Math. Proc. Camb. Phil. Soc. (1990), **107**, 261
Printed in Great Britain

A note on centrality in 3-manifold groups

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(*Received* 24 *April* 1989)

1. *Introduction and statement of results*

Centralizers in fundamental groups of 3-manifolds are well understood because of their relationship with Seifert fibre spaces. Jaco and Shalen's book [**4**] provides detailed information about Seifert fibre spaces in 3-manifolds, and consequently about centralizers in their fundamental groups. It is the purpose of this note to record two group-theoretic properties, both easily deduced from results of Jaco and Shalen. Doubtless many other authors could have established the same results had they needed them. Our motivation for writing this paper is that these properties can be used as a basis for group-theoretic proofs of certain fundamental results in 3-manifold theory: Proposition 1 below can be used as a basis for a proof of the Torus Theorem (cf. [**8**]) and Proposition 2 for the Torus Decomposition Theorem (cf. [**9**]).

Throughout the paper 3-manifolds are assumed to be connected. A (*compact/orientable*) 3-*manifold group* is any group isomorphic to the fundamental group of a (compact/orientable) 3-manifold.

The first result is a modest generalization of the theorem of Jaco and Shalen (theorem VI·2·1 of [**4**]) which asserts that the Baumslag–Solitar relation $y^{-1}x^m y = x^n$ cannot hold in the fundamental group of a sufficiently large 3-manifold unless $x^m = x^{\pm n}$. (Related results have also been proved by Kawauchi [**5**].) In fact, in many cases it is not necessary to assume the manifold sufficiently large:

Proposition 1. *If x is an element of infinite order in a 3-manifold group such that x^m, x^n are conjugate $(m, n \in \mathbb{Z})$ then $m = \pm n$.*

An argument for deducing this from theorem VI·2·1 of [**4**] is given here in Section 2. Using some of the methods underlying the original theorem of Jaco and Shalen, we are led to

Proposition 2. *Every 3-manifold group satisfies* Max-c, *the maximal condition on centralizers.*

This is proved in Section 4, after some preliminaries in Section 3. By a *centralizer* in a group G we mean any subgroup of the form

$$C_G(X) = \{g \in G; gx = xg \quad \text{for all} \quad x \in X\},$$

where X is a subset of G. The group G satisfies Max-c if and only if every ascending chain of centralizers breaks off. The most well known examples of groups satisfying this condition are linear groups: this is because every centralizer in the $n \times n$ matrix ring $M_n(K)$ over a field K is a linear subspace, so that chains of centralizers in $M_n(K)$ or in any subgroup of $\mathrm{GL}_n(K)$ have length at most n^2. It is conceivable, though not known, that all 3-manifold groups are linear.

https://doi.org/10.1017/S0305004100068523 Published online by Cambridge University Press

2. *Proof of Proposition* 1

It is very easy to prove Proposition 1 because the Baumslag–Solitar relation actually implies 'sufficiently large', and so one can appeal to the original theorem of Jaco and Shalen. To see this, let x be an element of infinite order in a group Γ, and suppose that

$$(*) \qquad y^{-1}x^m y = x^n$$

for some non-zero integers m, n and some group element y. Let $G = \langle x, y\rangle$ and let $H = \langle x\rangle$. Since H is infinite cyclic, (*) can be restated as

$$(**) \qquad H \quad \text{and} \quad H^y \quad \text{are commensurable.}$$

Put another way, this says that y belongs to the subgroup

$$\mathrm{Comm}_G(H) = \{g \in G; H \text{ and } H^g \text{ are commensurable}\}.$$

Since x clearly belongs to this subgroup we have

$$\mathrm{Comm}_G(H) = G.$$

Hence for any g in G there exist non-zero integers p, q such that

$$g^{-1}x^p g = x^q,$$

and we can define a function ϕ from G to the multiplicative group U of positive rational numbers by

$$\phi(g) = |p/q|.$$

It is at first sight rather surprising, but nevertheless very easy to check that

LEMMA 0. *ϕ is a well-defined group homomorphism.*

We leave the details to the reader. The idea can be generalized as, for example, in (3·4) of [**9**]. Proposition 1 now follows, for if Γ is a 3-manifold group then G is the fundamental group of a compact 3-manifold M say, by Scott's compact submanifold theorem [**14**], and the conclusion of Proposition 1 is equivalent to the assertion $\phi(y) = 1$. Therefore we assume that ϕ is non-trivial. Thus G has an infinite cyclic quotient by Lemma 0 and so $H_1(M)$ is infinite. If M is irreducible then it is sufficiently large and the result is known (theorem VI·2·1 of [**4**]). If M is not irreducible then G is either infinite cyclic or a non-trivial free product. In the second case G must be a free product of two cyclic groups because it is a 2-generator group. In either case it is easy to see that (*) can hold only if $m = \pm n$.

3. *Centrality in* PD^3*-groups*

Recall that a PD^n-group is a group G for which there are natural isomorphisms

$$H^i(G, -) = H_{n-i}(G, \tilde{\mathbb{Z}} \otimes -)$$

between cohomology and homology. Here, $\tilde{\mathbb{Z}}$ denotes the dualizing module and is a G-module with additive group $\mathbb{Z}$. The first result we need may be of independent interest.

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LEMMA 1. *Let G be a PD^n-group. Then the set $\Delta(G) = \{g \in G; [G : C_G(g)] < \infty\}$ of orbital elements of G, which is always torsion-free abelian, is either free abelian of rank n and of finite index in G or of rank at most $n-2$.*

Proof. The subgroup $\Delta(G)$ can be defined as above for any group, and a detailed discussion of its properties can be found in Passman's book [**11**]. Here, G is a torsion-free, and the fact that $\Delta(G)$ is abelian follows from lemma 4·1·6 and the remarks preceding 4·1·3 of [**11**]. Suppose that $\Delta(G)$ has rank at least $n-1$, and choose a free abelian subgroup A of this rank. Since A is generated by finitely many orbital elements, $C_G(A)$ has finite index in G, and so is again a PD^n-group. As in the proof of theorem 8·4 of [**1**], one can deduce that $C_G(A)/A$ has more than one end, and so contains an element of infinite order. By lifting such an element back it follows that $C_G(A)$ has a free abelian subgroup of rank n and this must have finite index; (cf. theorem 1 of [**3**] and the proof of lemma 3·2 of [**6**]). Thus G is abelian-by-finite and so $\Delta(G)$ is free abelian of rank n and has finite index.

Of more direct relevance to this paper is the following application of Lemma 1:

LEMMA 2. *Let G be a non-(abelian-by-finite) PD^3-group, or any group of cohomological dimension $\leqslant 2$. Then the index of any centralizer in G is either infinite or at most 2.*

Proof. We assume that G is a PD^3-group. For groups of cohomological dimension at most 2 there is a similar but easier argument, using theorem 8·8 of [**1**] in place of Lemma 1. Suppose that $C = C_G(X)$ has finite index in G. Then X belongs to $\Delta(G)$. Since G is not abelian-by-finite, $\Delta(G)$ must be torsion-free abelian of rank 1, by Lemma 1. Therefore $C_G(\Delta(G))$ has index at most 2 in G, and the result follows because $C_G(\Delta(G)) \subseteq C$.

In the next result we need to consider *coherent* groups, that is groups in which every finitely generated subgroup is finitely presented. Actually the proof only requires that there are no subgroups isomorphic to a direct product of two non-cyclic free groups. This can never happen in a coherent group, nor, by theorem C of [**7**], can it happen in a PD^3-group. In Section 4, where this result is applied, we shall of course know that the groups are coherent: this is true of all 3-manifold groups by Scott's theorem [**13**].

LEMMA 3. *Let G be a non-(abelian-by-finite) PD^3-group, or any coherent group of cohomological dimension $\leqslant 2$, and let C is a non-abelian centralizer of infinite index. Then* (i) *either C is a Klein bottle group or $[C, C]$ is a non-cyclic free group, and* (ii) *$C_G(C) = \zeta(C)$ is infinite cyclic.*

Proof. We give the proof for PD^3-groups, the argument for coherent groups of cohomological dimension at most 2 being similar. Set $D = C_G(C)$. If C has trivial centre then $C \cap D = 1$ and $\langle C, D\rangle \cong C \times D$. By theorem C of [**7**], C is a surface group and D is infinite cyclic. This implies that $D \subseteq C_G(D) = C$, a contradiction. Hence C has non-trivial centre. Since C has infinite index in G it has cohomological dimension at most 2, and hence $[C, C]$ is a free group by corollary 8·9 of [**1**]. If $[C, C]$ is infinite cyclic then it is easy to deduce that C is a Klein bottle group. Thus (i) is proved and it remains to show that D is infinite cyclic. If $F := [C, C]$ is a non-cyclic free group this follows from theorem C of [**7**], since $\langle C_G(F), F\rangle \cong C_G(F) \times F$, therefore $C_G(F)$ is cyclic,

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and D is infinite cyclic because it is a non-trivial subgroup of $C_G(F)$. Henceforth we assume that C is a Klein bottle group. If DC/C has an infinite finitely generated subgroup then arguments similar to the proof of Lemma 1 show that it has an element of infinite order, and then G is abelian-by-finite. Therefore DC/C is locally finite. Since $D \cap C$ is cyclic and D is torsion-free, it follows that D is locally cyclic. In particular D is abelian, and so it is contained in C, and must be infinite cyclic, as required.

4. *Proof of Proposition* 2

We shall prove a rather stronger result; namely that if G is a 3-manifold group then there is a bound on the length m of a strictly ascending chain

$$C_1 < C_2 < \ldots < C_m$$

of centralizers in G. The bound we obtain is 17, but we do not claim this is best possible. For the purposes of proving this it is convenient to have a shorthand notation: we shall simply say that G is *m-bad* if there is a chain of centralizers as above, of length m, and *n-good* if $m \leqslant n$ for every such chain. If G is both m-good and m-bad we might say that G has *centralizer dimension m.*

LEMMA 4. *Let G be a group which is m-bad. Then*

(i) *G has a finitely generated m-bad subgroup, and*

(ii) *G has a finitely generated freely indecomposable* $(m-1)$*-bad subgroup.*

Proof. (i) Let $C_1 < \ldots < C_m$ be a chain of centralizers in G, with $C_i = C_G(X_i)$. For $1 \leqslant i \leqslant m-1$ choose a non-commuting pair of elements $x_i \in X_i, g_i \in C_{i+1}$, and set

$$H = \langle x_1, g_1, \ldots, x_{m-1}, g_{m-1} \rangle, \quad Y_i = \{x_i, x_{i+1}, \ldots, x_{m-1}\}, \quad \text{and} \quad B_i = C_H(Y_i).$$

Set $B_m = H$. Then

$$B_1 < B_2 < \ldots < B_{m-1} < B_m$$

is a strictly ascending chain of centralizers in H, because g_i belongs to B_{i+1} but not B_i. Thus H is a subgroup of the required type.

(ii) In view of (i) we may assume that G is finitely generated. We can write G as a free product of finitely many freely indecomposable groups: $G = G_1 * G_2 * \ldots * G_k$. Now any proper centralizer in G is either self-centralizing and infinite cyclic or conjugate to a subgroup of some G_i. Therefore any strictly ascending chain of proper centralizers either consists of a single infinite cyclic group or can be conjugated into some G_i. The result follows.

This result is to be used to reduce the problem for a general 3-manifold group to that for the fundamental group of a compact irreducible 3-manifold. The various types of compact irreducible 3-manifold must be treated case by case. The following two Lemmas deal with two of the cases.

LEMMA 5. *Every finite closed* 3*-manifold group is* 16*-good.*

Proof. In [**10**], Milnor lists those finite groups which satisfy the group theoretic necessary conditions to be fundamental groups of closed 3-manifolds. All these groups have faithful representations in the unitary group $U(4)$, and so it is enough to check that $U(4)$ is 16-good. This is clear because it is a 4×4 matrix group.

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LEMMA 6. *Every abelian-by-finite* PD^3*-group is* 12*-good.*

Proof. These are precisely the torsion-free 3-dimensional crystallographic groups. They are all (discrete cocompact) subgroups of the group of isometries of Euclidean space, so it is sufficient to check that this full group of isometries is 12-good. The group is the split extension of $\mathbb{R}^3$ by $O(3)$, and can be represented as a 4×4 matrix group, from which the result is clear:

$$\left[\begin{array}{ccc|c} & O(3) & & \begin{matrix}\mathbb{R}\\ \mathbb{R}\\ \mathbb{R}\end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

LEMMA 7. *If G is an m-bad group then G has a strictly ascending chain of non-abelian centralizers of length $[\frac{1}{2}m]$.*

Proof. Let $C_1 < C_2 < \ldots < C_m$ be a strictly ascending chain of centralizers and set $D_i = C_G(C_i)$. Then $D_1 > D_2 > \ldots > D_m$ is a strictly descending chain of centralizers. Moreover if, for some i, both C_i and D_i are abelian then $C_i = D_i$, so this can happen for at most one i. Taking $i = [\frac{1}{2}m]$, we conclude that either D_i or C_{i+1} is non-abelian. Thus we find a chain of the required length, starting either at D_i or at C_{i+1}.

LEMMA 8. *Let M be a compact irreducible 3-manifold with infinite fundamental group G. Then G is* 12*-good.*

Proof. Note first that M is aspherical. This follows from the fact that the universal cover of M is simply connected, irreducible, and non-compact, and therefore contractible. Consequently G is either a PD^3-group, if M is closed, or of cohomological dimension at most 2, if M has non-empty boundary. In both cases G is coherent. If G is an abelian-by-finite PD^3-group then the result follows from Lemma 6. In all other cases, Lemma 3 can be applied, and in fact G is actually 11-good.

Suppose that G is 12-bad. Then there is a strictly ascending chain

$$C_0 < C_1 < C_2 < C_3 < C_4 < C_5$$

of non-abelian centralizers by Lemma 7. By Lemma 2, C_3 has infinite index in G, and by Lemma 3, the centralizers $Z_i = C_G(C_i)$ are all infinite cyclic, so that

$$Z_3 < Z_2 < Z_1 < Z_0 < C_0 < C_1 < C_2 < C_3$$

is a chain of centralizers in G. Note that C_i is actually a centralizer in C_{i+1} here. In particular, since proper centralizers in a Klein bottle group are abelian, it follows that C_1 is not a Klein bottle group, and hence it has a non-cyclic free subgroup.

The aim now is to choose a finitely generated subgroup H of C_3 which admits a similar chain of centralizers. This is achieved by choosing H so that it contains Z_1, does not contain C_2, so that $H \cap C_2$ is not contained in C_1, and so that $H \cap C_1$ has a non-cyclic free subgroup. For such an H, set $H_i = C_H(Z_i) = H \cap C_i$. Then

$$(\dagger) \qquad H_1 < H_2 < H_3$$

is a chain of non-abelian centralizers in $H = H_3$. Furthermore, by Scott's compact submanifold theorem, H is a compact 3-manifold group, and a subgroup of index at most two in H is a compact orientable 3-manifold group. Let K denote such a

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subgroup of H. Since K has index at most 2, it follows from the existence of the chain (†) that K has a proper non-abelian centralizer.

We shall reach a contradiction by using results on Seifert fibre spaces to show that every proper centralizer in K is abelian. First note that K is the fundamental group of a compact orientable 3-manifold with non-empty boundary. It is also a group of cohomological dimension 2 with infinite cyclic centre, Z, say, and it follows from corollary 12·8 of [2] and theorem 12·1 of [2] that K/Z is a free product of finitely many cyclic groups. Thus every proper centralizer in K/Z is cyclic, and every proper centralizer in K is abelian. This contradiction completes the proof.

The appeal to [2] at the end of this proof is really the heart of the matter. The point is that there are simple group theoretic criteria for Haken 3-manifolds to be Seifert fibre spaces. The way we use these criteria above is similar in spirit to Scott's proof of lemma 4·9 of [12].

PROPOSITION 2′. *Every strictly ascending chain of centralizers in a 3-manifold group has length at most* 17.

Proof. Let G be a 3-manifold group which is 18-bad. By Lemma 4, G has a finitely generated freely indecomposable 17-bad subgroup, H. However one can use Scott's compact submanifold theorem to see that H is the fundamental group of a compact irreducible 3-manifold. If H is finite, then it is 16-good by Lemma 5 and if H is infinite it is 12-good by Lemma 8, a contradiction.

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