal{P}$ has size at most $C_2 \delta^{-k}$.

Proposition 3.5 can be deduced from results of Alon, Fischer, and Newman from [2] (with slightly different bounds). In particular, Proposition 3.5 follows from an asymmetric version of Lemma 1.6 of [2], which the authors state also holds in an asymmetric setting via an identical proof.

For the sake of completeness, and because some amount of translation is required to move from [2] to Proposition 3.5, we also include a proof in this paper. The proof uses an analogue of the packing lemma for edge-coloured graphs, adapted from an application in [9]. The usual Haussler packing lemma says that for any graph $G = (V, E)$ of small VC-dimension, there exist a small number of vertices $x_1, \ldots, x_s$ so that every $v \in V$ looks like one of the x_i, in the sense that $N(v) \Delta N(x_i)$ is small (for more on this, see [9]). In the setting of bipartite edge-coloured graphs, something similar holds with respect to the "non-error" edge colours. A lemma to this effect was proved in [17].

Lemma 3.6 (Lemma 2.14 in [17]). *For all $k \geq 1$ there is a constants $c_1 = c_1(k) > 0$ and $c_2 = c_2(k) \geq 1$ so that the following holds. Suppose $\epsilon, \delta > 0$ satisfy $\epsilon \leq c_1 \delta^{2k+2}$. Assume $G = (A \cup B, E_0, E_1, E_2)$ is a bipartite edge-coloured graph such that there is no E_0/E_1-copy of $U(k)$ in G, and such that $|E_2| \leq \epsilon |A||B|$.*

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Then there is an integer $m \leq c_2\delta^{-k}$, *a subset* $U \subseteq A$ *with* $|U| \leq \sqrt{\epsilon}|A|$, *and vertices* $x_1, \dots, x_m \in A \setminus U$ *so that for all* $a \in A \setminus U$, $|N_{E_2}(a)| \leq \sqrt{\epsilon}|B|$ *and for some* $1 \leq i \leq m$, $\max\{|N_{E_1}(a)\Delta N_{E_1}(x_i)|, |N_{E_0}(a)\Delta N_{E_0}(x_i)|\} \leq \delta|B|$.

Using a proof similar to [9], we now use Lemma 3.6 to prove Proposition 3.5.

Proof of Proposition 3.5. Set $\delta_0 = (1/12)^{16/7}$. Let $c_1 = c_1(k)$ and $c_2 = c_2(k)$ be as in Lemma 3.6. Then set $C_1 = (c_1/2)^4 c_2^{-4}$ and $C_2 = c_2$. Fix $0 < \delta < \delta_0$ and $0 < \epsilon < C_1\delta^{4k+4}$. Suppose $G = (A \cup B, E_0, E_1, E_2)$ is a bipartite edge-coloured graph where $|E_2| \leq \epsilon|A||B|$, and such that there is no E_0/E_1-copy of $U(k)$ in G.

By Lemma 3.6, there is an integer $m \leq c_2\delta^{-k}$, a subset $U \subseteq A$ with $|U| \leq \sqrt{\epsilon}|A|$, and vertices $x_1, \dots, x_m \in A \setminus U$ so that for all $a \in A \setminus U$, $|N_{E_2}(a)| \leq \sqrt{\epsilon}|B|$ and for some $1 \leq i \leq m$, $\max\{|N_{E_1}(a)\Delta N_{E_1}(x_i)|, |N_{E_0}(a)\Delta N_{E_0}(x_i)|\} \leq \delta|B|$. Let

$$B_{\text{err}} = \bigcup_{i=1}^{m} N_{E_2}(x_i).$$

Note $|B_{\text{err}}| \leq m\sqrt{\epsilon}|B| < \delta|B|$, where the second inequality is due to the upper bound on m and our assumption on ϵ. To ease notation, let $B' = B \setminus B_{\text{err}}$ and $A' = A \setminus U$.

We now consider the restriction $G[A', B'] = (A' \cup B', E_0[A', B], E_1[A', B'], E_2[A', B'])$. Note that

$$|E_2[A', B']| \leq |E_2[A', B]| \leq \sum_{x \in A'} |N_{E_2}(a)| \leq \sqrt{\epsilon}|B| \leq 2\sqrt{\epsilon}|B'|,$$

where the last inequality is because $|B'| \geq (1-\delta)|B|$, and $\delta < 1/2$. Clearly $G[A', B']$ also contains no $E_0[A', B']/E_1[A', B']$-copy of $U(k)$. Therefore, we can apply Lemma 3.6 to $G[A', B']$ with parameters k, $2\sqrt{\epsilon}$ and δ. This yields an integer $p \leq c_2\delta^{-k}$, a subset $V \subseteq B'$ with $|V| \leq 2\epsilon^{1/4}|B'|$, and vertices $y_1, \dots, y_p \in B' \setminus V$ so that for all $b \in B' \setminus V$, $|N_{E_2}(b)| \leq 2\epsilon^{1/4}|A'|$ and for some $1 \leq i \leq p$,

$$\max\{|(N_{E_1}(b)\Delta N_{E_1}(y_i)) \cap A'|, |(N_{E_0}(b)\Delta N_{E_0}(y_i)) \cap A'|\} \leq \delta|A'|.$$

We now define partitions of A and B. First, for each $b \in B' \setminus V$, let

$$g(b) = \min\{j \in [p] : \max\{|N_{E_1}(b)\Delta N_{E_1}(y_j)|, |N_{E_0}(b)\Delta N_{E_0}(y_j)|\} \leq \delta|A'|\}.$$

In other words, $g(b)$ is the least index j so that the E_0 and E_1 neighbourhoods of b are close to those of y_j. Then let $B_0 = B_{\text{err}} \cup V$, and for each $j \in [p]$, define

$$B_j = \{b \in B \setminus B_0 : g(b) = j\}.$$

This yields a partition $\mathcal{P}^B = \{B_0, B_1, \dots, B_p\}$ of B. Note that by construction,

$$|B_0| \leq (\delta + 2\epsilon^{1/4})|B| < 2\delta|B|,$$

where the last inequality is by assumption on ϵ. Define

$$A_0 = U \cup \bigcup_{i=1}^{p} N_{E_2}(y_i).$$

Observe $|A_0| \leq \sqrt{\epsilon}|A| + 2p\epsilon^{1/4}|A'| \leq 2\delta|A|$, where the inequality is by the bound on p, and our assumptions on ϵ and δ. Then, for each $a \in A \setminus A_0$, let

$$f(a) = \min\{i \in [m] : \max\{|N_{E_1}(a)\Delta N_{E_1}(x_i)|, |N_{E_0}(a)\Delta N_{E_0}(x_i)|\} \leq \delta|B|\}.$$

In other words, $f(a)$ is the least index i so that the E_1 and E_0 neighbourhoods of a are close to those of x_i. Now for each $i \in [m]$, let

$$A_i = \{a \in A \setminus A_0 : f(a) = i\}.$$

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We now have a partition $\mathcal{P}^A = \{A_0, A_1, \ldots, A_m\}$ of A. We show $\mathcal{P} = (\mathcal{P}_A, \mathcal{P}_B)$ has the desired properties. To ease notation in the verification, we let $A'' = A \setminus A_0$ and $B'' = B \setminus B_0$. Note by above $|A''| \geq (1-2\delta)|A|$ and $|B''| \geq (1-2\delta)|B|$. For each $1 \leq i \leq m$ and $x \in A_i$, let

$$\Gamma_A(x) = \Big(N_{E_2}(x) \cup (N_{E_0}(x) \Delta N_{E_0}(x_i)) \cup (N_{E_1}(x) \Delta N_{E_1}(x_i))\Big) \cap B''.$$

By construction, for all $x \in A''$,

$$|\Gamma_A(x)| \leq \sqrt{\epsilon}|B| + 2\delta|B| \leq 8\delta|B''|,$$

where the second inequality is by the lower bound on B'' and our assumptions on ϵ, δ. Now let $\Gamma_A = \{(x,y) : x \in A'', y \in \Gamma_A(x)\}$. By above, $|\Gamma_A| \leq 8\delta|A''||B''|$. Similarly, for each $1 \leq j \leq p$ and $y \in B_j$, let

$$\Gamma_B(y) = \Big((N_{E_0}(y) \Delta N_{E_0}(b_j)) \cup (N_{E_1}(y) \Delta N_{E_1}(b_j))\Big) \cap A''.$$

By construction, we have that for all $y \in B''$, $|\Gamma_B(y)| \leq 2\delta|A''|$. Now set

$$\Gamma_B = \{(x,y) : y \in B'', x \in \Gamma_B(y)\},$$

and observe that $|\Gamma_B| \leq 2\delta|A''||B''|$. Setting, $\Gamma = \Gamma_A \cup \Gamma_B$, we have by above that $|\Gamma| \leq 10\delta|A''||B''|$. We now define three sets of pairs from $\mathcal{P}_A \times \mathcal{P}_B$, with the aim of collecting "bad" pairs of parts. Informally, Σ_1 contains the pairs involving an error part, Σ_2 contains the pairs involving too many "error edges", and Σ_3 contains pairs involving a part which is too small.

$$\Sigma_1 = \{(A_i, B_j) \in \mathcal{P}_A \times \mathcal{P}_B : i = 0 \text{ or } j = 0\},$$

$$\Sigma_2 = \{(A_i, B_j) \in \mathcal{P}_A \times \mathcal{P}_B : |\Gamma \cap K_2[A_i, B_j]| > 2\sqrt{\delta}|A_i||B_j|\} \text{ and}$$

$$\Sigma_3 = \left\{(A_i, B_j) \in \mathcal{P}_A \times \mathcal{P}_B : |A_i| < \frac{2\sqrt{\delta}|A|}{m} \text{ or } |B_j| < \frac{2\sqrt{\delta}|B|}{p}\right\}.$$

We observe that the upper bounds on $|A_0|$ and $|B_0|$ imply

$$\left|\bigcup_{(X,Y)\in\Sigma_1} K_2[X,Y]\right| \leq |A_0||B| + |A||B_0| \leq 4\delta|A||B|.$$

Since $|\Gamma| \leq 10\delta|A''||B''|$, Lemma 2.2 implies

$$\left|\bigcup_{(X,Y)\in\Sigma_2} K_2[X,Y]\right| \leq 5\sqrt{\delta}|A||B|.$$

Finally, it is straightforward to see that by definition of Σ_3,

$$\left|\bigcup_{(X,Y)\in\Sigma_3} K_2[X,Y]\right| \leq 4\sqrt{\delta}|A||B|.$$

Setting $\Sigma = (\mathcal{P}_A \times \mathcal{P}_B) \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$, we have by the inequalities above that

$$\left|\bigcup_{(X,Y)\in\Sigma} K_2[X,Y]\right| \geq (1 - 13\sqrt{\delta})|A||B| > (1 - \delta^{1/16})|A||B|,$$

where the last inequality is because δ is sufficiently small. Thus it suffices to show that for every $(X,Y) \in \Sigma$, there is some $\alpha \in \{0,1\}$ so that $|E_\alpha \cap K_2[X,Y]| > (1-\delta^{1/16})|X||Y|$.

To this end, fix some $(X,Y) \in \Sigma$. By construction, there are some $1 \leq i \leq m$ and $1 \leq j \leq p$ so that $X = A_i$ and $Y = B_j$. Define

$$Y_0 = N_{E_0}(x_i) \cap B_j \text{ and } Y_1 = N_{E_1}(x_i) \cap B_j.$$

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Note that by construction, $N_{E_2}(x_i) \subseteq B_0$), so $B_j = Y_0 \sqcup Y_1$. Since $(X, Y) \in \Sigma$, we have that for each $\alpha \in \{0, 1\}$,

$$|E_\alpha \cap K_2[A_i, Y_\alpha]| \geq |A_i||Y_\alpha| - |K_2[A_i, Y_\alpha] \cap \Gamma_A| \geq |A_i||Y_\alpha| - 2\delta^{1/2}|A_i||B_j|.$$

Consequently, if we know there is some $\alpha \in \{0, 1\}$ so that $|Y_\alpha| \geq (1 - \delta^{1/8})|B_j|$, then we are done, as the above would then imply

$$|E_\alpha \cap K_2[A_i, Y_\alpha]| \geq (1 - 2\delta^{1/2} - \delta^{1/8})|A_i||B_j| > (1 - \delta^{1/16})|A_i||B_j|.$$

Thus, it suffices to show $\max\{|Y_1|, |Y_0|\} \geq (1 - \delta^{1/8})|B_j|$. Assume towards a contradiction $\max\{|Y_1|, |Y_0|\} < (1 - \delta^{1/8})|B_j|$. Note this implies $\min\{|Y_1|, |Y_0|\} > \delta^{1/8}|B_j|$, since, as mentioned above, $B_j = Y_0 \sqcup Y_1$. Since $|K_2[A_i, B_j] \setminus \Gamma| \geq (1 - 2\delta^{1/2})|A_i||B_j|$, if we let

$$Y' = \{y \in B_j : |A_i \setminus N_\Gamma(y)| \geq (1 - \delta^{1/4})|A_i|\},$$

then by Lemma 2.1,

$$|Y'| \geq (1 - 2\delta^{1/4})|B_j|.$$

Combining with the lower bounds on the sizes of Y_1 and Y_0, the pigeon hole property tells us there exist some $b_0 \in Y' \cap Y_0$ and some $b_1 \in Y' \cap Y_1$. We now have that $x_i b_0 \in E_0$ and $x_i b_1 \in E_1$. However, we also have $x_i b_0, x_i b_1 \in K_2[A_i, B_j] \setminus \Gamma_B$, which implies

$$x_i b_0 \in E_0 \Leftrightarrow x_i y_j \in E_0 \Leftrightarrow x_i b_1 \in E_0.$$

Since $x_i b_1 \in E_0$ implies $x_i b_1 \notin E_1$, we have now arrived at a contradiction. □

4. Proof of the main theorem

In this section we will prove Theorem 1.4. It will follow from the following slightly less general statement.

Theorem 4.1. *For all $k \geq 1$ there are $K, \epsilon^* > 0$ so that for all $0 < \tau < \epsilon^*$ the following holds. Suppose $H = (U \cup V \cup W, F)$ is a sufficiently large tripartite 3-graph with $|U| = |V| = |W|$ and with slicewise VC-dimension less than k. Then there exists an τ-homogeneous partition $\mathcal{P}$ for H of size at most $2^{2^{\tau^{-K}}}$.*

Moreover, $\mathcal{P}$ has the form $\mathcal{P}_U \cup \mathcal{P}_V \cup \mathcal{P}_W$, where $\mathcal{P}_U, \mathcal{P}_V, \mathcal{P}_W$ are almost τ^{64}/5-good partitions of U, V, W, respectively.

We can easily deduce Theorem 1.4 from Theorem 4.1.

Proof of Theorem 1.4 from Theorem 4.1. Let K, ϵ^* be as in Theorem 4.1 and fix $0 < \epsilon < \min\{\sqrt{\epsilon^*}, 1/4\}$. Suppose $H = (V, E)$ is a sufficiently large 3-graph with slicewise VC-dimension less than k. Let H' be the 3-partite 3-graph $(A \cup B \cup C, E(H'))$, where $A = \{a_v : v \in V\}$, $B = \{b_v : v \in V\}$ and $C = \{c_v : v \in V\}$, and

$$E(H') = \{a_v b_{v'} c_{v''} : vv'v'' \in E\}.$$

Note H' has slicewise VC-dimension less than k, and $|A| = |B| = |C|$. By Theorem 4.1, there is an ϵ^2-homogeneous partition $\mathcal{P} = \mathcal{P}_A \cup \mathcal{P}_B \cup \mathcal{P}_C$ of H' of size $t \leq 2^{2^{\epsilon^{-2K}}}$. Say $\mathcal{P}_A = \{A_1, \ldots, A_{t_1}\}$, $\mathcal{P}_B = \{B_1, \ldots, B_{t_2}\}$ and $\mathcal{P}_C = \{C_1, \ldots, C_{t_3}\}$.

For each $A_i, B_j, C_s \in \mathcal{P}$, let $A_i' = \{v \in V : a_v \in A_i\}$, $B_j' = \{v \in V : b_v \in B_j\}$ and $C_s' = \{v \in V : c_v \in C_s\}$. Let $\mathcal{Q}$ be the partition of V which is the coarsest common refinement of the partitions $\{A_1', \ldots, A_{t_1}'\}$, $\{B_1', \ldots, B_{t_2}'\}$, and $\{C_1', \ldots, C_{t_3}'\}$. Clearly

$$|\mathcal{Q}| \leq t_1 t_2 t_3 \leq (2^{2^{\epsilon^{-2K}}})^3 = 2^{3 \cdot 2^{\epsilon^{-2K}}}.$$

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Since $\epsilon < 1/2$, there is clearly some $K' > 0$ depending only on K so that $2^{3\cdot 2^{\epsilon^{-2K}}} < 2^{2^{\epsilon^{-K'}}}$. Thus, it suffices to show $\mathcal{Q}$ is ϵ-homogeneous with respect to H.

For each $X \in \mathcal{Q}$, let $X^A = \{a_v : v \in X\}$, $X^B = \{b_v : v \in X\}$, $X^C = \{c_v : v \in X\}$. Then $\mathcal{P}' = \{X^A, X^B, X^C : X \in \mathcal{Q}\}$ is a partition of $V(H')$ refining $\mathcal{P}$. Since $\mathcal{P}$ is ϵ^2-homogeneous with respect to H', and $\mathcal{P}' \preceq \mathcal{P}$, Lemma 2.1 implies $\mathcal{P}'$ is ϵ-homogeneous with respect to H'. Now let

$$\Sigma = \{(X, Y, Z) \in \mathcal{Q}^3 : (X^A, Y^B, Z^C) \text{ is } \epsilon\text{-homogeneous with respect to } H'\}$$

For all $(X, Y, Z) \in \Sigma$, $d_H(X, Y, Z) = d_{H'}(X^A, Y^B, Z^C) \in [0, \epsilon) \cup (1 - \epsilon, 1]$. Further,

$$\left| \bigcup_{(X,Y,Z)\in\Sigma} X \times Y \times Z \right| \geq \left| \bigcup_{(X,Y,Z)\in\Sigma} X^A \times Y^B \times Z^C \right| \geq (1-\epsilon)|A||B||C| = (1-\epsilon)|V(H)|^3.$$

This shows $\mathcal{Q}$ is an ϵ-homogeneous partition of H, as desired. □

The rest of the paper will focus on proving Theorem 4.1. The proof of Theorem 4.1 is rather long, so we now give an outline of the overall strategy. The starting point is a 3-partite 3-graph $H = (U \cup V \cup W, E)$ with $|U| = |V| = |W|$ and slicewise VC-dimension less than k, and the goal is to construct a τ-homogeneous partition for H with few parts.

Step 1 (Reduced graphs for slice graphs): Since H has slicewise VC-dimension less than k, we have that for all $x \in U$, H_x has VC-dimension less than k (see Definition 2.11). By Theorem 2.10, there is an homogeneous equipartition $\mathcal{P}_x$ of H_x refining $\{V, W\}$ with a small number of parts. We then construct a bipartite edge-coloured graph $\mathcal{G}_x$ as discussed in Subsection 3. In particular, $\mathcal{G}_x$ will contain vertices $\{p_1, \ldots, p_{r_x}, q_1, \ldots, q_{s_x}\}$ which are in bijection with the parts of $\mathcal{P}_x$. The edge colours E_0^x, E_1^x, E_2^x will then correspond to the pairs from $\mathcal{P}_x$ which are dense, sparse, and non-homogeneous, respectively.

Step 2 (Group vertices based on reduced graphs): Partition U into equivalence classes, where $x \equiv x'$ if and only if $\mathcal{G}_x = \mathcal{G}_{x'}$.

Step 3 (Structure theorem on equivalence classes): For each $\equiv$-equivalence class X in U, find a partition $\mathcal{P}_X^V$ of V which is almost ϵ-good with respect to $H[X, V, W]$.

Step 4 (Refine): Let $\mathcal{P}^V$ be the partition of V obtained by taking the coarsest common refinement of the partitions $\mathcal{P}_X^V$ from step 3, as X ranges over the $\equiv$-equivalence classes in U. Show this yields an almost ϵ-good partition $\mathcal{P}_V$ of V with respect to H.

Step 5 (Repeat): Repeat Steps 1-4 with the roles of U, V, W permuted to obtain almost ϵ-good partitions $\mathcal{P}_W$ of W and $\mathcal{P}_U$ of U.

Step 7 (Symmetry): Use Proposition 2.16 to conclude $\mathcal{P} = \mathcal{P}_U \cup \mathcal{P}_V \cup \mathcal{P}_W$ is a homogeneous partition with respect to H.

As the reader can see, there are many steps to this argument. The hardest is Step 3. We choose to encapsulate this step into its own result, Proposition 4.2, which we will state and prove in the next subsection, Subsection 4.1. In the final subsection, Subsection 4.2, we will prove Theorem 4.1 using Proposition 4.2.

4.1 Accomplishing step 3

In this section, we prove the main ingredient, Proposition 4.2, which is required to execute Step 3 of the strategy outlined above. The goal of Proposition 4.2 is to give a structure theorem for any tripartite 3-graph $H = (U \cup V \cup W, F)$ with slicewise VC-dimension less than k, under the assumption that there exists a single bipartite edge-coloured graph $\mathcal{G}$ so that for all $x \in U$, $\mathcal{G}$ arises as the reduced graph associated to a homogeneous partition of H_x.

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Proposition 4.2. *For all $k \geq 1$, there exist constants $C = C(k) \geq 1$, $D = D(k) \geq 1$, and $\epsilon^* = \epsilon^*(k) > 0$ so that the following holds. Fix $0 < \epsilon < \epsilon^*$ and let r, s, ℓ be positive integers with $s \leq \ell$ and $r \leq \ell$, and assume the following hold.*

(1) $H = (U \cup V \cup W, E)$ is a sufficiently large tripartite 3-graph satisfying $\mathrm{SVC}(H) < k$ and $|U| = |V| = |W|$,

(2) $\mathcal{G} = (P \cup Q, F_0, F_1, F_2)$ is a bipartite edge-coloured graph with $P = \{p_0, \dots, p_r\}$ and $Q = \{q_0, \dots, q_s\}$,

(3) For all $x \in U$, there is a partition $V = B_x^0 \cup \cdots \cup B_x^r$ and $W = C_x^0 \cup \cdots \cup C_x^s$ so that the following hold, where $E(H)^1 := E(H)$ and $E(H)^0 := \binom{V(H)}{3} \setminus E(H)$.

  (a) For each $\alpha \in \{0, 1\}$ and $p_i q_j \in F_\alpha$, $|E(H_x)^\alpha \cap K_2[B_x^i, C_x^j]| \geq (1 - \epsilon)|B_x^i||C_x^j|$,

  (b) For each $1 \leq i \leq r$, $||B_x^i| - \frac{|V|}{\ell}| \leq 1$ and for all $1 \leq j \leq s$, $||C_x^j| - \frac{|W|}{\ell}| \leq 1$,

  (c) $|B_x^0| \leq \epsilon |V|$ and $|C_x^0| \leq \epsilon |W|$.

(4) For all $1 \leq i \leq r$ and $1 \leq j \leq s$, $|N_{F_2}(p_i)| \leq \epsilon \ell$ and $|N_{F_2}(q_j)| \leq \epsilon \ell$.

Then there exists a partition $\mathcal{P}_V$ of V which is almost $C\epsilon^{1/D}$-good with respect to H satisfying

$$|\mathcal{P}_V| \leq \epsilon^{-O_k(\ell^{\ell+1}\epsilon^{-\ell})}.$$

It is important to note that hypotheses (3b) and (3c) imply r and s are both close to ℓ. Consequently, one can check hypotheses (3) and (4) (along with the assumptions that $|V| = |W|$ are sufficiently large and ϵ is sufficiently small) imply that for each $x \in U$, the partition $\{B_x^0 \cup \cdots \cup B_x^r, C_x^0 \cup \cdots \cup C_x^s\}$ is a 2ϵ-homogeneous partition for the slice graph H_x. Before proving Proposition 4.2, we require two lemmas. The first, Lemma 4.3 below, tells us that if $G = (A \cup B, E)$ is a bipartite graph, then we can cover most of A with sets which are either contained the neighbourhood of some element in G, or incident to few edges in G.

Lemma 4.3. *Suppose $G = (A \cup B, E)$ is a bipartite graph and $\delta > 0$. Then there exists an integer $t \leq \delta^{-1/2}$ and a partition $A = A_{err} \cup A_0 \cup A_1 \cup \cdots \cup A_t$ of A so that the following hold.*

(i) $|A_{err}| \leq \delta^{1/8}|A|$,

(ii) For each $1 \leq i \leq t$, there is a vertex $b_i \in B$ so that $A_i \subseteq N_G(b_i)$ and $|A_i| = \lceil \sqrt{\delta}|A| \rceil$,

(iii) For all $x \in A_0$, $|N_G(x)| \leq \delta^{1/8}|B|$,

The proof of Lemma 4.3 a simple greedy algorithm, and appears in the appendix. The second lemma, Lemma 4.4, is about refinements of almost ϵ-good partitions. It is essentially a corollary of Lemma 2.1.

Lemma 4.4. *Suppose $H = (V_1 \cup V_2 \cup V_3, E)$ is a 3-partite 3-graph. Suppose $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. Suppose $\mathcal{P}$ is an almost ϵ-good partition of V_{i_1} and $\mathcal{Q} \preceq \mathcal{P}$. Then $\mathcal{Q}$ is an almost $\epsilon^{1/4}$-good partition of V_{i_1}.*

Proof. Let $\mathcal{P}_{\text{good}}$ be the set of almost ϵ-good sets in $\mathcal{P}$. Since, by assumption, $\mathcal{P}$ is almost ϵ-good with respect to H, we have $|\bigcup_{X \in \mathcal{P}_{\text{good}}} X| \geq (1 - \epsilon)|V_{i_1}|$. For each $X \in \mathcal{P}_{\text{good}}$, let

$$\Gamma_1(X) = \{yz \in K_3[V_{i_2}, V_{i_3}] : |N_H(yz) \cap X| \geq (1 - \epsilon)|X|\} \text{ and}$$
$$\Gamma_0(X) = \{yz \in K_3[V_{i_2}, V_{i_3}] : |N_H(yz) \cap X| \leq \epsilon |X|\}.$$

and then set

$$\Omega(X) = \left(\bigcup_{yz \in \Gamma_1(X)} K_3[\{y\}, \{z\}, N_H(yz) \cap X] \right) \cup \left(\bigcup_{yz \in \Gamma_0(X)} K_3[\{y\}, \{z\}, X \setminus N_H(yz)] \right).$$

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Note that for all $X \in \mathcal{P}_{\text{good}}$, $|\Omega(X)| \geq (1-\epsilon)|X||V_{i_2}||V_{i_3}|$. Consequently, if we let

$$\Omega = \bigcup_{X \in \mathcal{P}_{\text{good}}} \Omega(X),$$

then $|\Omega| \geq (1-\epsilon)|V_{i_1}||V_{i_2}||V_{i_3}|$. Define

$$\mathcal{S} = \{S \in \mathcal{Q} : |\Omega \cap K_3[S, V_{i_2}, V_{i_3}]| \geq (1-\epsilon^{1/2})|S||V_{i_2}||V_{i_3}|\}.$$

By Lemma 2.1, $|\bigcup_{S \in \mathcal{S}} S| \geq (1-\epsilon^{1/2})|V_{i_1}|$. We show every $S \in \mathcal{S}$ is almost $\epsilon^{1/4}$-good with respect to H, which will finish the proof. To this end, fix $S \in \mathcal{S}$. By Lemma 2.1, if we define

$$\mathcal{G}(S) := \{(y,z) \in V_{i_2} \times V_{i_3} : |N_\Omega(yz) \cap S| \geq (1-\epsilon^{1/4})|S|\},$$

then $|\mathcal{G}(S)| \geq (1-\epsilon^{1/4})|S|$. By definition of Ω, and since $S \subseteq X$ for some $X \in \mathcal{P}_{\text{good}}$, we know that for all $(y,z) \in \mathcal{G}(S)$, $N_\Omega(yz) \cap S \subseteq N_H(yz) \cap S$ or $N_\Omega(yz) \cap S \subseteq S \setminus N_H(yz)$. Thus, for every $(y,z) \in \mathcal{G}(S)$, we either have $|N_H(yz) \cap S| \geq (1-\epsilon^{1/4})|S|$ or $|S \setminus N_H(yz)| \geq (1-\epsilon^{1/4})|S|$. This finishes the proof. □

We now prove Proposition 4.2.

Proof of Proposition 4.2. Fix $k \geq 1$. Let δ_0, $C_1 = C_1(k)$ and $C_2 = C_2(k)$ be as in Proposition 3.5. Let $C, D \geq 1$ be the constants such that for all $x \in (0,1)$,

$$Cx^{1/D} = 5 \cdot 6^{1/256} \cdot (2C_1^{-1} \cdot (4x^{1/16})^{1/(4k+4)})^{1/16384}. \tag{4}$$

Let $\epsilon^* > 0$ be sufficiently small compared to C_1^k, C_2^{-k}, and δ_0^D (note these constants all depend only on k). Fix $0 < \epsilon < \epsilon^*$ and $1 \leq s, r \leq \ell$.

Assume $H = (U \cup V \cup W, E)$ is a sufficiently large tripartite 3-graph and $\mathcal{G} = (P \cup Q, F_0, F_1, F_2)$ is an edge-coloured bipartite graph so that H and $\mathcal{G}$ satisfy the hypotheses (1)–(4) of Proposition 4.2. Throughout, the edge-coloured bipartite graph serves a similar role as a "reduced graph" or "quotient graph" would in traditional applications of the regularity lemma. To emphasize this, we will refer to it as the auxiliary structure, $\mathcal{G}$. Observe that hypotheses (3b) and (3c), along with the assumption that $|V| = |W|$ is sufficiently large compared to ℓ, imply

$$\min\{r,s\} \geq (1-2\epsilon)\ell. \tag{5}$$

We start by defining partitions of $K_2[U,V]$ and $K_2[U,W]$ using the sets appearing in hypothesis (3). In particular, for each $0 \leq u \leq r$, $0 \leq v \leq s$, let

$$P_u = \{xy \in K_2[U,V] : y \in B_x^u\} \text{ and } Q_v = \{xy \in K_2[U,W] : y \in C_x^v\}.$$

Note that $K_2[U,V] = P_0 \cup \cdots \cup P_r$ and $K_2[U,W] = Q_0 \cup \cdots \cup Q_s$. By hypothesis (3c), we have that for all $x \in U$,

$$|N_{P_0}(x)| \leq \epsilon|V| \text{ and } |N_{Q_0}(x)| \leq \epsilon|W|,$$

and consequently,

$$|P_0| \leq \epsilon|U||V| \text{ and } |Q_0| \leq \epsilon|U||W|. \tag{6}$$

We will refer to the sets of the form P_u, Q_v as edge colours, and we will think of P_0 and Q_0 as the "error colours." Using these edge colours, we now define a tripartite (U,V,W)-decomposition $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2)$ of $V(H)$. Let $\mathcal{D}_1 = (\mathcal{D}_U, \mathcal{D}_V, \mathcal{D}_W)$, where $\mathcal{D}_U = \{U\}$, $\mathcal{D}_W = \{W\}$, $\mathcal{D}_V = \{V\}$. Then let $\mathcal{D}_2 = \{\mathcal{D}_{VW}, \mathcal{D}_{UV}, \mathcal{D}_{UW}\}$, where

$$\mathcal{D}_{VW} = \{K_2[V,W]\},\ \mathcal{D}_{UV} = \{P_u : 0 \leq u \leq r\},\ \text{and } \mathcal{D}_{UW} = \{Q_v : 0 \leq v \leq s\}.$$

A triad from $\mathcal{D}$ is then a 3-partite graph of the following form, where $0 \leq u \leq r$ and $0 \leq v \leq s$.

$$T_{uv} = (U \cup V \cup W, P_u \cup Q_v \cup K_2[V,W]).$$

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We now show that certain triads from $\mathcal{D}$ are homogeneous with respect to H in the sense of Definition 2.22.

Given $0 \le u \le r$ and $0 \le v \le s$, let $t(u,v) \in \{0,1,2\}$ be such that, in the auxiliary structure $\mathcal{G}$, $p_u q_v \in F_{t(u,v)}$. For $1 \le u \le r$ and $1 \le v \le s$, we have by hypothesis (3b) that

$$|U|\Big(\frac{|V|}{\ell}-1\Big)\Big(\frac{|W|}{\ell}-1\Big) \le |K_3^{(2)}(T_{uv})| = \sum_{x\in U} |N_{P_u}(x)||N_{Q_v}(x)| \le |U|\Big(\frac{|V|}{\ell}+1\Big)\Big(\frac{|W|}{\ell}+1\Big).$$

For any $0 \le u \le r$ and $0 \le v \le s$, if $t(u,v) \in \{0,1\}$, we have by hypothesis (3a),

$$|d_H(T_{uv}) - t(u,v)| = \Big|\frac{|E(H)\cap K_3^{(2)}(T_{uv})|}{|K_3^{(2)}(T_{uv})|} - t(u,v)\Big| \le \epsilon.$$

Using the notation $E(H)^1 = E(H)$ and $E(H)^0 = K_3[U,V,W] \setminus E(H)$, we can restate the above inequality as follows.

$$|E(H)^{t(u,v)} \cap K_3^{(2)}(T_{uv})| \ge (1-\epsilon)|K_3^{(2)}(T_{uv})|. \tag{7}$$

We now define a crucial object for the rest of the proof. Let Γ be the 3-partite 3-graph with vertex set $U \cup V \cup W$ and edge set

$$E(\Gamma) := \bigcup_{p_u q_v \in F_0 \cup F_1} \Big(K_3^{(2)}(T_{uv}) \cap E(H)^{t(u,v)}\Big).$$

Roughly speaking, $E(\Gamma)$ is the set of triples from $K_3[U,V,W]$ which are in a homogeneous triad from $\mathcal{D}$, and which are moreover "correct" for their triad. Note that for any $xyz \in E(\Gamma)$, we can decide whether or not xyz is an edge in H by looking at which triad T_{uv} contains xyz and then whether we have $p_u q_v \in F_0$ or $p_u q_v \in F_1$ in $\mathcal{G}$.

Our next goal is to prove a consequence of the assumption that $\mathrm{SVC}(H) < k$. To state this, we must define some auxiliary edge colours, which are obtained by grouping together the existing edge colours between U and V, based on properties of $\mathcal{G}$. Given $1 \le v \le s$, define

$$P_v^1 = \bigcup_{p_u \in N_{F_1}(q_v)} P_u,\ P_v^0 = \bigcup_{p_u \in N_{F_0}(q_v)} P_u,\ \text{and } P_v^2 = K_2[U,V] \setminus (P_v^1 \cup P_v^0), \tag{8}$$

where we note that the sets $N_{F_1}(q_v)$ and $N_{F_0}(q_v)$ are computed in auxiliary structure $\mathcal{G}$. This yields a new bipartite edge-coloured graph for each $v \in [r]$, namely

$$G_v := (U \cup V, P_v^0, P_v^1, P_v^2).$$

We now show there are restrictions on copies of $U(k)$ appearing in such a G_v (see Definition 3.3).

Observation 4.5. *Suppose $1 \le v \le s$, and $\{x_1,\dots,x_k\} \cup \{y_S : S \subseteq [k]\}$ form a P_v^1/P_v^0-copy of $U(k)$ in G_v. Then there DOES NOT exist $c \in W$ so that the following hold.*

(a) $(\{x_1,\dots,x_k\} \cup \{y_S : S \subseteq [k]\}) \cap U \subseteq N_{Q_v}(c)$ *and*

(b) For all $1 \le i \le k$ and $S \subseteq [k]$, $x_i y_S \in N_\Gamma(c)$.

Proof. Suppose towards a contradiction there exists some $c \in W$ so that (a) and (b) hold. Without loss of generality, we may assume the P_v^1/P_v^0-copy of $U(k)$ has right side in V, i.e. $x_1,\dots,x_k \in N_{Q_v}(c)$ and $\{y_S : S \subseteq [k]\} \subseteq V$ (the other case is similar). By assumption, we now have that for all $i \in [k]$ and $S \subseteq [k]$, $i \in S$ implies $x_i y_S \in P_v^1 \cap N_\Gamma(c)$ and $i \notin S$ implies $x_i y_S \in P_v^0 \cap N_\Gamma(c)$. By definition of Γ, this means that for all $i \in S$, $c x_i y_S \in E(H)$ and for all $i \notin S$, $c x_i y_S \notin E(H)$. This contradicts that H has slicewise VC-dimension less than k. □

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The goal of the rest of the proof to create situations where we can use Observation 4.5. We give an informal (and slightly inaccurate) outline of the steps we will take. The reader may choose to skip the outline, and merely refer back to it for reference.

(I) Throw away edges and vertices which do not behave well with respect to Γ. This is necessary for part (b) in Observations 4.5.

(II) Build a partition $\mathcal{X}$ of U so that for all $A \in \mathcal{X}$, and any $1 \leq \nu \leq s$, there exists some $c \in W$ so that $A \subseteq N_{Q_\nu}(c)$. This will allow us to satisfy part (a) of Observation 4.5 when we restrict ourselves to $H[A \cup V \cup W]$ for each $A \in \mathcal{X}$.

(III) For each $A \in \mathcal{X}$ and most $\nu \in [s]$, use Observation 4.5 to define a bipartite edge-coloured graph $(A \cup V, \mathbf{P}^0_{\nu,A} \cup \mathbf{P}^1_{\nu,A} \cup \mathbf{P}^2_{\nu,A})$ so that $\mathbf{P}^2_{\nu,A}$ is small, so that there is no $\mathbf{P}^0_{\nu,A}/\mathbf{P}^1_{\nu,A}$-copy of $U(k)$, and so that the triads of the form

$$(A \cup V \cup W, Q_\nu \cup \mathbf{P}^0_{\nu,A} \cup K_2[V, W]) \text{ and } (A \cup V \cup W, Q_\nu \cup \mathbf{P}^1_{\nu,A} \cup K_2[V, W])$$

are homogeneous with respect to H in the sense of Definition 2.22.

(IV) For each $A \in \mathcal{X}$ and $\nu \in [s]$ as in the previous step, apply Proposition 3.5 to $(A \cup V, \mathbf{P}^0_{\nu,A}, \mathbf{P}^1_{\nu,A}, \mathbf{P}^2_{\nu,A})$ to obtain vertex partitions $\mathcal{P}^A_{\nu,A}$ of A and $\mathcal{P}^V_{\nu,A}$ of V so that $(\mathcal{P}^A_{\nu,A}, \mathcal{P}^V_{\nu,A})$ is homogeneous with respect to $\{\mathbf{P}^0_{\nu,A}, \mathbf{P}^1_{\nu,A}\}$ in the sense of Definition 3.4.

(V) For each $A \in \mathcal{X}$, define a tripartite decompositions $\mathcal{D}^A$ of $A \cup V \cup W$ as follows. Let $\mathcal{D}^A_W = \{W\}$ be the trivial partition, and let $\mathcal{D}^A_A$ and $\mathcal{D}^A_V$ be the partitions of A and V respectively, generated by taking the coarsest common refinements of the partitions of the form $\mathcal{P}^A_{\nu,A}$ and $\mathcal{P}^V_{\nu,A}$ from the preceding step, as ν ranges over most elements of $[s]$. For each $X \in \mathcal{D}^A_A$, this decomposition will include partitions of $K_2[X, W]$ roughly equal to $\{Q_\nu \cap K_2[X, W] : 1 \leq \nu \leq s\}$. All other elements in $\mathcal{D}^A_2$ are trivial partitions. Show each $\mathcal{D}^A$ is homogenous with respect to $H[A, V, W]$ in the sense of Definition 2.22.

(VI) Show each partition $\mathcal{D}^A_V$ of V from the previous step is almost good with respect to $H[A, V, W]$ in the sense of Definition 2.5.

(VII) Take the coarsest common refinement of the partitions $\mathcal{D}^A_V$ of V, as A ranges over the elements in $\mathcal{X}$. Conclude the result is a partition of V which is almost good with respect to H using Lemma 4.4.

Step I. In this step, we restrict our vertex set and adjust the edge colours to ensure good behaviour with respect to Γ. We first observe that by definition of Γ and equation (7),

$$\begin{aligned}
|E(\Gamma) \cap K_3[U, V, W]| &= \sum_{p_u q_v \in F_0 \cup F_1} |K_3^{(2)}(T_{uv}) \cap E(H)^{t(u,v)}| \\
&\geq (1 - \epsilon) \sum_{p_u q_v \in F_0 \cup F_1} |K_3^{(2)}(T_{uv})| \\
&= (1 - \epsilon)(|U||V||W| - \sum_{p_u q_v \in F_2} |K_3^{(2)}(T_{uv})|). \qquad (9)
\end{aligned}$$

Note that by inequality equation (6) and hypotheses (3b) and (4), we have that

$$\begin{aligned}
\sum_{p_u q_v \in F_2} |K_3^{(2)}(T_{uv})| &\leq |P_0||W| + |Q_0||V| + \sum_{x \in U} \sum_{u=1}^{r} \sum_{q_v \in N_{F_2}(p_u)} |N_{P_u}(x)||N_{Q_v}(x)| \\
&\leq 2\epsilon|U||V||W| + r\epsilon\ell|U|\Big(\frac{|V|}{\ell} + 1\Big)\Big(\frac{|W|}{\ell} + 1\Big) \\
&\leq 6\epsilon|U||V||W|,
\end{aligned}$$

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where the last inequality uses that $r \leq \ell$ and $|W|$ and $|V|$ are large. Combining with equation (9) yields

$$|E(\Gamma) \cap K_3[U, V, W]| \geq (1-\epsilon)(1-6\epsilon)|U||V||W| \geq (1-7\epsilon)|U||V||W|. \tag{10}$$

Crucially, this means Γ covers almost all triples from $K_3[U, V, W]$. We will restrict our attention to vertices which are sufficiently well behaved with respect to Γ. First, we set

$$V_{\text{good}} = \{b \in V : |N_\Gamma(b)| \geq (1-\epsilon^{1/2})|U||W|\}, \text{ and}$$

$$W_{\text{good}} = \{c \in W : |N_\Gamma(c)| \geq (1-\epsilon^{1/2})|U||V|\}.$$

Inequality equation (10) along with Lemma 2.1 implies

$$|V_{\text{good}}| \geq (1-7\epsilon^{1/2})|V| \text{ and } |W_{\text{good}}| \geq (1-7\epsilon^{1/2})|W|. \tag{11}$$

We now define G^{good} to be the tripartite graph with vertex set $U \cup V_{\text{good}} \cup W_{\text{good}}$, and edge set

$$\begin{aligned} E(G^{\text{good}}) =& \{xy \in K_2[U, V_{\text{good}}] : |N_\Gamma(xy)| \geq (1-\epsilon^{1/4})|W_{\text{good}}|\} \\ &\cup \{xz \in K_2[U, W_{\text{good}}] : |N_\Gamma(xz)| \geq (1-\epsilon^{1/4})|V_{\text{good}}|\}. \end{aligned}$$

Using the definition of V_{good} and equation (11), we have

$$\begin{aligned} |E(\Gamma) \cap K_3[U, V_{\text{good}}, W_{\text{good}}]| &\geq \sum_{v \in V_{\text{good}}} |N_\Gamma(v) \cap K_2[U, W_{\text{good}}]| \\ &\geq \sum_{v \in V_{\text{good}}} |N_\Gamma(v)| - \sum_{v \in V_{\text{good}}} |K_2[U, W \setminus W_{\text{good}}]| \\ &\geq |V_{\text{good}}|(1-\sqrt{\epsilon})|U||W| - |V_{\text{good}}||U||W \setminus W_{\text{good}}| \\ &\geq |V_{\text{good}}|(1-\sqrt{\epsilon})|U||W| - 7\epsilon^{1/2}|V_{\text{good}}||U||W| \\ &= (1-8\sqrt{\epsilon})|V_{\text{good}}||U||W| \\ &\geq (1-8\sqrt{\epsilon})|V_{\text{good}}||U||W_{\text{good}}|. \end{aligned}$$

Combining this with the definition of $E(G^{\text{good}})$ and Lemma 2.1 yields

$$|E(G^{\text{good}}) \cap K_2[U, V_{\text{good}}]| \geq (1-8\epsilon^{1/4})|U||V|. \tag{12}$$

A symmetric argument shows

$$|E(G^{\text{good}}) \cap K_2[U, W_{\text{good}}]| \geq (1-8\epsilon^{1/4})|U||W|. \tag{13}$$

We now collect the remaining the pairs outside $E(G^{\text{good}})$ and add them to P_0, Q_0 to make slightly larger "error colours." In particular, let

$$P^{\text{bad}} = P_0 \cup \left(K_2[U, V] \setminus E(G^{\text{good}})\right) \text{ and } Q^{\text{bad}} = Q_0 \cup \left(K_2[U, W] \setminus E(G^{\text{good}})\right).$$

By inequalities equations (6), (12), and (13),

$$|P^{\text{bad}}| \leq 9\epsilon^{1/4}|U||V| \text{ and } |Q^{\text{bad}}| \leq 9\epsilon^{1/4}|U||W|. \tag{14}$$

We now restrict ourselves to vertices in U which mostly avoid Q^{bad} and P^{bad}. Let

$$U_{\text{good}} = \{x \in U : |N_{Q^{\text{bad}}}(x)| \leq \epsilon^{1/8}|W_{\text{good}}| \text{ and } |N_{P^{\text{bad}}}(x)| \leq \epsilon^{1/8}|V_{\text{good}}|\}. \tag{15}$$

By inequality equation (14) and Lemma 2.1, $|U_{\text{good}}| \geq (1-9\epsilon^{1/8})|U|$. Finally, we will restrict the edge colours between U and W to the "good" pairs. Specifically, for each $1 \leq \nu \leq s$, let

$$Q_\nu^{\text{good}} = Q_\nu \cap E(G^{\text{good}}).$$

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Note that by construction, $K_2[U, W] = Q^{\text{bad}} \cup Q_1^{\text{good}} \cup \cdots \cup Q_s^{\text{good}}$. This finishes Step I.

Step II. Our next goal is to partition most of U_{good} into pieces that are covered by neighbourhoods of the form $Q_v^{\text{good}}(c)$ for some $c \in W_{\text{good}}$. We will do this for each edge colour Q_v^{good} individually and then intersect the resulting partitions. We begin by defining

$$\delta = \frac{\epsilon}{(2\ell)^8}. \tag{16}$$

Now, for each $1 \le v \le s$, apply Lemma 4.3, to $(U_{\text{good}} \cup W_{\text{good}}, Q_v^{\text{good}})$ to obtain an integer $0 \le t_v \le \delta^{-1/2}$ and a partition $\mathcal{X}_{Q_v} = \{X_v(\text{err}), X_v(0), \ldots, X_v(t_v)\}$ of U_{good} so that the following hold.

(i) $|X_v(\text{err})| \le \delta^{1/8}|U_{\text{good}}|$,

(ii) For each $1 \le j \le t_v$, there is some $c_{v,j} \in W_{\text{good}}$ so that $X_v(j) \subseteq N_{Q_v^{\text{good}}}(c_{v,j})$ and $|X_v(j)| = \lceil \delta^{1/2}|U_{\text{good}}| \rceil$,

(iii) For all $x \in X_v(0)$, $|N_{Q_v^{\text{good}}}(x) \cap W_{\text{good}}| \le \delta^{1/8}|W_{\text{good}}|$.

Let $\mathcal{X}$ be the partition of U_{good} obtained by taking the coarsest common refinement the partitions of the form $\mathcal{X}_{Q_v}$ for $v \in [s]$. Note

$$|\mathcal{X}| \le (\delta^{-1/2} + 2)^s \le (\delta^{-1})^s \le \delta^{-\ell} \le (2\ell)^\ell \epsilon^{-\ell}, \tag{17}$$

where the second inequality is because $\delta < \epsilon$ and ϵ is sufficiently small, the third is because $s \le \ell$, and the last is by definition of δ. We now collect certain elements from $\mathcal{X}$ which we need to throw away. First, set

$$\mathcal{X}_{\text{err}} = \{A \in \mathcal{X} : |A| \le \epsilon |U_{\text{good}}|/|\mathcal{X}|\} \cup \{A \in \mathcal{X} : \text{ for some } v \in [s], A \subseteq X_v(\text{err})\}.$$

Now let

$$U_{\text{err}} = \Big(U \setminus U_{\text{good}}\Big) \cup \Big(\bigcup_{A \in \mathcal{X}_{\text{err}}} A\Big) \text{ and } U'_{\text{good}} = U \setminus U_{\text{err}}. \tag{18}$$

Since $|U_{\text{good}}| \ge (1 - 9\epsilon^{1/8})|U|$, and by definition of $\mathcal{X}_{\text{err}}$, we have the following upper bound on $|U_{\text{err}}|$.

$$|U_{\text{err}}| \le 9\epsilon^{1/8}|U| + \epsilon|U| + \sum_{v \in [s]} |X_v(\text{err})| \le 9\epsilon^{1/8}|U| + \epsilon|U| + \ell\delta^{1/8}|U| \le 10\epsilon^{1/8}|U|, \tag{19}$$

where the last inequality is by definition of δ (see equation (16)) and because ϵ is sufficiently small. This finishes Step II.

Step III. We now prove a structure theorem for $H[A, V, W]$ for each $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$. We do this in Claim 4.6 below. Roughly speaking, Claim 4.6 says we can group together edge colours to make several homogeneous decompositions for $H[A, V, W]$, one for most elements of $[s]$.

Claim 4.6. *For all $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$, there exist the following.*

- *A set $\mathcal{S}^A \subseteq [s]$,*
- *A partition $K_2[A, W] = \mathbf{Q}_0^A \cup \bigcup_{v \in \mathcal{S}^A} \mathbf{Q}_v^A$,*
- *For each $v \in \mathcal{S}^A$, a partition*

$$K_2[A, V] = \mathbf{P}_{A,v}^0 \cup \mathbf{P}_{A,v}^1 \cup \mathbf{P}_{A,v}^2,$$

so that the following hold.

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(1) *For all* $x \in A$, $|N_{\mathbf{Q}_0^A}(x)| \leq 3\epsilon^{1/8}|W|$,

(2) *For all* $v \in \mathcal{S}^A$ *and* $x \in A$, $|N_{\mathbf{P}^2_{A,v}}(x)| \leq 4\epsilon^{1/16}|V|$,

(3) *For all* $v \in \mathcal{S}^A$, *there is no* $\mathbf{P}^1_{A,v}/\mathbf{P}^0_{A,v}$*-copy of* $U(k)$ *in* $(A \cup V, \mathbf{P}^0_{A,v}, \mathbf{P}^1_{A,v}, \mathbf{P}^2_{A,v})$,

(4) *Given* $v \in \mathcal{S}^A$ *and* $\tau \in \{0,1,2\}$, *let* $T(\mathbf{P}^\tau_{A,v}, \mathbf{Q}^A_v)$ *denote the triad*

$$T(\mathbf{P}^\tau_{A,v}, \mathbf{Q}^A_v) := (A \cup W \cup V, \mathbf{P}^\tau_{A,v} \cup \mathbf{Q}^A_v \cup K_2[V,W]).$$

Then $T(\mathbf{P}^\tau_{A,v}, \mathbf{Q}^A_v)$ *is* $2\epsilon^{3/16}$*-homogeneous with respect to* H *whenever* $\tau \neq 2$ *and* $\mathbf{P}^\tau_{A,v} \neq \emptyset$.

Proof. Fix $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$. We begin by defining the distinguished sets of edge colours $\mathcal{S}^A$ for A. In particular, define

$$\mathcal{S}^A = \{v \in [s] : A \cap X_v(0) = \emptyset\} \text{ and } \mathcal{S}^A_{\text{err}} = \{v \in [s] : A \subseteq X_v(0)\}.$$

Since $A \notin \mathcal{X}_{\text{err}}$, we have $\mathcal{S}^A \cup \mathcal{S}^A_{\text{err}} = [s]$. For each $v \in \mathcal{S}^A$, define

$$\mathbf{Q}^A_v = Q^{\text{good}}_v \cap K_2[A,W],$$

and then let $\mathbf{Q}^A_0 = K_2[A,W] \setminus (\bigcup_{v\in[s]} \mathbf{Q}^A_v)$. Using that $A \subseteq U'_{\text{good}} \subseteq U_{\text{good}}$ (recall their definitions from equations (18) and (15)), the definition of $\mathcal{S}^A_{\text{err}}$, the bound for W_{good} from equation (11), and property (iii) from the construction of the sets $X_v(0)$ in Step II, we have that for all $x \in A$,

$$\begin{aligned}
|N_{\mathbf{Q}^A_0}(x)| &\leq |N_{Q^{\text{bad}}}(x)| + \Big|\sum_{v\in\mathcal{S}^A_{\text{err}}} N_{Q^{\text{good}}_v}(x) \cap W_{\text{good}}\Big| + |W \setminus W_{\text{good}}| \\
&\leq \epsilon^{1/8}|W| + |\mathcal{S}^A_{\text{err}}|\delta^{1/8}|W| + 7\epsilon^{1/2}|W| \\
&\leq (2\epsilon^{1/8} + \ell\delta^{1/8})|W| \\
&\leq 3\epsilon^{1/8}|W|, \qquad (20)
\end{aligned}$$

where the last inequality is by definition of δ (see equation (16)), and the second to last inequality uses that ϵ is sufficiently small. Thus we have shown conclusion (1) of Claim 4.6. Note this implies $|\mathbf{Q}^A_0| \leq 3\epsilon^{1/8}|A||W|$, and therefore, we now know that most of $K_2[A,W]$ is covered by the sets of the form $\mathbf{Q}^A_v$ for $v \in \mathcal{S}^A$.

Our next goal is to define the edge colours $\mathbf{P}^0_{v,A}, \mathbf{P}^1_{v,A}, \mathbf{P}^2_{v,A}$ for each $v \in \mathcal{S}^A$. By definition of $\mathcal{S}^A$, we have that for each $v \in \mathcal{S}^A$, $A \subseteq X_v(j_v) \subseteq N_{Q^{\text{good}}_v}(c_{v,j_v})$ for $1 \leq j_v \leq t_v$. We let $\mathcal{C}^A$ denote the set of these vertices c_{v,j_v}, i.e. define

$$\mathcal{C}^A := \{c_{v,j_v} : v \in \mathcal{S}^A\}.$$

We now define two special subsets of $\mathcal{C}^A$ based on behaviour in the auxiliary structure $\mathcal{G}$.

$$\mathcal{C}^A_0 = \{c_{v,j_v} \in \mathcal{C}^A : |N_{F_1}(q_v)| \leq \epsilon^{1/16}r\}, \text{ and } \mathcal{C}^A_1 = \{c_{v,j_v} \in \mathcal{C}^A : |N_{F_0}(q_v)| \leq \epsilon^{1/16}r\}.$$

We do not expect $\mathcal{C}^A_0 \cup \mathcal{C}^A_1$ to necessarily cover all of $\mathcal{C}^A$. Recall that by hypothesis (4), we know that for every $v \in [s]$, $|N_{F_2}(q_v)| \leq \epsilon\ell$. Consequently, since ϵ is sufficiently small, and by inequality equation (5), we can conclude that $\mathcal{C}^A_0 \cap \mathcal{C}^A_1 = \emptyset$. We will next define the partition $K_2[A,V] = \mathbf{P}^0_{v,A} \cup \mathbf{P}^1_{v,A} \cup \mathbf{P}^2_{v,A}$ for each $v \in \mathcal{S}^A$. How we define this partition depends on whether c_{v,j_v} is in one of $\mathcal{C}^A_1$, $\mathcal{C}^A_0$, or neither (cases (a), (b), and (c) below). This definition will also make use of the edge-colours P^0_v, P^1_v, P^2_v which were defined in equation equation (8).

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(a) If $c_{v,j_v} \in \mathcal{C}_1^A$, let $\mathbf{P}_{v,A}^0 = \emptyset$, and define

$$\mathbf{P}_{v,A}^1 = P_v^1 \cap E(G^{\text{good}}) \cap N_\Gamma(c_{v,j_v}), \text{ and}$$
$$\mathbf{P}_{v,A}^2 = K_2[A, V] \setminus \mathbf{P}_{v,A}^1.$$

(b) If $c_{v,j_v} \in \mathcal{C}_0^A$, let $\mathbf{P}_{v,A}^1 = \emptyset$ and define

$$\mathbf{P}_{v,A}^0 = P_v^0 \cap E(G^{\text{good}}) \cap N_\Gamma(c_{v,j_v}), \text{ and}$$
$$\mathbf{P}_{v,A}^2 = K_2[A, V] \setminus \mathbf{P}_{v,A}^0.$$

(c) If $c_{v,j_v} \in \mathcal{C}^A \setminus (\mathcal{C}_0^A \cup \mathcal{C}_1^A)$, let

$$\mathbf{P}_{v,A}^0 = P_v^0 \cap E(G^{\text{good}}) \cap N_\Gamma(c_{v,j_v}),$$
$$\mathbf{P}_{v,A}^1 = P_v^1 \cap E(G^{\text{good}}) \cap N_\Gamma(c_{v,j_v}), \text{ and}$$
$$\mathbf{P}_{v,A}^2 = K_2[A, V] \setminus \Big(\mathbf{P}_{v,A}^1 \cup \mathbf{P}_{v,A}^0\Big).$$

To show these definitions work to satisfy the conclusions of Claim 4.6, we will need to compute the sizes of the edge colours above. For this, we will use the the following fact. By definition of P_v^τ (see equality equation (8)) and hypothesis (3b), we have that for all $x \in A$, $v \in \mathcal{S}^A$ and $\tau \in \{0, 1, 2\}$,

$$\Big(\frac{|V|}{\ell} - 1\Big)|N_{F_\tau}(q_v)| \le |N_{P_v^\tau}(x)| \le |N_{F_\tau}(q_v)|\Big(\frac{|V|}{\ell} + 1\Big), \tag{21}$$

where $N_{F_\tau}(q_v)$ is computed in the auxiliary structure $\mathcal{G}$. We show that in all three cases, (a), (b), and (c), conclusion (2) of Claim 4.6 holds. Suppose first we are in case (a). Then by construction we have that for all $x \in A$,

$$|N_{\mathbf{P}_{v,A}^2}(x)| \le |V \setminus N_\Gamma(xc_{v,j_v})| + |N_{P^{\text{bad}}}(x)| + \sum_{p_u \in N_{F_0}(q_v) \cup N_{F_2}(q_v)} |N_{P_u}(x)|$$
$$\le |V \setminus N_\Gamma(xc_{v,j_v})| + |N_{P^{\text{bad}}}(x)| + |N_{F_0}(q_v) \cup N_{F_2}(q_v)|\Big(\frac{|V|}{\ell} + 1\Big),$$

where the last inequality is by equation (21). Since $A \subseteq N_{Q_v^{\text{good}}}(c_{v,j_v})$, we have that for all $x \in A$,

$$|V \setminus N_\Gamma(xc_{v,j_v})| \le \epsilon^{1/4}|V|.$$

Since $A \subseteq U'_{\text{good}}$, we have for all $x \in A$, $|N_{P^{\text{bad}}}(x)| \le \epsilon^{1/8}|V_{\text{good}}|$. Since we are in case (a), $c_{v,j_v} \in \mathcal{C}_A^1$ implies $|N_{F_0}(q_v)| \le \epsilon^{1/16}r$. Hypothesis (4) implies $|N_{F_2}(q_v)| \le \epsilon\ell$. Combining these facts, we have that

$$|N_{\mathbf{P}_{v,A}^2}(x)| \le \epsilon^{1/4}|V| + \epsilon^{1/8}|V_{\text{good}}| + (\epsilon\ell + \epsilon^{1/16}r)\Big(\frac{|V|}{\ell} + 1\Big) \le 4\epsilon^{1/16}|V|,$$

where the last inequality uses that $r \le \ell$, the fact that $|V|$ is sufficiently large, and the fact that ϵ is sufficiently small. A similar computation shows that in case (b), we have for all $x \in A$, $|N_{\mathbf{P}_{v,A}^2}(x)| \le 4\epsilon^{1/16}|V|$. Suppose now we are in case (c). Then via a similar argument to the above, we have that for all $x \in A$,

$$|N_{\mathbf{P}_{v,A}^2}(x)| \le |V \setminus N_\Gamma(xc_{v,j_v})| + |N_{P^{\text{bad}}}(x)| + |N_{F_2}(q_v)|\Big(\frac{|V|}{\ell} + 1\Big) \le \epsilon^{1/4}|V| + \epsilon^{1/8}|V| + 2\epsilon|V|$$
$$\le 4\epsilon^{1/16}|V|,$$

where the second to last inequality uses that $r \le \ell$ and $|V|$ is sufficiently large. This finishes our verification of conclusion (2) of Claim 4.6.

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We next verify conclusion (3) of Claim 4.6. Let $v \in \mathcal{S}^A$. We want to show there is no $\mathbf{P}^0_{v,A}/\mathbf{P}^1_{v,A}$-copy of $U(k)$ in $(A \cup V, \mathbf{P}^0_{v,A}, \mathbf{P}^1_{v,A}, \mathbf{P}^2_{v,A})$. If we are in cases (a) or (b), then $c_{v,j_v} \in \mathcal{C}^A_0 \cup \mathcal{C}^A_1$ implies one of $\mathbf{P}^0_{v,A}$ or $\mathbf{P}^1_{v,A}$ is empty, so there is clearly no $\mathbf{P}^0_{v,A}/\mathbf{P}^1_{v,A}$ copy of $U(k)$ in $(A \cup V, \mathbf{P}^0_{v,A}, \mathbf{P}^1_{v,A}, \mathbf{P}^2_{v,A})$.

Assume now we are in case (c), so $c_{v,j_v} \in \mathcal{C}^A \setminus (\mathcal{C}^A_0 \cup \mathcal{C}^A_1)$. Suppose towards a contradiction there is a $\mathbf{P}^0_{v,A}/\mathbf{P}^1_{v,A}$ copy of $U(k)$ in $(A \cup V, \mathbf{P}^0_{v,A}, \mathbf{P}^1_{v,A}, \mathbf{P}^2_{v,A})$. This means there exist vertices $\{x_1, \dots, x_k\} \cup \{y_S : S \subseteq [k]\} \subseteq A \cup V$ such that for all $1 \leq j \leq k$ and $S \subseteq [k]$, if $j \in S$, then $x_j y_S \in P^1_v \cap N_\Gamma(c_{v,j_v})$ and if $j \notin S$, then $x_j y_S \in P^0_v \cap N_\Gamma(c_{v,j_v})$. However, since $A \subseteq N_{Q_v}(c_{v,j_v})$, this contradicts Observation 4.5. This finishes our verification of conclusion (3) of Claim 4.6.

We just have left to show conclusion (4) of Claim 4.6. For each $v \in \mathcal{S}^A$ and $\tau \in \{0,1,2\}$ define the triad

$$T(\mathbf{P}^\tau_{v,A}, \mathbf{Q}^A_v) = (A \cup V \cup W, \mathbf{P}^\tau_{v,A} \cup \mathbf{Q}^A_v \cup K_2[V,W]).$$

Fix $v \in \mathcal{S}^A$ and $\tau \in \{0,1\}$, and assume $\mathbf{P}^\tau_{v,A} \neq \emptyset$. We want to show $T(\mathbf{P}^\tau_{v,A}, \mathbf{Q}^A_v)$ is $2\epsilon^{3/16}$-homogeneous with respect to H. Since $A \subseteq N_{Q^{\text{good}}_v}(c_{v,j_v})$, we have that for all $x \in A$, $xc_{v,j_v} \in E(G^{\text{good}})$, and consequently,

$$|N_\Gamma(c_{v,j_v}) \cap K_2[\{x\}, V]| \geq (1 - \epsilon^{1/4})|V|.$$

Combining this with the fact $A \subseteq U'_{\text{good}}$, we have that for all $x \in A$,

$$\begin{aligned} |N_{\mathbf{P}^\tau_{v,A}}(x)| \geq |N_{P^\tau_v}(x)| - |N_{P^{\text{bad}}}(x)| - |V \setminus N_\Gamma(xc_{v,j_v})| &\geq |N_{F_\tau}(q_v)|\left(\frac{|V|}{\ell} - 1\right) - \epsilon^{1/8}|V_{\text{good}}| - \epsilon^{1/4}|V| \\ &\geq \epsilon^{1/16} r\left(\frac{|V|}{\ell} - 1\right) - 3\epsilon^{1/8}|V| \\ &\geq \frac{\epsilon^{1/16}|V|}{2}, \end{aligned} \tag{22}$$

where the inequalities use equation (21), the fact that $\mathbf{P}^\tau_{v,A} \neq \emptyset$ implies $|N_{F_\tau}(q_v)| \geq \epsilon^{1/16} r$, inequality equation (5), the fact that $|V|$ is sufficiently large, and the fact that ϵ is sufficiently small. We now have the following, where we recall $E(H)^1 = E(H)$ and $E(H)^0 = \binom{V(H)}{3} \setminus E(H)$.

$$\begin{aligned} |E(H)^\tau \cap K^{(2)}_3(T(\mathbf{P}^\tau_{v,A}, \mathbf{Q}^A_v))| &\geq \sum_{x \in A} |N_\Gamma(x) \cap K_2[N_{\mathbf{P}^\tau_{v,A}}(x), N_{\mathbf{Q}^A_v}(x)]| \\ &= \sum_{x \in A}\left(\sum_{w \in N_{\mathbf{Q}^A_v}(x)} |N_\Gamma(xw) \cap N_{\mathbf{P}^\tau_{v,A}}(x)|\right) \\ &\geq \sum_{x \in A}\left(\sum_{w \in N_{\mathbf{Q}^A_v}(x)} (|N_{\mathbf{P}^\tau_{v,A}}(x)| - |V \setminus N_\Gamma(xw)|)\right) \\ &\geq \sum_{x \in A}\left(\sum_{w \in N_{\mathbf{Q}^A_v}(x)} (|N_{\mathbf{P}^\tau_{v,A}}(x)| - \epsilon^{1/4}|V|)\right), \end{aligned}$$

where the last inequality uses $\mathbf{Q}^A_v \subseteq Q^{\text{good}}_v$. Using equation (22), this is at least

$$\begin{aligned} \sum_{x \in A}\sum_{w \in N_{\mathbf{Q}^A_v}(x)} |N_{\mathbf{P}^\tau_{v,A}}(x)|(1 - 2\epsilon^{3/16}) &= (1 - 2\epsilon^{3/16}) \sum_{x \in A} |N_{\mathbf{Q}^A_v}(x)||N_{\mathbf{P}^\tau_{v,A}}(x)| \\ &= (1 - 2\epsilon^{3/16})|K^{(2)}_3(T(\mathbf{P}^\tau_{v,A}, \mathbf{Q}^A_v))|, \end{aligned}$$

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where the last inequality uses equation (22). This finishes our verification of conclusion (4), and thus we have finished proving Claim 4.6. □

Step IV. In this step, we will apply Proposition 3.5 to the edge-coloured graphs constructed in Claim 4.6. Now define

$$\epsilon' = 4\epsilon^{1/16} \text{ and } \mu = 2C_1^{-1}(\epsilon')^{1/(4k+4)}. \tag{23}$$

First, note that since ϵ is sufficiently small, we may assume $\mu < C_2^{-1}$ (this will be useful later). Second, as ϵ is sufficiently small, $\mu < \delta_0$, and by definition, $\epsilon' < C_1\mu^{4k+4}$. Thus, for each $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$ and $\nu \in \mathcal{S}^A$, by Claim 4.6, we can apply Proposition 3.5 to the bipartite edge-coloured graph $(A \cup V, \mathcal{P}^0_{A,\nu}, \mathcal{P}^1_{A,\nu}, \mathcal{P}^2_{A,\nu})$, with parameters k, ϵ', and μ to obtain a partition $\mathcal{P}^V_{A,\nu}$ of V and $\mathcal{P}^A_{A,\nu}$ of A so that the following hold.

- $\max\{|\mathcal{P}^V_{A,\nu}|, |\mathcal{P}^A_{A,\nu}|\} \le C_2\mu^{-k}$,
- For each $\tau \in \{0, 1\}$,

$$\Big|\bigcup_{(X,Y)\in\Sigma^0_{A,\nu}\cup\Sigma^1_{A,\nu}} (X \times Y)]\Big| \ge (1-\mu^{1/16})|A||V|,$$

  where we define

$$\Sigma^\tau_{A,\nu} = \{(X, Y) \in \mathcal{P}^A_{A,\nu} \times \mathcal{P}^V_{A,\nu} : |\mathbf{P}^\tau_{\nu,A} \cap K_2[X, Y]| \ge (1-\mu^{1/16})|X||Y|\}.$$

Step V. In this step, we will define an (A, V, W)-tripartite decomposition $\mathcal{D}^A$ of $A \cup V \cup W$ for each $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$, which we will then show is homogeneous with respect to $H[A, V, W]$ in the sense of Definition 2.22. Fix $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$. We begin by defining the tripartite (A, V, W)-decomposition $\mathcal{D}^A$ of $A \cup V \cup W$. First, let

$$\mathcal{D}^A_1 = (\mathcal{D}^A_A, \mathcal{D}^A_V, \mathcal{D}^A_W),$$

where $D^A_W = \{W\}$, and $\mathcal{D}^A_A$ and $\mathcal{D}^A_V$ are the partitions of A and V obtained by taking the coarsest common refinements of the partitions $\mathcal{P}^A_{\nu,A}$ and $\mathcal{P}^V_{\nu,A}$, respectively, as ν ranges over $\mathcal{S}^A$. Observe that

$$|\mathcal{D}^A_V| \le \prod_{\nu\in\mathcal{S}^A} |\mathcal{P}^V_{\nu,A}| \le (C_2\mu^{-k})^\ell \le \mu^{-2k\ell},$$

where the last inequality is by our observation that $\mu < C_2^{-1}$. We now define $\mathcal{D}^A_2$. For $X \in \mathcal{D}^A_A$ and $Y \in \mathcal{D}^A_V$, let $\mathcal{D}^A_{XY} = \{K_2[X, Y]\}$. For $Y \in \mathcal{D}^A_V$, let $\mathcal{D}^A_{YW} = \{K_2[Y, W]\}$. Finally, for each $X \in \mathcal{D}^A_A$, define

$$\mathcal{D}^A_{XW} = \{\mathbf{Q}^A_\nu \cap K_2[X, W] : \nu \in \{0\} \cup \mathcal{S}^A\}.$$

We then let

$$\mathcal{D}^A_2 = \bigcup_{X\in\mathcal{D}^A_A} \bigcup_{Y\in\mathcal{D}^A_V} \{\mathcal{D}^A_{XY}, \mathcal{D}^A_{YW}, \mathcal{D}^A_{XW}\}.$$

We have now defined a tripartite (A, V, W)-decomposition $\mathcal{D}^A = (\mathcal{D}^A_1, \mathcal{D}^A_2)$.

Our next goal is to show $\mathcal{D}^A$ is $3\mu^{1/32}$-homogeneous with respect to $H[A, V, W]$, in the sense of Definition 2.22. The triads of $\mathcal{D}^A$ have the following form, for some $X \in \mathcal{D}^A_A$, $Y \in \mathcal{D}^A_V$, and $\nu \in \{0\} \cup \mathcal{S}^A$:

$$T_{A,\nu}(X, Y) := (X \cup Y \cup W, K_2[Y, W] \cup (\mathbf{Q}^A_\nu \cap K_2[A, W]) \cup K_2[X, Y]).$$

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Note that by construction, this gives us a partition

$$K_3[A,V,W] = \bigcup_{v\in\{0\}\cup\mathcal{S}^A}\ \bigcup_{X\in\mathcal{P}^A_{A,v}}\ \bigcup_{Y\in\mathcal{P}^V_{A,v}} K_3^{(2)}(T_{A,v}(X,Y)).$$

We show that most triples in $K_3[A,V,W]$ are covered by a *homogeneous* triad of the form $T_{A,v}(X,Y)$. The rough idea is to show that most such triads inherit homogeneity from a triad of the form $T(\mathbf{P}^\tau_{v,A},\mathbf{Q}^A_v)$, for some $v\in\mathcal{S}^A$ and $\tau\in\{0,1\}$ (recall from Claim 4.6 that all such triads are $2\epsilon^{3/16}$-homogeneous with respect to H).

Our next several definitions are aimed at defining a set of triads from $\mathcal{D}^A$ which we will show are homogeneous. First, define

$$\Gamma_A = \bigcup_{v\in\mathcal{S}^A}\ \bigcup_{\tau\in\{0,1\}}\Big(E(H)^\tau\cap K_3^{(2)}(T(\mathbf{P}^\tau_{v,A},\mathbf{Q}^A_v))\Big).$$

In other words, Γ_A is the set of triples $xyz\in K_3[A,V,W]$ which come from a homogeneous triad of the form $T(\mathbf{P}^\tau_{v,A},\mathbf{Q}^A_v)$, for some $v\in\mathcal{S}^A$ and $\tau\in\{0,1\}$, and so that xyz is "correct" with respect to this triad. We now define the set of triads from $\mathcal{D}^A$ which are mostly covered by Γ_A.

$$\mathcal{T}_A = \{G\in\mathrm{Tri}(\mathcal{D}^A) : |\Gamma_A\cap K_3^{(2)}(G)|\geq(1-3\epsilon^{1/32})|K_3^{(2)}(G)|\}.$$

This set will be useful because, if $G\in\mathcal{T}_A$ and G is mostly contained in a *single* triad of the form $T(\mathbf{P}^\tau_{v,A},\mathbf{Q}^A_v)$ with $v\in\mathcal{S}^A$ and $\tau\in\{0,1\}$, then we can show G is also somewhat homogeneous. We now work towards making this precise. First, given $v\in\mathcal{S}^A$, define

$$\Omega^1_{A,v} = \{(X,Y)\in\mathcal{D}^A_A\times\mathcal{D}^A_V : |\mathbf{P}^1_{v,A}\cap K_2[X,Y]|\geq(1-\mu^{1/32})|X||Y|\}\text{ and}$$
$$\Omega^0_{A,v} = \{(X,Y)\in\mathcal{D}^A_A\times\mathcal{D}^A_V : |\mathbf{P}^0_{v,A}\cap K_2[X,Y]|\geq(1-\mu^{1/32})|X||Y|\}.$$

Since $\mathcal{P}^A_{v,A}\cup\mathcal{P}^V_{v,A}$ is a $\mu^{1/16}$-homogeneous partition of $(A\cup V,\mathbf{P}^0_v,\mathbf{P}^1_v,\mathbf{P}^2_v)$ (see Definition 3.4), we have by Lemma 2.1 and the definitions of $\Omega^0_{A,v},\Omega^1_{A,v}$ that for each $v\in\mathcal{S}^A$,

$$\Big|\bigcup_{(X,Y)\in\Omega^0_{A,v}\cup\Omega^1_{A,v}} K_2[X,Y]\Big|\geq(1-\mu^{1/32})|A||V|. \tag{24}$$

The sets $\Omega^0_{A,v},\Omega^1_{A,v}$ are important because, given a triad of the form $T_{A,v}(X,Y)$, if we know $(X,Y)\in\Omega^\tau_{A,v}$ for some $\tau\in\{0,1\}$, then $T_{A,v}(X,Y)$ is mostly contained in the homogeneous triad $T(\mathbf{P}^\tau_{v,A},\mathbf{Q}^A_v)$ (we will make this precise shortly). We now give a name to the set of the triads of the form $T_{A,v}(X,Y)$ where $v\in\mathcal{S}^A$ and $(X,Y)\in\Omega^0_{A,v}\cup\Omega^1_{A,v}$.

$$\mathcal{T}'_A = \{T_{A,v}(X,Y) : v\in\mathcal{S}^A,(X,Y)\in\Omega^0_{A,v}\cup\Omega^1_{A,v}\}.$$

We will also need a name for the following set of triads which are not too small.

$$\mathcal{T}''_A = \{T\in\mathrm{Tri}(\mathcal{D}^A) : |K_3^{(2)}(T)|\geq\frac{(1-\epsilon^{1/32})|X||Y||W|}{\ell}\}.$$

Finally, we define

$$\mathcal{T}^{\mathrm{good}}_A = \mathcal{T}_A\cap\mathcal{T}'_A\cap\mathcal{T}''_A.$$

We will show in Claim 4.7 below that the triads in $\mathcal{T}^{\mathrm{good}}_A$ are homogeneous with with respect to $H[A,V,W]$.

Claim 4.7. *Every $T\in\mathcal{T}^{\mathrm{good}}_A$ is $3\mu^{1/32}$-homogeneous with respect to $H[A,V,W]$.*

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Proof. Fix $T \in \mathcal{T}_A^{\text{good}}$. Then T has the form $T_{A,\nu}(X, Y)$ some $\nu \in \mathcal{S}^A$ and $(X, Y) \in \Omega^0_{A,\nu} \cup \Omega^1_{A,\nu}$. Without loss of generality, say $(X, Y) \in \Omega^1_{A,\nu}$ (the other case is similar). Observe that

$$\begin{aligned}|E(H) \cap K_3^{(2)}(T)| &\geq |K_3^{(2)}(T) \cap K_3^{(2)}(T_{A,\nu}(\mathbf{P}^1_{\nu,A}, \mathbf{Q}^A_\nu)) \cap \Gamma_A| \\ &\geq |K_3^{(2)}(T)| - |K_3^{(2)}(T) \setminus \Gamma_A| - |K_3^{(2)}(T) \setminus K_3^{(2)}(T(\mathbf{P}^1_{\nu,A}, \mathbf{Q}^A_\nu))|.\end{aligned}$$

By definition of $\mathcal{T}_A^{\text{good}}$ as a subset of $\mathcal{T}_A$, and since ϵ is sufficiently small compared to μ, we know that

$$|K_3^{(2)}(T) \setminus \Gamma_A| \leq 3\epsilon^{1/32}|K_3^{(2)}(T)| < \mu^{1/32}|K_3^{(2)}(T)|.$$

Using that $(X, Y) \in \Omega^1_{A,\nu}$, hypothesis (3b) and the fact that $\mathbf{Q}^A_\nu \subseteq Q_\nu$, we have

$$\begin{aligned}|K_3^{(2)}(T) \setminus K_3^{(2)}(T_{A,\nu}(\mathbf{P}^1_{\nu,A}, \mathbf{Q}^A_\nu))| &= \sum_{x \in X} |N_{\mathbf{Q}^A_\nu}(x)||Y \setminus N_{\mathbf{P}^1_{\nu,A}}(x)| \\ &\leq \sum_{x \in X} \Big(\frac{|W|}{\ell} + 1\Big)|Y \setminus N_{\mathbf{P}^1_{\nu,A}}(x)| \\ &= \Big(\frac{|W|}{\ell} + 1\Big)|K_2[X, Y] \setminus \mathbf{P}^1_{\nu,A}| \\ &\leq \mu^{1/32}|X||Y|\Big(\frac{|W|}{\ell} + 1\Big) \\ &\leq 2\mu^{1/32}|K_3^{(2)}(T)|,\end{aligned}$$

where the last inequality is since $|W|$ is sufficiently large, and since $T \in \mathcal{T}_A^{\text{good}}$ implies $|K_3^{(2)}(T)| \geq (1 - \mu^{1/32})|X||Y||W|/\ell$ (again this uses that $\epsilon < \mu$). Combining, we have that $|E(H) \cap K_3^{(2)}(T)| \geq (1 - 3\mu^{1/32})|K_3^{(2)}(T)|$, as desired. □

We now show that most triples in $K_3[A, V, W]$ come from a triad in $\mathcal{T}_A^{\text{good}}$.

Claim 4.8. *We have* $|\bigcup_{T \in \mathcal{T}_A^{\text{good}}} K_3^{(2)}(T)| \geq (1 - 3\mu^{1/32})|A||V||W|$.

Proof. We first bound the number of triples from $K_3[A, V, W]$ not in a $\mathcal{T}_A$ triad. By Claim 4.6 parts (1), (2), and (4), and since the $\mathbf{Q}^A_\nu$ are pairwise disjoint, we have

$$\begin{aligned}|K_3[A, V, W] \setminus \Gamma_A| &\leq |\mathbf{Q}^A_0||V| + \sum_{\nu \in \mathcal{S}^A} \sum_{\tau \in \{0,1\}} |E(H)^\tau \setminus K_3^{(2)}(T(\mathbf{P}^\tau_{\nu,A}, \mathbf{Q}^A_\nu))| \\ &\quad + \sum_{\nu \in \mathcal{S}^A} |K_3^{(2)}(T(\mathbf{P}^2_{\nu,A}, \mathbf{Q}^A_\nu))| \\ &\leq 3\epsilon^{1/8}|A||V||W| + \sum_{\nu \in \mathcal{S}^A} \sum_{\tau \in \{0,1\}} 2\epsilon^{3/16}|K_3^{(2)}(T(\mathbf{P}^\tau_{\nu,A}, \mathbf{Q}^A_\nu))| \\ &\quad + \sum_{\nu \in \mathcal{S}^A} |K_3^{(2)}(T(\mathbf{P}^2_{\nu,A}, \mathbf{Q}^A_\nu))| \\ &\leq 3\epsilon^{1/8}|A||V||W| + 2\epsilon^{3/16}|V| \sum_{\nu \in \mathcal{S}^A} |\mathbf{Q}^A_\nu| + \sum_{x \in A} \sum_{\nu \in \mathcal{S}^A} |N_{\mathbf{P}^2_{\nu,A}}(x)||N_{\mathbf{Q}^A_\nu}(x)| \\ &\leq 3\epsilon^{1/8}|A||V||W| + 2\epsilon^{3/16}|A||V||W| + \sum_{x \in A} \sum_{\nu \in \mathcal{S}^A} 4\epsilon^{1/16}|V||N_{\mathbf{Q}^A_\nu}(x)|\end{aligned}$$

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$$\leq 3\epsilon^{1/8}|A||V||W| + 2\epsilon^{3/16}|A||V||W| + 4\epsilon^{1/16}|A||V||W|$$
$$\leq 9\epsilon^{1/16}|A||V||W|.$$

Consequently, by Lemma 2.1 we have

$$\Big|\bigcup_{T\in\mathcal{T}_A} K_3^{(2)}(T)\Big| \geq (1-3\epsilon^{1/32})|A||V||W|.$$

We now consider triads from $\mathcal{T}_A'$. By equation (24),

$$\begin{aligned}\Big|K_3[A,V,W]\setminus\Big(\bigcup_{T\in\mathcal{T}_A'} K_3^{(2)}(T)\Big)\Big| &\leq |\mathbf{Q}_0^A||V| + \sum_{v\in\mathcal{S}^A}\Bigg(\sum_{\{(X,Y)\in(\mathcal{P}_U\times\mathcal{P}_V)\setminus(\Omega^0_{A,v}\cup\Omega^1_{A,v})\}}\Big(\sum_{x\in X}|Y||N_{\mathbf{Q}_v^A}(x)|\Big)\Bigg)\\ &\leq \Big(\frac{|W|}{\ell}+1\Big)\sum_{v\in\mathcal{S}^A}\Bigg(\sum_{\{(X,Y)\in(\mathcal{P}_U\times\mathcal{P}_V)\setminus(\Omega^0_{A,v}\cup\Omega^1_{A,v})\}}|X||Y|\Bigg)\\ &\leq \Big(\frac{|W|}{\ell}+1\Big)\mu^{1/32}|A||V|\\ &\leq 2\mu^{1/32}|W||A||V|,\end{aligned}$$

where the last inequality uses that $|W|$ is sufficiently large. We now turn to $\mathcal{T}_A''$. Given $X\in\mathcal{D}_A^A$, define

$$\mathcal{S}^A_{\text{small}}(X) = \Big\{v\in\mathcal{S}^A : |\mathbf{Q}_v^A\cap K_2[X,W]| < \frac{(1-\epsilon^{1/32})|X||W|}{\ell}\Big\}.$$

For $v\in\mathcal{S}^A$ and $Y\in\mathcal{D}_V^A$, we observe $T_{A,v}(X,Y)\notin\mathcal{T}_A''$ if and only if $v\in\mathcal{S}^A_{\text{small}}(X)$, since by definition,

$$|K_3^{(2)}(T_{A,v}(X,Y))| = |\mathbf{Q}_v^A\cap K_2[X,W]||Y|.$$

By Claim 4.6(1), we have

$$(1-3\epsilon^{1/8})|X||W| \leq |K_2[X,W]\setminus\mathbf{Q}_0^A|. \tag{25}$$

By definition of $\mathcal{S}^A_{\text{small}}(X)$, we have

$$\begin{aligned}|K_2[X,W]\setminus\mathbf{Q}_0^A| &\leq \frac{|\mathcal{S}^A_{\text{small}}(X)|(1-\epsilon^{1/32})|X||W|}{\ell} + |\mathcal{S}^A\setminus\mathcal{S}^A_{\text{small}}(X)||X|\Big(\frac{|W|}{\ell}+1\Big)\\ &\leq |X|(|\mathcal{S}^A| - \epsilon^{1/32}|\mathcal{S}^A_{\text{small}}(X)|)\Big(\frac{|W|}{\ell}+1\Big)\\ &\leq |X||W|\Big(1-\frac{\epsilon^{1/32}|\mathcal{S}^A_{\text{small}}(X)|}{\ell}\Big)\Big(1+\frac{\ell}{|W|}\Big)\\ &\leq |X||W|\Big(1-\frac{2\epsilon^{1/32}|\mathcal{S}^A_{\text{small}}(X)|}{\ell}\Big),\end{aligned}$$

where the second to last inequality uses that $|\mathcal{S}^A|\leq\ell$, and the last uses that $|W|$ is sufficiently large. Combining with equation (25), we have that

$$|X||W|\Big(1-\frac{2\epsilon^{1/32}|\mathcal{S}^A_{\text{small}}(X)|}{\ell}\Big) \geq (1-3\epsilon^{1/8})|X||W|,$$

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so

$$|\mathcal{S}^A_{\text{small}}(X)| \le \frac{3\epsilon^{1/8-1/32}\ell}{2} \le \epsilon^{1/16}\ell,$$

where the last inequality is since ϵ is sufficiently small. Since ϵ is sufficiently small and $|W|$ is sufficiently large, this implies

$$\begin{aligned}\Big|K_3[A,V,W]\setminus\Big(\bigcup_{T\in\mathcal{T}''_A} K_3^{(2)}(T)\Big)\Big| &\le \sum_{X\in\mathcal{D}^A_A}\sum_{v\in\mathcal{S}^A_{\text{small}}(X)} |\mathbf{Q}^A_v\cap K_2[X,W]||V| \\ &\le \sum_{X\in\mathcal{D}^A_A}\sum_{v\in\mathcal{S}^A_{\text{small}}(X)} |X||V|\Big(\frac{|W|}{\ell}+1\Big) \\ &\le 2\epsilon^{1/16}|A||V||W|.\end{aligned}$$

We now have that

$$\Big|K_3[A,V,W]\setminus\Big(\bigcup_{T\in\mathcal{T}^{\text{good}}_A} K_3^{(2)}(T)\Big)\Big| \le (3\epsilon^{1/16}+2\mu^{1/32}+2\epsilon^{1/16})|A||V||W| \le 3\mu^{1/32}|A||V||W|,$$

where the last inequality uses that ϵ is sufficiently smaller than μ. This finishes the proof of Claim 4.8. □

Step VI. We show that for all $A\in\mathcal{X}\setminus\mathcal{X}_{\text{err}}$, the partition $\mathcal{D}^A_V$ arising from Step V is almost $6\mu^{1/64}$-good with respect to $H[A,V,W]$ (recall Definition 2.5). First, define

$$\mathcal{V}_A = \Big\{Y\in\mathcal{D}^A_V : \Big|\Big(\bigcup_{T\in\mathcal{T}^{\text{good}}_A} K_3^{(2)}(T)\Big)\cap K_3[A,Y,W]\Big| \ge (1-3\mu^{1/64})|A||Y||W|\Big\}.$$

Now set $\mathcal{G}_A=\bigcup_{Y\in\mathcal{V}_A} Y$. By Claim 4.8 and Lemma 2.1, $|\mathcal{G}_A|\ge\mu^{1/64}|V|$. We show every $Y\in\mathcal{V}_A$ is almost $6\mu^{1/64}$-good with respect to $H[A,V,W]$. Given $T_{A,v}(X,Y)\in\mathcal{T}^{\text{good}}_A$, let

$$B(T_{A,v}(X,Y)) = \Big\{xw\in\mathbf{Q}^A_v\cap K_2[X,W] : \frac{|N_H(xw)\cap Y|}{|Y|}\in[0,3\mu^{1/64})\cup(1-3\mu^{1/64},1]\Big\}.$$

Since any $T_{A,v}(X,Y)\in\mathcal{T}^{\text{good}}_A$ is $3\mu^{1/32}$-homogeneous with respect to $H[A,V,W]$ (by Claim 4.7), Lemma 2.1 implies

$$|B(T_{A,v}(X,Y))|\ge(1-\mu^{1/64})|\mathbf{Q}^A_v\cap K_2[X,W]|.$$

Given $Y\in\mathcal{D}^A_V$, let

$$\mathcal{S}(Y)=\{xw\in K_2[X,W] : |N_H(xw)\cap Y|/|Y|\in[0,3\mu^{1/64})\cup(1-3\mu^{1/64},1]\}.$$

For every $Y\in\mathcal{D}^A_V$ and $v\in\mathcal{S}^A$, we have

$$|\mathcal{S}(Y)\cap\mathbf{Q}^A_v| \ge \Big|\bigcup_{T_{A,v}(X,Y)\in\mathcal{T}^{\text{good}}_A} B(T_{A,v}(X,Y))\Big| \ge (1-\mu^{1/64})\Big|\bigcup_{T_{A,v}(X,Y)\in\mathcal{T}^{\text{good}}_A}\mathbf{Q}^A_v\cap K_2[X,W]\Big|.$$

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Thus, for every $Y \in \mathcal{D}_V^A$,

$$|\mathcal{S}(Y)| \geq \left| \bigcup_{\nu \in \mathcal{S}^A} \bigcup_{T_{A,\nu}(X,Y) \in \mathcal{T}_A^{\text{good}}} B(T_{A,\nu}(X,Y)) \right| \geq (1-\mu^{1/64}) \left| \bigcup_{\nu \in \mathcal{S}^A} \bigcup_{T_{A,\nu}(X,Y) \in \mathcal{T}_A^{\text{good}}} \mathbf{Q}_\nu^A \cap K_2[X,W] \right|. \tag{26}$$

By definition of $\mathcal{V}_A$ and definition of the triads in $\mathcal{T}_A^{\text{good}}$, we have that for all $Y \in \mathcal{V}_A$,

$$\begin{aligned} |Y| \left| \bigcup_{\nu \in \mathcal{S}^A} \bigcup_{T_{A,\nu}(X,Y) \in \mathcal{T}_A^{\text{good}}} \mathbf{Q}_\nu^A \cap K_2[X,W] \right| &= \left| \left(\bigcup_{T \in \mathcal{T}_A^{\text{good}}} K_3^{(2)}(T) \right) \cap K_3[A,Y,W] \right| \\ &\geq (1-3\mu^{1/64})|A||Y||W|, \end{aligned}$$

where the last inequality is by Claim 4.8. Dividing by $|Y|$ yields

$$\left| \bigcup_{\nu \in \mathcal{S}^A} \bigcup_{T_{A,\nu}(X,Y) \in \mathcal{T}_A^{\text{good}}} \mathbf{Q}_\nu^A \cap K_2[X,W] \right| \geq (1-3\mu^{1/64})|A||W|.$$

Combining with equation (26), we have that for every $Y \in \mathcal{V}_A$, $|\mathcal{S}(Y)| \geq (1-3\mu^{1/64})^2|A||W|$, and consequently, every $Y \in \mathcal{V}_A$ is almost $6\mu^{1/64}$-good with respect to $H[A,V,W]$. Since $|\mathcal{G}_A| \geq \mu^{1/64}|V|$, this finishes our verfication that $\mathcal{D}_V^A$ is almost $6\mu^{1/64}$-good with respect to $H[A,V,W]$.

Step VII. Define $\mathcal{P}_V$ to be the coarsest common refinement of the $\mathcal{D}_V^A$ as A ranges over the elements in $\mathcal{X} \setminus \mathcal{X}_{\text{err}}$. Note, using equation (17),

$$|\mathcal{P}_V| \leq (\mu^{-2k\ell})^{|\mathcal{X}|} \leq \mu^{-2k\ell(2\ell)^\ell \epsilon^{-\ell}} = \epsilon^{-O_k(2^\ell \ell^{\ell+1} \epsilon^{-\ell})},$$

where the last equality uses the definition of μ (see equation (23)) and the fact ϵ is sufficiently small compared to C_1^{-1}. By Lemma 4.4, $\mathcal{P}_V$ is almost $6^{1/4}\mu^{1/256}$-good with respect to $H[A,V,W]$ for every $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$. By Proposition 2.16, $\mathcal{P}_V$ is then $5 \cdot 6^{1/256} \cdot \mu^{1/16384}$-homogeneous with respect to $H[A,V,W]$ for every $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$. By definition of μ, C, and D (see equations (23) and (4)), $5 \cdot 6^{1/256} \cdot \mu^{1/16384} = C\epsilon^{1/D}$, so $\mathcal{P}_V$ is $C\epsilon^{1/D}$-homogeneous with respect to $H[A,V,W]$ for all $A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}$. Using equation (19), we have

$$\left| V \setminus \left(\bigcup_{A \in \mathcal{X} \setminus \mathcal{X}_{\text{err}}} A \right) \right| \leq |U_{\text{err}}| \leq 10\epsilon^{1/8}|U| < C\epsilon^{1/D}|V|,$$

where the last inequality is clear from the definition of C, D (see equation (4)). Thus $\mathcal{P}_V$ is $C\epsilon^{1/D}$-homogeneous with respect to H. This concludes the proof of Proposition 4.2. □

4.2 Proof of Theorem 4.1

In this section, we prove Theorem 4.1.

Proof of Theorem 4.1. Fix $k \geq 1$, and let $C_1 = C_1(k)$ and $D_1 = D_1(k)$ be from Theorem 2.10. Let $C = C(k)$, $D = D(k)$, and $\epsilon_0 = \epsilon_0(k)$ be from Proposition 4.2. Choose ϵ^* sufficiently small compared to $\epsilon_0, C_1^{-1}, D_1^{-1}, C^{-1}, D^{-1}$.

Fix $0 < \tau < \epsilon^*$. Let $\epsilon > 0$ be such that $5(C\epsilon^{1/D})^{1/64} = \tau$ and let $\ell = C_1\epsilon^{-2D_1}$.

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Suppose n is sufficiently large, and $H = (U \cup V \cup W, F)$ is a tripartite 3-graph with $|U| = |V| = |W| = n$, and with slicewise VC-dimension less than k. We will show H has a τ-homogeneous partition of size at most $2^{2^{\tau - O_k(1)}}$. The first main ingredient in the proof is the following claim.

Claim 4.9. *There exists a partition $\mathcal{P}_V$ of V which is almost $C\epsilon^{1/D}$-good with respect to H and so that*

$$|\mathcal{P}_V| \leq \epsilon^{-O_k(2^\ell 3^{\ell^2} \ell^{\ell+2} \epsilon^{-\ell})}.$$

We now give a sketch for how we will prove Claim 4.9. We start by obtaining regular partitions $\mathcal{P}_x$ for all the slice graphs H_x for $x \in U$. We then group together elements $x \in U$ based on the auxiliary structures arising from the regular partition $\mathcal{P}_x$ of H_x. This will set us up to apply Proposition 4.2 on H, restricted to each of these equivalence classes. We will then take the coarsest common refinements of the resulting partitions of V, and show this yields the conclusion of Claim 4.9.

Proof of Claim 4.9. Fix $x \in U$. We define several auxiliary objects associated to the slice-graph H_x. By Theorem 2.10, there exists an ϵ^2-homogeneous equipartition $\mathcal{P}_x = \{V_x^1, \dots, V_x^{\ell_x}, W_x^1, \dots, W_x^{\ell_x}\}$ of H_x, for some $\ell_x \leq \ell$, where $V = \bigcup_{i=1}^{\ell_x} V_x^i$ and $W = \bigcup_{i=1}^{\ell_x} W_x^i$. We emphasize to the reader that $\mathcal{P}_x$ is an *equipartition*. Let $\mathrm{Irr}(\mathcal{P}_x)$ be the set of pairs (V_x^i, W_x^j) which are not ϵ^2-homogeneous for H_x. By assumption, and since $\mathcal{P}_x$ is an equipartition, we have $|\mathrm{Irr}(\mathcal{P}_x)| \leq \epsilon^2 \ell_x^2$. Define

$$\mathcal{P}_x^{\mathrm{err},V} = \{V_i^x : |\{W_x^j : (V_x^i, W_x^j) \in \mathrm{Irr}(\mathcal{P}_x)\}| \geq \epsilon \ell_x\} \text{ and}$$
$$\mathcal{P}_x^{\mathrm{err},W} = \{W_i^x : |\{V_x^j : (V_x^j, W_x^i) \in \mathrm{Irr}(\mathcal{P}_x)\}| \geq \epsilon \ell_x\}.$$

Now let $B_x^0 = \bigcup_{X \in \mathcal{P}_x^{\mathrm{err},V}} X$ and $C_x^0 = \bigcup_{X \in \mathcal{P}_x^{\mathrm{err},W}} X$. Clearly

$$|B_x^0| \leq \epsilon |V| \text{ and } |C_x^0| \leq \epsilon |W|.$$

Fix enumerations

$$\{B_1^x, \dots, B_{r_x}^x\} = \{V_1^x, \dots, V_{\ell_x}^x\} \setminus \mathcal{P}_x^{\mathrm{err},V} \text{ and}$$
$$\{C_1^x, \dots, C_{s_x}^x\} = \{W_1^x, \dots, W_{\ell_x}^x\} \setminus \mathcal{P}_x^{\mathrm{err},W}.$$

We define an auxiliary edge-coloured graph $\mathcal{G}_x$ as follows. Let $\mathcal{G}_x$ have vertex set

$$\{b_0, c_0\} \cup \{b_i : i \in [r_x]\} \cup \{c_i : i \in [s_x]\}$$

and edge colours F_x^1, F_x^0, F_x^2, where

$$F_x^1 = \{b_i c_j : i \in [r_x], j \in [s_x], |E(H_x) \cap K_2[B_x^i, C_x^j]| \geq (1 - \epsilon^2)|B_x^i||C_x^j|\}$$
$$F_x^0 = \{b_i c_j : i \in [r_x], j \in [s_x], |E(H_x) \cap K_2[B_x^i, C_x^j]| \leq \epsilon^2 |B_x^i||C_x^j|\}$$
$$F_x^2 = \{b_i c_j : i = 0, j = 0, \text{ or } \epsilon^2|B_x^i||C_x^j| \leq |E(H_x) \cap K_2[B_x^i, C_x^j]| \leq (1 - \epsilon^2)|B_x^i||C_x^j|\}.$$

Note that for all $1 \leq i \leq r_x$ and $1 \leq j \leq s_x$,

$$|N_{F_x^2}(b_i)| \leq \frac{|W|}{\ell_x} |\{w \in W : w \in W_x^j \text{ for some } (V_x^i, W_x^j) \in \mathrm{Irr}(\mathcal{P}_x)\}| \leq \epsilon \ell \text{ and}$$
$$|N_{F_x^2}(c_j)| \leq \frac{|V|}{\ell_x} |\{v \in V : v \in V_x^i \text{ for some } (V_x^i, W_x^j) \in \mathrm{Irr}(\mathcal{P}_x)\}| \leq \epsilon \ell.$$

Given $x, x' \in U$, write $x \equiv x'$ if and only if $\ell_x = \ell_{x'}$ and $\mathcal{G}_x = \mathcal{G}_{x'}$. Clearly $\equiv$ is an equivalence relation. Let $\mathcal{E}$ be the set of equivalence classes of $\equiv$ in U. Note $|\mathcal{E}| \leq \ell 3^{\ell^2}$. To ease notation, given $X \in \mathcal{E}$, let ℓ_X, r_X, s_X, and $\mathcal{G}_X$ be such that for all $x \in X$, $\ell_x = \ell_X$, $r_x = r_X$, $s_x = s_X$, and $\mathcal{G}_x = \mathcal{G}_X$.

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By relabelling, we may assume $\mathcal{G}_X$ has vertex set $\{p_0, \dots, p_{r_X}, q_0, \dots, q_{s_X}\}$ and edge sets F_0^X, F_1^X, F_2^X. Let

$$\mathcal{E}_{\text{err}} = \left\{X \in \mathcal{E} : |X| < \frac{\epsilon|V|}{|\mathcal{E}|}\right\} \text{ and } U_{\text{err}} = \bigcup_{X \in \mathcal{E}_{\text{err}}} X.$$

Clearly $|U_{\text{err}}| \leq \epsilon|U|$. For each $X \in \mathcal{E} \setminus \mathcal{E}_{\text{err}}$, we apply Proposition 4.2 to $\mathcal{G}_X$ and $H[X, V, W]$ with parameters r_X, s_X, ℓ_X, and ϵ. This yields, for all $X \in \mathcal{E} \setminus \mathcal{E}_{\text{err}}$, a partition $\mathcal{P}_V^X$ of V so that the following hold.

- $\mathcal{P}_V^X$ is almost $C\epsilon^{1/D}$-good with respect to $H[X, V, W]$,
- $\mathcal{P}_V^X$ has size at most $\epsilon^{-O_k(2^\ell \ell^{\ell+1} \epsilon^{-\ell})}$.

We now let $\mathcal{P}_V$ be the coarsest common refinement of the $\mathcal{P}_V^X$ as X ranges over the elements of $\mathcal{E} \setminus \mathcal{E}_{\text{err}}$. Note that since $|\mathcal{E}| \leq \ell 3^{\ell^2}$, we have

$$|\mathcal{P}_V| \leq (\epsilon^{-O_k(2^\ell \ell^{\ell+1} \epsilon^{-\ell})})^{\ell 3^{\ell^2}} = \epsilon^{-O_k(2^\ell 3^{\ell^2} \ell^{\ell+2} \epsilon^{-\ell})}$$

This ends the proof of Claim 4.9. □

We now finish the proof of the theorem. By symmetry, analogues of Claim 4.9 hold with the roles of U, V, W permuted. Say $\mathcal{P}_U, \mathcal{P}_V$, and $\mathcal{P}_W$ are the resulting partitions of U, V, and W. Let $\mathcal{P} = \mathcal{P}_U \cup \mathcal{P}_V \cup \mathcal{P}_W$. Note

$$|\mathcal{P}| \leq 3\epsilon^{-O_k(2^\ell 3^{\ell^2} \ell^{\ell+2} \epsilon^{-50\ell})}. \tag{27}$$

By Proposition 2.16, $\mathcal{P}$ is $5(C\epsilon^{1/D})^{1/64}$-homogeneous with respect to H. By definition of ϵ, this means $\mathcal{P}$ is τ-homogeneous with respect to H. We now consider the bound on the size of $\mathcal{P}$. Note that by definition, $\epsilon = \tau^{O_k(1)}$, and thus $\ell = \tau^{O_k(1)}$. Combining with equation (27), we have $|\mathcal{P}| = 2^{2^{\tau^{O_k(1)}}}$. □

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Appendix A

We begin with a proof of Theorem 2.10, which we restate here for convenience.

Theorem A.1. *For all $k \geq 1$ there exist constants $C = C(k)$ and $D(k) = 4k + 4$ so that for all sufficiently small $\epsilon > 0$ the following holds. Suppose $H = (A \cup B, E)$ is a bipartite graph with VC-dimension less than k and $|A| = |B|$. Then there exists an equipartition $\mathcal{P}$ of $A \cup B$ refining $\{A, B\}$ which is ϵ-homogeneous for H and so that $\epsilon^{-1} \leq |\mathcal{P}| \leq C\epsilon^{-D}$.*

Proof. Let $c = c(k)$ be as in Theorem 1.1. Set $D' = 2k + 2$ and $C' = 2c$. Fix $\epsilon \in (0, 1/25)$ and assume $H = (A \cup B, E)$ is a bipartite graph of VC-dimension less than k with $|A| = |B|$. We show H has a $2\sqrt{\epsilon}$-homogeneous equipartition of size at least $8\epsilon^{-1}$ and most $C'\epsilon^{-D'}$. Theorem 2.10 then follows, as this means that for any $\mu \in (0, 2/5)$, H has a μ-homogeneous equipartition with at size at least $32\mu^{-2}$ and at most $C\mu^{-D}$ where $C = 4C'$ and $D = 2D' = 4k + 4$.

By Theorem 1.1, there is an ϵ-homogeneous equipartition $\mathcal{P} = \{V_1, \ldots, V_t\}$ of $V(H)$ with $8\epsilon^{-1} \leq t \leq c\epsilon^{-2k-1}$. We begin by finding an equipartition of $V(H)$ which refines $\{A, B\}$. We start by defining, for each $i \in [t]$, let $A_i = A \cap V_i$ and $B_i = B \cap V_i$. These sets will not all have the same size, so we will now collect together the ones which are too small, and split the remaining parts into equally sized pieces. In particular, define

$$\mathcal{A}_0 = \{A_i : i \in [t], |A_i| < \epsilon|A|/t\} \text{ and } \mathcal{B}_0 = \{B_i : i \in [t], |B_i| < \epsilon|B|/t\}.$$

For each $A_i \notin \mathcal{A}_0$, let $A_{i0}, A_{i1}, \ldots, A_{ij_i}$ be a partition of A_i so that for each $1 \leq u \leq j_i$, $|A_{iu}| = \lceil \epsilon|A|/t \rceil$ and $|A_{i0}| \leq \epsilon|A|/t$. Similarly, for each $B_i \notin \mathcal{B}_0$, let $B_{i0}, B_{i1}, \ldots, B_{is_i}$ be a partition of B_i so that for each $1 \leq u \leq s_i$, $|B_{iu}| = \lceil \epsilon|B|/t \rceil$ and $|B_{i0}| \leq \epsilon|A|/t$. Fix new enumerations

$$\{A'_1, \ldots, A'_{t_1}\} = \{A_{iu} : A_i \notin \mathcal{A}_0, 1 \leq u \leq j_i\} \text{ and}$$
$$\{B'_1, \ldots, B'_{t_2}\} = \{B_{iu} : B_i \notin \mathcal{B}_0, 1 \leq u \leq s_i\}.$$

Note $t_1, t_2 \leq \epsilon^{-1}t$. Without loss of generality, assume $t_1 \leq t_2$ (the other case is similar). Now define

$$A_0 = \Big(\bigcup_{A_i \in \mathcal{A}_0} A_i\Big) \cup \Big(\bigcup_{A_i \notin \mathcal{A}_0} A_{i0}\Big) \text{ and } B_0 = \Big(\bigcup_{B_i \in \mathcal{B}_0} B_i\Big) \cup \Big(\bigcup_{B_i \notin \mathcal{B}_0} B_{i0}\Big) \cup \Big(\bigcup_{t_1 < i \leq t_2} B'_i\Big).$$

Clearly $|A_0| \leq \epsilon|A|$. Since each $|A'_j|$ and $|B'_i|$ have the same size,

$$|B_0| = |B| - t_1\lceil \epsilon|B|/t \rceil = |A| - t_1\lceil \epsilon|A|/t \rceil = |A_0| \leq 2\epsilon|A| = 2\epsilon|B|.$$

Now let $\mathcal{P}' = \{A_0, B_0\} \cup \{A'_i : i \in [t_1]\} \cup \{B'_i : i \in [t_1]\}$. Let Σ_{reg} be the set of ϵ-homogeneous pairs from $\mathcal{P}$, and let Σ'_{reg} be the set of $\sqrt{\epsilon}$-homogeneous pairs from $\mathcal{P}'$. By assumption, $\sum_{(X,Y) \in \Sigma_{\text{reg}}} |X||Y| \geq (1 - \epsilon)|V(H)|^2$. Since $\mathcal{P}'$ refines $\mathcal{P}$, we have by Lemma 2.1 that for any

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$(X, Y) \in \Sigma_{\text{reg}}$,

$$\Big| \bigcup_{\{(X',Y') \in \Sigma'_{\text{reg}} : X' \subseteq X, Y' \subseteq Y\}} X' \times Y'] \Big| \geq (1 - \sqrt{\epsilon})|X||Y|.$$

Thus

$$\Big| \bigcup_{(X',Y') \in \Sigma'_{\text{reg}}} X' \times Y' \Big| \geq (1 - \sqrt{\epsilon}) \Big| \bigcup_{(X,Y) \in \Sigma_{\text{reg}}} X \times Y \Big| \geq (1 - \sqrt{\epsilon})(1 - \epsilon)|V(H)|^2.$$

Now choose arbitrary equipartitions $A_0 = A_{01} \cup \cdots \cup A_{0t_1}$ and $B_0 = B_{01} \cup \cdots \cup B_{0t_1}$, and for each $i \in [t]$, let $A''_i = A'_i \cup A_{0i}$ and $B''_i = B'_i \cup B_{0i}$. Note that for each $i \in [t_1]$, $|A_{0i}| < 2\epsilon|A|/t_1 = 2\epsilon|A''_i|$ and $|B_{0i}| < 2\epsilon|B|/t_1 = 2\epsilon|B''_i|$. Consequently, if (A'_i, B'_j) is in Σ'_{reg}, is is straightforward to show (A''_i, B''_j) is $2\sqrt{\epsilon}$-homogeneous (this uses the ϵ is sufficiently small). Finally, we set

$$\mathcal{P}'' = \{A''_i : i \in [t_1]\} \cup \{B''_i : i \in [t_1]\},$$

and we let Σ''_{reg} be the set of $2\sqrt{\epsilon}$-homogeneous pairs from $\mathcal{P}''$. We then have

$$\begin{aligned} \Big| \bigcup_{(X'',Y'') \in \Sigma''_{\text{reg}}} X' \times Y' \Big| &\geq \Big| \bigcup_{(X',Y') \in \Sigma'_{\text{reg}}} X' \times Y' \Big| - |A_0||B| - |A||B_0| \\ &\geq (1 - \sqrt{\epsilon})|A||B| - 4\epsilon|A||B| \\ &\geq (1 - 2\sqrt{\epsilon})|A||B|, \end{aligned}$$

where the last inequality uses that ϵ is sufficiently small. We have now constructed $\mathcal{P}''$, a $2\sqrt{\epsilon}$-homogeneous equipartition of $A \cup B$ refining $\{A, B\}$. Note $|\mathcal{P}''| = 2t_1 \leq 2\epsilon^{-1}t \leq 2c\epsilon^{-2k-2} = C'\epsilon^{-D'}$. This finishes the proof. □

We next prove Lemma 4.3, which we restate here for convenience.

Lemma A.2. *Suppose $G = (A \cup B, E)$ is a bipartite graph and $\delta > 0$. Then there exists an integer $t \leq \delta^{-1/2}$ and a partition $A = A_{err} \cup A_0 \cup A_1 \cup \cdots \cup A_t$ of A so that the following hold.*

- *(i)* $|A_{err}| \leq \delta^{1/8}|A|$,
- *(ii) For each $1 \leq i \leq t$, there is a vertex $b_i \in B$ so that $A_i \subseteq N_G(b_i)$ and $|A_i| = \lceil \sqrt{\delta}|A| \rceil$,*
- *(iii) For all $x \in A_0$, $|N_G(x)| \leq \delta^{1/8}|B|$,*

Proof. We build the partition by induction as follows.

Step 1: If there exists some $b_1 \in B$ with $|N_G(b_1)| \geq \lceil \delta^{1/2}|A| \rceil$, let A_1 be any subset of $N_G(b_1)$ of size $\lceil \delta^{1/2}|A| \rceil$, and go to step 2. Otherwise, no such b_1 exists, meaning that for all $b \in B$, $|N_G(b)| \leq \lfloor \delta^{1/2}|A| \rfloor$. Consequently, $|E| \leq \delta^{1/2}|A||B|$, so by Lemma 2.1, there exists $A_0 \subseteq A$ of size at least $(1 - \delta^{1/4})|A|$ so that for all $x \in A_0$, $|N_G(x)| \leq \delta^{1/4}|B|$. Now set $t = 0$, let $A_{\text{err}} = A \setminus A_0$, and end the construction.

Step j+1: Suppose by induction we have defined $A_1, \ldots, A_j$. If $|A \setminus (\bigcup_{j'=1}^{j} A_{j'})| \leq \delta^{1/4}|A|$, set $t = j$, $A_0 = \emptyset$, $A_{\text{err}} = A \setminus (\bigcup_{j'=1}^{j} A_{j'})$ and end the construction.

Assume now $|A \setminus (\bigcup_{j'=1}^{j} A_{j'})| > \delta^{1/4}|A|$. If there exists $b_{j+1} \in B$ with

$$|N_G(b_{j+1}) \cap \Big(A \setminus \Big(\bigcup_{j'=1}^{j} A_{j'}\Big)\Big)| \geq \lceil \delta^{1/2}|A| \rceil,$$

then let A_{j+1} be any subset of $N_G(b_{j+1}) \cap (A \setminus (\bigcup_{j'=1}^{j} A_{j'}))$ of size $\lceil \delta^{1/2}|A| \rceil$, and go to step $j + 2$.

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We are left with the case where $|A \setminus (\bigcup_{j'=1}^{j} A_{j'})| > \delta^{1/4}|A|$, and for all $b \in B$,

$$|N_G(b) \cap \Big(A \Big\backslash \Big(\bigcup_{j'=1}^{j} A_{j'}\Big)\Big)| \leq \lfloor \delta^{1/2}|A| \rfloor.$$

In this case we have

$$\Big|E \cap K_2\Big[A \Big\backslash \Big(\bigcup_{j'=1}^{j} A_{j'}\Big), B\Big]\Big| \leq \delta^{1/2}|A||B| < \delta^{1/4}\Big|A \Big\backslash \Big(\bigcup_{j'=1}^{j} A_{j'}\Big)\Big||B|.$$

By Lemma 2.1, the set $A_0 := \{a \in A \setminus (\bigcup_{j'=1}^{j} A_{j'}) : |N_G(x)| \leq \delta^{1/8}|B|\}$ has size at least $(1-\delta^{1/8})|A \setminus (\bigcup_{j'=1}^{j} A_{j'})|$. Now set $t = j$, $A_{\mathrm{err}} = A \setminus ((\bigcup_{j'=1}^{j} A_{j'}) \cup A_0)$, and end the construction.

Clearly this algorithm will halt at some stage $t \leq \delta^{-1/2}$, producing the desired partition. □

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