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Coefficient inequalities for analytic functions in H^1

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Abstract

Improving earlier result of Hardy and Littlewood[1] and McGehee, Pigno and Smith[2] we show for analytic functions $F(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc that

$$\sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n} |a_i|^2}{(\sum_{i=0}^{n} |a_i|)^2} |a_n| \leqslant 66 \|F\|_{H^1}, \quad \sum_{n=0}^{\infty} \frac{|a_n|^2}{\sum_{i=0}^{n} |a_i|} \leqslant 660 \|F\|_{H^1}.$$

For any analytic function F on the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$ we define, as usual,

$$\|F\|_{H^1} = \sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{it})| \, dt.$$

The Hardy space H^1 consists of all those functions, for which this norm is finite. Belonging to H^1 implies the following consequence for the coefficients of the power series expansion.

THEOREM 1 (Hardy and Littlewood[1]). *For any analytic function* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *on* D *we have*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leqslant \pi \|F\|_{H_1}$$

If some coefficients a_n are zero, one can do better.

THEOREM 2 (McGehee, Pigno and Smith[2]). *Let* (n_i) *be an increasing sequence in* $\mathbf{N} \cup \{0\}$ *and* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *on* D *such that* $a_n = 0$ *if* $n \notin \{n_i : i \in \mathbf{N}\}$. *Then we have*

$$\sum_{i=0}^{\infty} \frac{|a_{n_i}|}{i} \leqslant C \|F\|_{H_1},$$

where the constant C *is independent of* (n_i) *and* F.

Theorem 2 immediately proves the so-called Littlewood conjecture. If there are many 'small' but non-zero coefficients, it is natural to suppose a similar improvement of Theorem 1 as in the case of many zero coefficients. This is the subject of this paper. We have the following

THEOREM 3. *For any analytic function* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *on* D *we have*

$$\sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n} |a_i|^2}{(\sum_{i=0}^{n} |a_i|)^2} |a_n| \leqslant 66 \|F\|_{H^1},$$

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where we put $\frac{0}{0} = 0$.

Let now F and (n_i) as in Theorem 2. Then, by the Cauchy–Schwarz inequality, we have

$$\sum_{i=1}^{k} |a_{n_i}| \leqslant \left(\sum_{i=1}^{k} |a_{n_i}|^2\right)^{1/2} k^{1/2}$$

and therefore

$$\frac{1}{k} \leqslant \frac{\sum_{i=1}^{k} |a_{n_i}|^2}{(\sum_{i=1}^{k} |a_{n_i}|)^2} = \frac{\sum_{i=0}^{n_k} |a_i|^2}{\sum_{i=0}^{n_k} |a_i|)^2}.$$

Thus Theorem 3 contains Theorem 2.

Rather than proving Theorem 3 we prove the following slightly stronger result.

THEOREM 4. *For any analytic function* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *on* D *we have*

$$\sum_{n=0}^{\infty} c_n |a_n| \leqslant 66 \|F\|_{H^1},$$

where

$$c_n = \sup_{m \geqslant n} \frac{\sum_{i=0}^{m} |a_i|^2}{(\sum_{i=0}^{m} |a_i|)^2}.$$

Proof. Replacing $F(z)$ by $F(rz)$ and letting r tend to 1 at the end of the proof, we may assume that

$$\sum_{n=0}^{\infty} |a_n| < \infty. \tag{1}$$

We now recall some standard notation. For a complex valued measurable function f on $[0, 2\pi]$ we define

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p \, dt\right)^{1/p}$$

and L^p is the class of all functions, for which this norm is finite. For $f \in L^1$ we define the Fourier coefficients by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt \quad (-\infty < n < \infty).$$

For $k \geqslant 1$ we now define $n_k = \sup\{n \geqslant 0 : c_n \geqslant \alpha_1^{2k}\}$ (this is well defined, because $c_0 = 1$) and set $n_0 = -1$. The constant $0 < \alpha_1 < 1$ will be specified at the end of the proof. Because of (1), there are only finitely many k with $n_k < \infty$. Let $k_0 = \inf\{k \geqslant 0 : n_k = \infty\}$. We have

$$c_{n_k} = \frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \geqslant \alpha_1^{2k} \quad (1 \leqslant k < k_0), \tag{2}$$

otherwise we would have

$$c_{n_k+1} = \sup_{m > n_k} \frac{\sum_{i=0}^{m} |a_i|^2}{(\sum_{i=0}^{m} |a_i|)^2} = \sup_{m \geqslant n_k} \frac{\sum_{i=0}^{m} |a_i|^2}{(\sum_{i=0}^{m} |a_i|)^2} = c_{n_k} \geqslant \alpha_1^{2k}$$

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contradicting the definition of n_k. From (2) we get

$$\alpha_1^2 c_j < \alpha_1^{2k} \leqslant \frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \quad (1 \leqslant k \leqslant k_0, n_{k-1} < j). \tag{3}$$

If $k = k_0$ then using (1) we have the following analogue of (2)

$$\alpha_1^{2k} \leqslant \lim_{n\to\infty} c_n = \limsup_{n\to\infty} \frac{\sum_{i=0}^{n} |a_i|^2}{(\sum_{i=0}^{n} |a_i|)^2} = \frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \quad (k = k_0). \tag{4}$$

Defining

$$d_k = \alpha_1^{-2k} \sum_{i=0}^{n_k} |a_i|^2, \quad f_k = d_k^{-\frac{1}{2}} \sum_{i=0}^{n_k} \overline{a_n}\, e^{int}$$

for any $1 \leqslant k \leqslant k_0$, we have

$$\|f\|_2^2 = d_k^{-1} \sum_{i=0}^{n_k} |a_i|^2 = \alpha_1^{2k} \quad (1 \leqslant k \leqslant k_0), \tag{5}$$

$$\left(\sum_{i=0}^{n_k} |a_i|\right)^2 \leqslant d_k \quad (1 \leqslant k \leqslant k_0), \tag{6}$$

$$\alpha_1^2 d_k \leqslant \left(\sum_{i=0}^{n_k} |a_i|\right)^2 \quad (1 \leqslant k \leqslant k_0), \tag{7}$$

$$|f_k| \leqslant 1 \quad (1 \leqslant k \leqslant k_0) \tag{8}$$

(5) is an immediate consequence of Parseval's identity. (6) follows from (2) if $k < k_0$ and from (4) if $k = k_0$. From the definition of n_k we get for $1 \leqslant k \leqslant k_0$ with $n_k > n_{k-1}$

$$\frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \leqslant \alpha_1^{2(k-1)},$$

whence, using the definition of d_k, (7) follows from

$$\alpha_1^2 d_k = \alpha_1^{-2(k-1)} \sum_{i=0}^{n_k} |a_i|^2 \leqslant \left(\sum_{i=0}^{n_k} |a_i|\right)^2.$$

(8) follows from (6) and the definition of f_k. Defining

$$h_k(t) = \widehat{|f_k|}\,(0) + 2 \sum_{m=-\infty}^{-1} \widehat{|f_k|}\,(m)\, e^{imt}$$

we have

$$\operatorname{Re} h_k = |f_k| \quad (1 \leqslant k \leqslant k_0), \tag{9}$$

because

$$\begin{aligned} \operatorname{Re} h_k(t) &= \tfrac{1}{2}(h_k(t) + \overline{h}_k(t)) \\ &= \tfrac{1}{2}\widehat{|f_k|}\,(0) + \sum_{m=-\infty}^{-1} \widehat{|f_k|}\,(m)\, e^{imt} + \overline{\tfrac{1}{2}\widehat{|f_k|}\,(0)} + \sum_{m=-\infty}^{-1} \overline{\widehat{|f_k|}\,(m)}\, e^{-imt} \\ &= \sum_{m=-\infty}^{-\infty} \widehat{|f_k|}\,(m)\, e^{imt} = |f_k(t)|, \end{aligned}$$

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where the second equality follows from $\widehat{|f_k|}(m) = \widehat{|f_k|}(-m)$. From the last fact and Parseval's identity we also get

$$\|h_k\|_2^2 = |\widehat{|f_k|}(0)|^2 + 4\sum_{m=-\infty}^{-1} |\widehat{|f_k|}(m)|^2 = -|\widehat{|f_k|}(0)|^2 + 2\sum_{m=-\infty}^{\infty} |\widehat{|f_k|}(m)|^2 \leqslant 2\|f_k\|_2^2. \quad (10)$$

We now fix $0 < \alpha_2 < 2$ and put $\alpha_3 = \alpha_2 - \alpha_2^2/2$. Maximizing the function $\alpha_2 - \alpha_2^2/2$ we see that $0 < \alpha_3 \leqslant \frac{1}{2}$ for any such choice of α_2. For any $0 \leqslant x \leqslant 1$ we have

$$e^{-\alpha_2 x} + \alpha_3 x \leqslant 1 - \alpha_2 x + \tfrac{1}{2}(\alpha_2 x)^2 + \alpha_3 x \leqslant 1. \quad (11)$$

Defining inductively

$$g_1 = \alpha_3 f_1, \quad g_{k+1} = \alpha_3 f_{k+1} + g_k e^{-\alpha_2 h_{k+1}} \quad (1 \leqslant k < k_0),$$

we have

$$\|g_k\|_\infty \leqslant 1 \quad (1 \leqslant k \leqslant k_0). \quad (12)$$

Indeed, because $0 \leqslant \operatorname{Re} h_k = |f_k(x)| \leqslant 1$ by (9) and (8), we have $\|g_1\|_\infty \leqslant \alpha_3 \leqslant \frac{1}{2}$ and using (11) the assertion (12) follows inductively from

$$|g_{k+1}(x)| \leqslant |g_k(x)|\, e^{-\alpha_2 \operatorname{Re} h_{k+1}(x)} + \alpha_3 |f_{k+1}(x)| \leqslant e^{-\alpha_2 |f_{k+1}(x)|} + \alpha_3 |f_{k+1}(x)| \leqslant 1.$$

Because $\hat{h}_k(n) = 0$ for $n > 0$, we have also $(e^{-\alpha_2 h_k})\hat{}\,(n) = 0$ for any $n > 0$ and therefore

$$\hat{g}_k(n) = 0 \quad (1 \leqslant k \leqslant k_0, n > n_k).$$

Putting

$$u = g_{k_0}, \quad v = \alpha_3 \sum_{j=1}^{k_0} f_j,$$

we have for $1 \leqslant k \leqslant k_0$ and $n > n_{k-1}$

$$\hat{u}(n) - \hat{v}(n) = \left(\sum_{j=k}^{k_0} \alpha_3 f_j \left(\exp\left(-\alpha_2 \sum_{l=j+1}^{k_0} h_l\right) - 1\right)\right)\hat{}\,(n). \quad (13)$$

By the mean value theorem, we have

$$|e^{-z} - 1| \leqslant |z| \sup_{0<t<1} |e^{-tz}| \leqslant |z| \quad (\operatorname{Re} z \geqslant 0),$$

whence, using (5) and (10), we obtain

$$\begin{aligned}\left\| f_j \left(\exp\left(-\alpha_2 \sum_{l=j+1}^{k_0} h_l\right) - 1\right)\right\|_1 &\leqslant \|f_j\|_2 \left\|\exp\left(-\alpha_2 \sum_{l=j+1}^{k_0} h_l\right) - 1\right\|_2 \\ &\leqslant \|f_j\|_2 \left\|\alpha_2 \sum_{l=j+1}^{k_0} h_l\right\|_2 \leqslant \|f_j\|_2 \alpha_2 \sum_{l=j+1}^{k_0} \sqrt{2}\|f_l\|_2 \\ &= \sqrt{2}\alpha_2 \alpha_1^j \sum_{l=j+1}^{k_0} \alpha_1^l \leqslant \frac{\sqrt{2}\alpha_2 \alpha_1^{2j+1}}{1-\alpha_1}.\end{aligned}$$

Combining this with (13) and the inequality $|\hat{w}(n)| \leqslant \|w\|_1 (w \in L^1)$ we get

$$|\hat{u}(n) - \hat{v}(n)| \leqslant \sum_{j=k}^{k_0} \alpha_3 \frac{\sqrt{2}\alpha_2 \alpha_1^{2j+1}}{1-\alpha_1} \leqslant \frac{\sqrt{2}\alpha_1 \alpha_2 \alpha_3}{(1-\alpha_1)(1-\alpha_1^2)} \alpha_1^{2k} \quad (n > n_{k-1}).$$

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Hence using (5), (6), we obtain

$$\begin{aligned}
\sum_{i=0}^{\infty} |\hat{u}(i)-\hat{v}(i)|\,|a_i| &= \sum_{k=1}^{k_0} \sum_{i=n_{k-1}+1}^{n_k} |\hat{u}(i)-\hat{v}(i)|\,|a_i| \\
&\leqslant \sum_{k=1}^{k_0} \frac{\sqrt{2}\alpha_1\alpha_2\alpha_3}{(1-\alpha_1)(1-\alpha_1^2)} \alpha_1^{2k} \sum_{i=n_{k-1}+1}^{n_k} |a_i| \\
&\leqslant \frac{\sqrt{2}\alpha_1\alpha_2\alpha_3}{(1-\alpha_1)(1-\alpha_1^2)} \sum_{k=1}^{k_0} \alpha_1^{2k} \sum_{i=0}^{n_k} |a_i| \\
&= \frac{\sqrt{2}\alpha_1\alpha_2\alpha_3}{(1-\alpha_1)(1-\alpha_1^2)} \sum_{k=1}^{k_0} d_k^{-1} \sum_{i=0}^{n_k} |a_i|^2 \sum_{i=0}^{n_k} |a_i| \\
&\leqslant \frac{\sqrt{2}\alpha_1\alpha_2\alpha_3}{(1-\alpha_1)(1-\alpha_1^2)} \sum_{k=1}^{k_0} d_k^{-\frac{1}{2}} \sum_{i=0}^{n_k} |a_i|^2 \\
&= \frac{\sqrt{2}\alpha_1\alpha_2\alpha_3}{(1-\alpha_1)(1-\alpha_1^2)} \sum_{k=1}^{k_0} \sum_{i=0}^{n_k} \hat{f}_k(i)\, a_i \\
&= \frac{\sqrt{2}\alpha_1\alpha_2}{(1-\alpha_1)(1-\alpha_1^2)} \sum_{i=0}^{\infty} \hat{v}(i)\, a_i
\end{aligned}$$

and therefore

$$\begin{aligned}
\left|\sum_{i=0}^{\infty} \hat{u}(i)\, a_i\right| &\geqslant \left|\sum_{i=0}^{\infty} \hat{v}(i)\, a_i\right| - \sum_{i=0}^{\infty} |\hat{u}(i)-\hat{v}(i)|\,|a_i| \\
&\geqslant \left(1-\frac{\sqrt{2}\alpha_1\alpha_2}{(1-\alpha_1)(1-\alpha_1^2)}\right) \sum_{i=0}^{\infty} \hat{v}(i)\, a_i. \qquad (14)
\end{aligned}$$

On the other hand, by the definition of n_k together with (3) and (7) we have

$$\begin{aligned}
\sum_{i=0}^{\infty} \hat{v}(i)\, a_i &= \alpha_3 \sum_{k=1}^{k_0} \sum_{i=0}^{n_k} \hat{f}_k(i)\, a_i = \alpha_3 \sum_{k=1}^{k_0} d_k^{-\frac{1}{2}} \sum_{i=0}^{n_k} |a_i|^2 \\
&\geqslant \alpha_3 \sum_{k=1}^{k_0} \frac{\alpha_1}{\sum_{i=0}^{n_k} |a_i|} \sum_{i=0}^{n_k} |a_i|^2 = \alpha_1\alpha_3 \sum_{k=1}^{k_0} \frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \sum_{i=0}^{n_k} |a_i| \\
&\geqslant \alpha_1\alpha_3 \sum_{k=1}^{k_0} \frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \sum_{i=n_{k-1}+1}^{n_k} |a_i| \\
&\geqslant \alpha_1\alpha_3 \sum_{k=1}^{k_0} \sum_{i=n_{k-1}+1}^{n_k} \alpha_1^2 c_i |a_i| = \alpha_1^3\alpha_3 \sum_{i=0}^{n_k} c_i |a_i|.
\end{aligned}$$

Combining this with (14) we obtain

$$\left|\sum_{i=0}^{\infty} \hat{u}(i)\, a_i\right| \geqslant \alpha_1^3\alpha_3 \left(1-\frac{\sqrt{2}\alpha_1\alpha_2}{(1-\alpha_1)(1-\alpha_1^2)}\right) \sum_{i=0}^{n_k} c_i |a_i|. \qquad (15)$$

We have not yet used $\|F\|_{H^1}$. This norm will enter the proof through the following convolution:

$$w(t) = \frac{1}{2\pi} \int_0^{2\pi} q(t-s)\, u(s)\, ds,$$

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where $q(t) = \sum_{n=0}^{\infty} a_n e^{int}$, which defines a periodic continuous function $\mathbf{R}$ by (1). The following properties are well known for such convolutions:

$$\|w\|_\infty \leqslant \|u\|_\infty \|q\|_1 = \|u\|_\infty \|F\|_{H^1}, \tag{16}$$

$$\hat{w}(n) = \hat{u}(n)\,\hat{q}(n) = \begin{cases} \hat{u}(n)\,a_n & \text{if } n \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since q is continuous, w is continuous as well and therefore, by (16), (12) and $u = g_{k_0}$, we have

$$w(0) \leqslant \|u\|_\infty \|F\|_{H^1} \leqslant \|F\|_{H^1}. \tag{18}$$

By (1) and (17) we have

$$w(0) = \sum_{n=0}^{\infty} \hat{u}(n)\,a_n. \tag{19}$$

Combining now (15), (18) and (19) we finally obtain

$$\alpha_1^3 \alpha_3 \left(1 - \frac{\sqrt{2}\alpha_1 \alpha_2}{(1-\alpha_1)(1-\alpha_1^2)}\right) \sum_{i=0}^{n_k} c_i |a_i| \leqslant \|F\|_{H^1}. \tag{20}$$

Inserting $\alpha_1 = 0{\cdot}58$ and $\alpha_2 = 0{\cdot}16$ into this expression, the assertion follows.

COROLLARY 5. *For any analytic function* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *on* D *we have*

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{\sum_{i=0}^{n} |a_i|} \leqslant 660\,\|F\|_{H^1},$$

where we put $\frac{0}{0} = 0$.

Proof. Let $\alpha > 1$. Then, by Wittmann [**4**], lemma 3·3, there exist an increasing sequence $(n_k)_{k \in \mathbf{N}}$ such that

$$\alpha \sum_{i=0}^{n_k} |a_i| \leqslant \sum_{i=0}^{n_{k+1}} |a_i| \leqslant \alpha^3 \sum_{i=0}^{n_k+1} |a_i| \quad (k \in \mathbf{N}). \tag{1}$$

An inspection of the proof in [**4**] shows that we can also fulfil

$$\sum_{i=0}^{n_1} |a_i| \leqslant |a_0|\,\alpha^2. \tag{2}$$

Let (c_n) be as in Theorem 4. Using (1), (2) and putting $n_0 = -1$ we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{|a_n|^2}{\sum_{i=0}^{n} |a_i|} &= \sum_{k=1}^{\infty} \sum_{j=n_{k-1}+1}^{n_k} \frac{|a_j|^2}{\sum_{i=0}^{j} |a_i|} \leqslant \sum_{k=0}^{\infty} \sum_{j=n_{k-1}+1}^{n_k} \frac{|a_j|^2}{\alpha^{-3} \sum_{i=0}^{n_k} |a_i|} \\
&\leqslant \alpha^3 \sum_{k=0}^{\infty} \sum_{j=0}^{n_k} \frac{|a_j|^2}{\sum_{i=0}^{n_k} |a_i|} = \alpha^3 \sum_{k=0}^{\infty} \frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \sum_{j=0}^{n_k} |a_j| \\
&\leqslant \alpha^3 \sum_{k=0}^{\infty} \frac{\sum_{i=0}^{n_k} |a_i|^2}{(\sum_{i=0}^{n_k} |a_i|)^2} \frac{1}{1-\alpha^{-1}} \sum_{j=n_{k-1}+1}^{n_k} |a_j| \\
&\leqslant \frac{\alpha^3}{1-\alpha^{-1}} \sum_{k=0}^{\infty} \sum_{j=n_{k-1}+1}^{n_k} c_j |a_j| = \frac{\alpha^3}{1-\alpha^{-1}} \sum_{j=0}^{\infty} c_j |a_j|.
\end{aligned}$$

Choosing $\alpha = \frac{4}{3}$ the assertion follows from Theorem 4.

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Remarks. (*a*) We have not aimed at the best constant in Theorem 4 and Corollary 5. Substantial optimizations of the constant in the Littlewood conjecture have been done by Stegeman[3] and Yabuta[5].

(*b*) The estimates in Theorem 4 and Corollary 5 are not monotone, i.e. there exist analytic functions $F(z) = \sum_{n=0}^{\infty} a_n z^n, G(z) = \sum_{n=0}^{\infty} b_n z^n$ such that $|a_n| \leqslant |b_n|$ and

$$\sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n} |a_i|^2}{(\sum_{i=0}^{n} |a_i|)^2} |a_n| > \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n} |b_i|^2}{(\sum_{i=0}^{n} |b_i|)^2} |b_n|$$

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{\sum_{i=0}^{n} |a_i|} > \sum_{n=0}^{\infty} \frac{|b_n|^2}{\sum_{i=0}^{n} |b_i|}.$$

On the other hand, we may also have $\|F\|_{H^1} > \|G\|_{H^1}$.

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