Ergod. Th. & Dynam. Sys., (2024), **44**, 2975–3001 © The Author(s), 2024. Published by Cambridge University Press.
doi:10.1017/etds.2024.3

Dimension estimates and approximation in non-uniformly hyperbolic systems

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(*Received* 14 *November* 2023 *and accepted in revised form* 23 *December* 2023)

Abstract. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and μ a hyperbolic ergodic f-invariant probability measure. This paper obtains an upper bound for the stable (unstable) pointwise dimension of μ, which is given by the unique solution of an equation involving the sub-additive measure-theoretic pressure. If μ is a Sinai–Ruelle–Bowen (SRB) measure, then the Kaplan–Yorke conjecture is true under some additional conditions and the Lyapunov dimension of μ can be approximated gradually by the Hausdorff dimension of a sequence of hyperbolic sets $\{\Lambda_n\}_{n\geq 1}$. The limit behaviour of the Carathéodory singular dimension of Λ_n on the unstable manifold with respect to the super-additive singular valued potential is also studied.

Key words: dimension, hyperbolic measure, hyperbolic set
2020 Mathematics Subject Classification: 37C45, 37D25 (Primary); 37D20 (Secondary)

1. *Introduction*

Hyperbolic approximation plays a fundamental role in the study of smooth dynamical systems. Roughly speaking, for a hyperbolic ergodic measure μ of positive entropy, one can always find a sequence of horseshoes $\{\Lambda_n\}_{n\geq 1}$ so that the dynamical quantities on them

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are close to the corresponding ones of the measure μ. Such results can be traced back to the landmark work by Katok [18] or Katok and Hasselblatt [19]. An earlier related work was obtained by Misiurewicz and Szlenk [25] for piecewise continuous and monotone maps of interval. For more results of this type, we would like to refer the reader to [2, 8, 10, 14, 15, 27, 30, 34, 35] and the references therein.

From the point of dimension theory of dynamical systems, it is natural and non-trivial to use Hausdorff dimension to estimate how large that part of the dynamics described by these horseshoes is. If μ is an ergodic hyperbolic Sinai–Ruelle–Bowen (SRB) measure of a surface diffeomorphism, Mendoza [24] proved that the Hausdorff dimension of the horseshoes on the unstable manifolds approaches to one. For the higher dimensional case, Sánchez-Salas [31] proved that the measure μ can be approximated in the weak topology by ergodic measures supported on the horseshoes $\{\Lambda_n\}_{n\geq 1}$. Moreover, he established some interesting results concerning the Hausdorff dimension of the horseshoes. Using Cao, Pesin and Zhao's ideas [8], Wang, Qu and Cao [34] generalized Mendoza's result [24] for diffeomorphisms on a higher dimensional manifold. In fact, the authors proved that the Hausdorff dimension of the horseshoes $\{\Lambda_n\}_{n\geq 1}$ on the unstable manifold tends to the dimension of the unstable manifold. Furthermore, if the stable direction is one dimension, then the Hausdorff dimension of the measure μ can be approximated by the Hausdorff dimension of $\{\Lambda_n\}_{n\geq 1}$. The first result in this paper shows that the Lyapunov dimension of μ (see equation (1) for the definition) can be approximated gradually by the Hausdorff dimension of a sequence of hyperbolic sets $\{\Lambda_n\}_{n\geq 1}$, provided that the stable direction is one or μ satisfies the Pesin's entropy formula in the stable direction.

The main motivation of our first result is the study of the Kaplan–Yorke conjecture [13]. To be more precise, let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and let μ be a hyperbolic ergodic f-invariant probability measure. For $x \in M$, *the pointwise dimension of* μ *at* x is defined by

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r},$$

provided the limit exists, where $B(x, r)$ denotes the ball of radius r centred at x. A measure μ is called *exact dimensional* if $d_\mu(x)$ is constant almost everywhere and let $\dim_H \mu$ denote the *Hausdorff dimension* of the measure μ (see [29] for the detailed definition). Young [36] proved that almost all the known characteristics of dimension type of a measure μ coincide if μ is exact dimensional. This indicates that it is very important to show the exactness of a measure in dimension theory of dynamical systems.

Let Γ be the set of points which are regular in the sense of Oseledec multiplicative ergodic theorem [26]. For every $x \in \Gamma$, denote the Lyapunov exponents of f at x by

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_u(\mu) > 0 > \lambda_{u+1}(\mu) \geq \cdots \geq \lambda_{m_0}(\mu),$$

where u and $s := m_0 - u$ are the dimension of the unstable and stable subspaces of $T_x M$, respectively.

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The *Lyapunov dimension of* μ is defined as follows:

$$\dim_L \mu = \begin{cases} m_0 & \text{if } \ell = m_0; \\ \ell + \dfrac{\lambda_1(\mu) + \cdots + \lambda_u(\mu) + \cdots + \lambda_\ell(\mu)}{|\lambda_{\ell+1}(\mu)|} & \text{otherwise,} \end{cases} \tag{1}$$

where $\ell = \max\{i : \lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_i(\mu) \geq 0\}$. It is not difficult to show that $\dim_H \mu \leq \dim_L \mu$, e.g., see [32, Proposition 4.2] for details. It was conjectured in [13] that if μ is an SRB measure, which is absolutely continuous along the unstable leaves, then generically,

$$\dim_H \mu = \dim_L \mu. \tag{2}$$

By Young's dimension formula in [36], the conjecture is true if M is a surface. This paper proves the conjecture in the higher dimensional case under the assumption that the stable direction is one or μ satisfies the 'Pesin's entropy formula in the stable direction'. Moreover, the measure μ is exact dimensional in this case (see Theorem A).

To summarize, let $h_\mu(f)$ denote the metric entropy of f with respect to μ (see Walters' book [33] for details of metric entropy), the first result is stated as the following theorem.

THEOREM A. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M and μ a hyperbolic ergodic SRB measure on M. Assume that either one of the following properties holds:*

(i) *μ has a one-dimensional stable manifold;*

(ii) *μ satisfies $h_\mu(f) = -\lambda_{u+1}(\mu) - \lambda_{u+2}(\mu) - \cdots - \lambda_{m_0}(\mu)$,*

then $\dim_H \mu = \dim_L \mu$. *Furthermore, there exists a sequence of hyperbolic sets $\{\Lambda_n\}$ such that*

$$\dim_H \Lambda_n \to \dim_L \mu \ \ (n \to \infty).$$

Example 1.1. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M. Assume that the volume measure ϱ is f-invariant ergodic and hyperbolic. Let

$$\lambda_1(\varrho) \geq \lambda_2(\varrho) \geq \cdots \geq \lambda_u(\varrho) > 0 > \lambda_{u+1}(\varrho) \geq \cdots \geq \lambda_{m_0}(\varrho)$$

denote the Lyapunov exponents of f with respect to ϱ. By Pesin's entropy formula [28] (see also [23] for a simple proof), one has that

$$h_\varrho(f) = \lambda_1(\varrho) + \lambda_2(\varrho) + \cdots + \lambda_u(\varrho) = -\lambda_{u+1}(\varrho) - \cdots - \lambda_{m_0}(\varrho),$$

where the second equality holds since f is volume-preserving. By Theorem A, there exists a sequence of hyperbolic sets $\{\Lambda_n\}$ such that

$$\dim_H \Lambda_n \to m_0 \ \ (n \to \infty),$$

since $\dim_L \mu = m_0$ in this case.

Ledrappier [20] proved the existence of the pointwise dimension of each SRB measure. For a hyperbolic invariant measure μ of a C^2 (or $C^{1+\alpha}$) diffeomorphism f of a smooth compact Riemannian manifold M without boundary, Ledrappier and Young [22] proved the existence of dimension of μ on stable/unstable manifolds, and that the *upper pointwise*

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dimension of μ is upper bounded by the sum of the dimension of μ on stable and unstable manifolds. Later, Barreira, Pesin and Schmeling [4] proved that the *lower pointwise dimension* of μ is also lower bounded by the sum of the dimension of μ on stable and unstable manifolds. This showed that the measure μ is exact dimensional, which finally solves the Eckmann–Ruelle conjecture.

Motivated by the work in [12], where it is proved that the unique solution of the measure-theoretic pressure is exactly the dimension of an invariant measure supported on an average conformal repeller, the second result in this paper shows that the unique solution of measure-theoretic pressure gives an upper bound of the dimension of a hyperbolic ergodic measure μ on stable/unstable manifolds. To be more precise, we introduce some notation first. For each $x \in M$ and $n \geq 1$, consider the differentiable operator $D_x f^n : T_x M \to T_{f^n(x)} M$ and denote the singular values of $D_x f^n$ in the decreasing order by

$$\alpha_1(x, f^n) \geq \alpha_2(x, f^n) \geq \cdots \geq \alpha_u(x, f^n) \geq \cdots \geq \alpha_{m_0}(x, f^n).$$

Recall that u and s are the dimension of the unstable and stable subspace of $T_x M$, respectively. For every $t \in [0, u]$, define

$$\phi^t(x, f^n) := \sum_{i=1}^{[t]} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{[t]+1}(x, f^n)$$

and

$$\psi^t(x, f^n) := \sum_{i=u-[t]+1}^{u} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{u-[t]}(x, f^n).$$

For every $t \in [0, s]$, define

$$\varphi^t(x, f^n) := \sum_{i=u+1}^{u+[t]} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{u+[t]+1}(x, f^n).$$

Since f is smooth, the functions $x \mapsto \alpha_i(x, f^n)$, $x \mapsto \phi^t(x, f^n)$, $x \mapsto \psi^t(x, f^n)$ and $x \mapsto \varphi^t(x, f^n)$ are continuous. It is easy to see that the sequences of functions

$$\Phi_f(t) := \{-\phi^t(\cdot, f^n)\}_{n \geq 1} \tag{3}$$

are super-additive and

$$\Psi_f(t) := \{-\psi^t(\cdot, f^n)\}_{n \geq 1}, \quad \Xi_f(t) := \{\varphi^t(\cdot, f^n)\}_{n \geq 1} \tag{4}$$

are sub-additive. Ledrappier and Young [22] proved the existence of stable and unstable pointwise dimension $d^s_\mu(x)$, $d^u_\mu(x)$ of a hyperbolic ergodic measure μ for μ-almost every (a.e.) x. The following theorem shows that the unique solution of the sub-additive measure-theoretic pressure equation

$$P_\mu(f, \Psi_f(t)) = 0 \quad (P_\mu(f, \Xi_f(t)) = 0)$$

is an upper bound for the unstable (stable) dimension of μ, see §2 for the definitions of measure-theoretic pressure and stable and unstable dimension of an invariant measure.

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THEOREM B. *Suppose $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M and μ is a hyperbolic ergodic measure on M. Then one has*

$$d_\mu^u(x) \le t_u^* \quad \text{and} \quad d_\mu^s(x) \le t_s^* \ \ \mu\text{-a.e. } x,$$

where t_u^* and t_s^* are the unique solutions of the equations $P_\mu(f, \Psi_f(t)) = 0$ and $P_\mu(f, \Xi_f(t)) = 0$, respectively.

For each hyperbolic ergodic measure μ of positive entropy, there exists a sequence of hyperbolic sets $\{\Lambda_n\}_{n\ge1}$ such that the dynamical quantities on Λ_n gradually approach to those of the measure μ (see Theorem 2.4). Since the hyperbolic sets $\{\Lambda_n\}_{n\ge1}$ are non-conformal, it is difficult to compute their Hausdorff dimension. Following the approach described in [8], this paper introduces the concept of Carathéodory singular dimension of a hyperbolic set on unstable manifolds (see §2 for the detailed definition). The third result of this paper shows that the zero of the super-additive/sub-additive measure-theoretic pressure $P_\mu(f, \Phi_f(t))/P_\mu(f, \Psi_f(t))$ gives a lower/upper bound of the Carathéodory singular dimension of Λ_n on the unstable manifold. In addition, if μ is an SRB measure, then the Carathéodory singular dimension of Λ_n on the unstable manifold tends to the dimension of the unstable manifold, and the Lyapunov dimension of μ is exactly the sum of t_s^* and the dimension of the unstable manifold, where t_s^* is the unique root of the equation $P_\mu(f, \Xi_f(t)) = 0$.

THEOREM C. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M, and let μ be a hyperbolic ergodic measure on M. Then there exists a sequence of hyperbolic sets $\{\Lambda_\varepsilon\}_{\varepsilon\ge0}$ such that the following properties hold:*

(i) $\liminf_{\varepsilon\to0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W_{\mathrm{loc}}^u(x, f)) \ge t_{u*}$ *for every $x \in \Lambda_\varepsilon$, where t_{u*} is the unique root of the equation $P_\mu(f, \Phi_f(t)) = 0$;*

(ii) $\limsup_{\varepsilon\to0} \dim_C^{\Psi_f}(\Lambda_\varepsilon \cap W_{\mathrm{loc}}^u(x, f)) \le t_u^*$ *for every $x \in \Lambda_\varepsilon$, where t_u^* is the unique root of the equation $P_\mu(f, \Psi_f(t)) = 0$.*

Furthermore, if μ is an SRB measure, then $\dim_L \mu = u + t_s^*$ and

$$\lim_{\varepsilon\to0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W_{\mathrm{loc}}^u(x, f)) = u$$

for every $x \in \Lambda_\varepsilon$, where u is the dimension of the unstable manifold and t_s^* is the unique root of the equation $P_\mu(f, \Xi_f(t)) = 0$.

The paper is organized as follows. Section 2 gives some basic notions and properties, including Hausdorff dimension, hyperbolic set, pressure and singular dimension. All the proofs of the main results will be given in §3.

2. *Preliminaries*

In this section, we will recall some definitions and preliminary results which are used in the proofs of the main results.

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2.1. *Hyperbolic set.* Let f be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M. We say an f-invariant compact subset $\Lambda \subset M$ is a *hyperbolic set* if for any $x \in \Lambda$, the tangent space admits a decomposition $T_x M = E^s(x) \oplus E^u(x)$ such that the following properties hold:

(1) the splitting is Df-invariant, that is, for every $x \in \Lambda$, $D_x f E^\sigma(x) = E^\sigma(f(x))$ for $\sigma = s, u$;

(2) the stable subspace $E^s(x)$ is uniformly contracting and the unstable subspace $E^u(x)$ is uniformly expanding in the sense that there are constants $C \geq 1$ and $0 < \chi < 1$ such that for every $n \geq 0$ and $v^\sigma \in E^\sigma(x)$ ($\sigma = s$ or u), we have

$$\|D_x f^n v^s\| \leq C\chi^n \|v^s\| \quad \text{and} \quad \|D_x f^{-n} v^u\| \leq C\chi^n \|v^u\|.$$

Recall that a hyperbolic set Λ is *locally maximal* if there exists an open neighbourhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and a diffeomorphism f is called *topologically transitive* on Λ if for every two non-empty (relative) open subsets $U, V \subset \Lambda$, there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. Given a point $x \in \Lambda$, for each small $\beta > 0$, the *local stable and unstable manifolds* at the point x are defined as follows:

$$W^s_{\mathrm{loc}}(x, f) = \{y \in M : d(f^n(x), f^n(y)) \leq \beta \text{ for all } n \geq 0\},$$

and

$$W^u_{\mathrm{loc}}(x, f) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \beta \text{ for all } n \geq 0\}.$$

The *global stable and unstable sets* of $x \in \Lambda$ are given as follows:

$$W^s(x, f) = \bigcup_{n\geq 0} f^{-n}(W^s_{\mathrm{loc}}(f^n(x), f)), \quad W^u(x, f) = \bigcup_{n\geq 0} f^n(W^u_{\mathrm{loc}}(f^{-n}(x), f)).$$

Let d^s/d^u be the metric induced by the Riemannian structure on the stable/unstable manifold W^s/W^u.

2.2. *Dimension.* Let X be a compact Riemannian manifold with a Riemannian metric. Given a subset Z of X, for $s \geq 0$ and $\delta > 0$, define

$$\mathcal{H}^s_\delta(Z) := \inf\left\{\sum_i |U_i|^s : Z \subset \bigcup_i U_i,\ |U_i| \leq \delta \text{ for all } i\right\},$$

where $|\cdot|$ denotes the diameter of a subset. The quantity

$$\mathcal{H}^s(Z) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(Z)$$

is called the *s-dimensional Hausdorff measure* of Z. It is easy to show that there is a jump-up value

$$\dim_H Z := \inf\{s : \mathcal{H}^s(Z) = 0\} = \sup\{s : \mathcal{H}^s(Z) = \infty\},$$

which is called the *Hausdorff dimension* of Z.

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Given a Borel probability measure μ on X, *the Hausdorff dimension of the measure* μ is defined as

$$\dim_H \mu = \inf\{\dim_H Y : Y \subset X,\ \mu(Y) = 1\}.$$

The *lower and upper pointwise dimension of* μ at point $x \in X$ are defined respectively by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ denotes the ball of radius r centred at x. If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$, then we denote the common value by $d_\mu(x)$. In particular, Barreira and Wolf [5] proved that

$$\dim_H \mu = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in X\}, \tag{5}$$

where the essential supremum is taken with respect to μ. The following well-known result gives the relation between the Hausdorff dimension and the lower pointwise dimension.

PROPOSITION 2.1. *The following properties hold:*

(1) *if* $\underline{d}_\mu(x) \geq \alpha$ *for* μ*-a.e.* $x \in X$, *then* $\dim_H \mu \geq \alpha$;
(2) *if* $\underline{d}_\mu(x) \leq \alpha$ *for every* $x \in Z \subseteq X$, *then* $\dim_H Z \leq \alpha$.

Let $f : X \to X$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact Riemannian manifold X, and let μ be a hyperbolic ergodic measure on X. Let Γ be the set of points which are regular in the sense of Oseledets [26]. A measurable partition ξ^u/ξ^s of X is said to be *subordinate to the unstable/stable manifold* if for μ-almost every x, $\xi^u(x) \subset W^u(x,f)/\xi^s(x) \subset W^s(x,f)$ and contains an open neighbourhood of x in $W^u(x,f)/W^s(x,f)$. Let $\{\mu_x^u\}$ and $\{\mu_x^s\}$ be the collections of conditional measures associated with ξ^u and ξ^s, respectively. For every $x \in \Gamma$, Ledrappier and Young [22] proved the existence of the following limits:

$$d_\mu^u(x) := \lim_{r\to 0} \frac{\log \mu_x^u(B^u(x,r))}{\log r} \quad \text{and} \quad d_\mu^s(x) := \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r}, \tag{6}$$

which are called the stable and unstable dimension of the measure μ, respectively. Here $B^\sigma(x,r) := \{y \in W^\sigma(x,f) : d^\sigma(x,y) < r\}$ with $\sigma \in \{u,s\}$. Since we consider the limit $r \to 0$ in equation (6), the definition of $d_\mu^u(x)$ will remain unchanged if we consider the global metric d in the dynamical ball $B^\sigma(x,r)$ instead.

2.3. *Pressure.* Let (M,f) be a topological dynamical system (TDS for short), that is, $f : M \to M$ is a continuous map on a compact metric space M equipped with the metric d. Denote by $\mathcal{M}_{\text{inv}}(f|_M)$ and $\mathcal{M}_{\text{erg}}(f|_M)$ the set of all f-invariant and ergodic Borel probability measures on M, respectively. Given $n \in \mathbb{N}$ and $x, y \in M$, let

$$d_n(x,y) = \max\{d(f^k(x), f^k(y)) : 0 \leq k < n\}.$$

Given $\varepsilon > 0$, denote by $B_n(x,\varepsilon) = \{y : d_n(x,y) < \varepsilon\}$ the *Bowen's ball* of radius ε centred at x of length n. A subset $E \subset M$ is called (n,ε)-*separated* if $d_n(x,y) > \varepsilon$ for any two

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distinct points $x, y \in E$. A sequence of continuous functions $\Psi = \{\psi_n\}_{n\geq 1}$ on M is called *sub-additive* if

$$\psi_{m+n} \leq \psi_n + \psi_m \circ f^n \quad \text{for all } m, n \geq 1.$$

Similarly, one calls a sequence of continuous functions $\Phi = \{\phi_n\}_{n\geq 1}$ on M *super-additive* if $-\Phi = \{-\phi_n\}_{n\geq 1}$ is sub-additive.

Let $\Psi = \{\psi_n\}_{n\geq 1}$ be a sub-additive sequence of continuous potentials on M, set

$$P_n(f, \Psi, \varepsilon) = \sup\left\{\sum_{x\in E} e^{\psi_n(x)} : E \text{ is an } (n, \varepsilon)\text{-separated subset of } M\right\}.$$

The quantity

$$P(f, \Psi) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(f, \Psi, \varepsilon)$$

is called the *sub-additive topological pressure* of Ψ.

The sub-additive topological pressure satisfies the following variational principle, see [6] for more details.

THEOREM 2.1. *Let $\Psi = \{\psi_n\}_{n\geq 1}$ be a sub-additive sequence of continuous potentials on M. Then*

$$P(f, \Psi) = \sup\{h_\mu(f) + \mathcal{F}_*(\Psi, \mu) |\ \mu \in \mathcal{M}_{\text{inv}}(f|_M), \mathcal{F}_*(\Psi, \mu) \neq -\infty\},$$

where $h_\mu(f)$ is the measure theoretic entropy of f with respect to the measure μ and $\mathcal{F}_*(\Psi, \mu) = \lim_{n\to\infty}(1/n) \int \psi_n \, d\mu$.

Remark 2.1. If $\Psi = \{\psi_n\}_{n\geq 1}$ is *additive* in the sense that $\psi_n(x) = \psi(x) + \psi(fx) + \cdots + \psi(f^{n-1}x) := S_n\psi(x)$ for some continuous function $\psi : M \to \mathbb{R}$, we simply denote the topological pressure $P(f, \Psi)$ as $P(f, \psi)$.

Next we recall the super-additive topological pressure introduced in [8] by the variational relation for topological pressure, although it is unknown whether the variational principle holds for super-additive topological pressure defined via separated sets. Given a sequence of super-additive continuous potentials $\Phi = \{\phi_n\}_{n\geq 1}$ on M, the *super-additive topological pressure* of Φ is defined as

$$P(f, \Phi) := \sup\{h_\mu(f) + \mathcal{F}_*(\Phi, \mu) : \mu \in \mathcal{M}_{\text{inv}}(f|_M)\},$$

where

$$\mathcal{F}_*(\Phi, \mu) = \lim_{n\to\infty} \frac{1}{n} \int \phi_n \, d\mu = \sup_{n\in\mathbb{N}} \frac{1}{n} \int \phi_n \, d\mu.$$

The second equality is due to the standard sub-additive argument. The following result gives the relation between the sub-additive (super-additive) topological pressure and the topological pressure for additive potentials.

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PROPOSITION 2.2. *Let $\Phi = \{\phi_n\}_{n\geq 1}$ be a sequence of continuous potentials on M. Then the following properties hold:*

(1) *if Φ is sub-additive and the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then*

$$P(f, \Phi) = \lim_{n\to\infty} P(f, \phi_n/n) = \lim_{n\to\infty} (1/n)P(f^n, \phi_n);$$

(2) *if Φ is super-additive, then*

$$P(f, \Phi) = \lim_{n\to\infty} P(f, \phi_n/n) = \lim_{n\to\infty} (1/n)P(f^n, \phi_n).$$

The first statement is proved in [3], where the sub-additive topological pressure is defined via separated sets, so one requires that the entropy map be upper semi-continuous. The second statement is proved in [8], and one does not need any additional condition since the super-additive topological pressure is defined via the variational relations.

Following the approach described in [29], we recall the topological pressure on an arbitrary subset of unstable manifolds. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M and let $\Lambda \subset M$ be a hyperbolic set. Let $\Psi = \{\psi_n\}_{n\geq 1}$ be a sub-additive sequence of continuous functions on Λ. For every $x \in \Lambda$, denote $Z = \Lambda \cap W^u_{\mathrm{loc}}(x, f)$. Given $s \in \mathbb{R}$, set

$$m(Z, \Psi, s, \delta) := \lim_{N\to\infty} \inf\left\{\sum_i \exp\Big(-sn_i + \sup_{y\in B^u_{n_i}(x_i,\delta)} \psi_{n_i}(y)\Big)\right\}, \tag{7}$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda, n_i \geq N$ that cover Z, and

$$B^u_{n_i}(x_i, \delta) := \{y \in W^u(x, f) : d^u(f^j(x_i), f^j(y)) < \delta \text{ for } j = 0, 1, \ldots, n_i - 1\}.$$

It is easy to show that there is a jump-up value

$$P_Z(f, \Psi, \delta) := \inf\{s : m(Z, \Psi, s, \delta) = 0\} = \sup\{s : m(Z, \Psi, s, \delta) = +\infty\}.$$

The quantity

$$P_Z(f, \Psi) := \lim_{\delta\to 0} P_Z(f, \Psi, \delta)$$

is called *the topological pressure of Ψ on the subset Z*. It is not difficult to show that $P_\Lambda(f, \Psi) = P(f|_\Lambda, \Psi)$ (see [6, Proposition 4.4]).

Let μ be an f-invariant Borel probability measure on M. Given a sub-additive potential $\Phi = \{\phi_n\}_{n\geq 1}$ on M, for $0 < \delta < 1, n \geq 1$ and $\varepsilon > 0$, a subset $F \subset M$ is called *an (n, ε, δ)-spanning set* if the union $\bigcup_{x\in F} B_n(x, \varepsilon)$ has μ-measure more than or equal to $1 - \delta$. Put

$$P_\mu(f, \Phi, n, \varepsilon, \delta) := \inf\left\{\sum_{x\in F} \exp\Big(\sup_{y\in B_n(x,\varepsilon)} \phi_n(y)\Big) : F \text{ is an } (n, \varepsilon, \delta)\text{-spanning set}\right\}$$

and let further that

$$P_\mu(f, \Phi, \varepsilon, \delta) := \limsup_{n\to\infty} \frac{1}{n} \log P_\mu(f, \Phi, n, \varepsilon, \delta),$$

$$P_\mu(f, \Phi, \delta) := \liminf_{\varepsilon\to 0} P_\mu(f, \Phi, \varepsilon, \delta),$$

$$P_\mu(f, \Phi) := \lim_{\delta\to 0} P_\mu(f, \Phi, \delta),$$

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and we call $P_\mu(f, \Phi)$ *the sub-additive measure-theoretic pressure* of (f, Φ) with respect to μ. If one considers a super-additive potential $\Phi = \{\phi_n\}_{n\geq 1}$ on M, replacing $\sup_{y\in B_n(x,\varepsilon)} \phi_n(y)$ by $\phi_n(x)$ in $P_\mu(f, \Phi, n, \varepsilon, \delta)$, then the corresponding quantity $P_\mu(f, \Phi)$ is called *the super-additive measure theoretic pressure* of (f, Φ) with respect to μ.

Remark 2.2.

(i) It is easy to see that $P_\mu(f, \Phi, \delta)$ increases with δ decreasing to zero. So the limit in the last formula exists. Moreover, it is proved in [7] that $P_\mu(f, \Phi, \delta)$ is independent of δ. Hence, the limit of $\delta \to 0$ is redundant in the definition.

(ii) If $\Phi = \{\phi_n\}_{n\geq 1}$ is an additive potential on M, that is, $\phi_n(x) = \sum_{i=0}^{n-1} \phi_1(f^i x)$ for some continuous function ϕ_1, then we simply write $P_\mu(f, \Phi)$ as $P_\mu(f, \phi_1)$.

In the following, we recall some properties of sub-additive/super-additive measure-theoretic pressure which are proved in [7].

THEOREM 2.2. [7, Theorem A] *Let (M, f) be a TDS and $\Phi = \{\phi_n\}_{n\geq 1}$ a sub-additive potential on M. For every $\mu \in \mathcal{M}_{\mathrm{erg}}(f|_M)$ with $\mathcal{F}_*(\Phi, \mu) \neq -\infty$, we have that*

$$P_\mu(f, \Phi) = h_\mu(f) + \mathcal{F}_*(\Phi, \mu).$$

THEOREM 2.3. [7, Proposition 3.2] *Let (M, f) be a TDS and $\Phi = \{\phi_n\}_{n\geq 1}$ a super-additive potential on M. For every $\mu \in \mathcal{M}_{\mathrm{erg}}(f|_M)$, we have that*

$$P_\mu(f, \Phi) = h_\mu(f) + \mathcal{F}_*(\Phi, \mu).$$

Remark 2.3. In Theorem 2.2, to avoid the indeterminate form $\infty - \infty$, the condition $\mathcal{F}_*(\Phi, \mu) \neq -\infty$ is necessary. However, we do not need this condition in Theorem 2.3. If $\Phi = \{\phi_n\}_{n\geq 1}$ is an additive potential on M, that is, $\phi_n(x) = S_n\phi(x)$ for some continuous function ϕ, then we have

$$P_\mu(f, \phi) = h_\mu(f) + \int \phi \, d\mu \quad \text{for all } \mu \in \mathcal{M}_{\mathrm{erg}}(f|_M).$$

The above formula is also proven in [16].

2.4. *Singular dimension.* Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and $\Lambda \subset M$ a hyperbolic set. Consider the sub-additive singular valued potential $\Psi_f(t) = \{-\psi^t(\cdot, f^n)\}_{n\geq 1}$ given by equation (4). Fix $x \in \Lambda$ and let $Z = \Lambda \cap W^u_{\mathrm{loc}}(x, f)$. Following the approach described in [8], we introduce the Carathéodory singular dimension of Z. Put

$$m(Z, \Psi_f(t), \delta) := \lim_{N\to\infty} \inf \left\{ \sum_i \exp \left[\sup_{y\in B^u_{n_i}(x_i,\delta)} -\psi^t(y, f^{n_i}) \right] \right\}, \tag{8}$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda, n_i \geq N$ that cover Z. It is easy to see that there is a jump-up value

$$\begin{aligned} \dim^{\Psi_f}_{C,\delta} Z &:= \inf\{t : m(Z, \Psi_f(t), \delta) = 0\} \\ &= \sup\{t : m(Z, \Psi_f(t), \delta) = +\infty\}. \end{aligned} \tag{9}$$

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The quantity

$$\dim_C^{\Psi_f} Z := \lim_{\delta\to 0} \dim_{C,\delta}^{\Psi_f} Z \tag{10}$$

is called *the Carathéodory singular dimension of Z with respect to the sub-additive singular valued potential Ψ_f.*

Consider the super-additive singular valued potential $\Phi_f(t) = \{-\phi^t(\cdot, f^n)\}_{n\geq 1}$ given by equation (3), replacing $\sup_{y\in B_{n_i}^u(x_i,\delta)} -\psi^t(y, f^{n_i})$ by $-\phi^t(x_i, f^{n_i})$ in equation (8), one can define $m(Z, \Phi_f(t), \delta)$ and $\dim_{C,\delta}^{\Phi_f} Z$ in a similar fashion as equations (8) and (9). The corresponding quantity $\dim_C^{\Phi_f} Z$ as in equation (10) is called *the Carathéodory singular dimension of Z with respect to the super-additive singular valued potential Φ_f.*

2.5. *Approximation of hyperbolic measures by hyperbolic sets with dominated splitting.* First we recall the definition of the dominated splitting. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M. Suppose $\Lambda \subset M$ is a compact f-invariant set. We say Λ admits a *dominated splitting* if there is a continuous invariant splitting $T_\Lambda M = E \oplus F$ and constants $C > 0, \lambda \in (0, 1)$ such that for each $x \in \Lambda, n \in \mathbb{N}, 0 \neq u \in E(x)$ and $0 \neq v \in F(x)$, it holds that

$$\frac{\|D_x f^n(u)\|}{\|u\|} \leq C\lambda^n \frac{\|D_x f^n(v)\|}{\|v\|}.$$

We say F dominates E and write it as $E \preceq F$. Furthermore, given $0 < \ell \leq m_0$, we say a continuous invariant splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_\ell$ dominates if there are numbers $\chi_1 < \chi_2 < \cdots < \chi_\ell$, constants $C > 0$ and $0 < \varepsilon < \min_{1\leq i\leq \ell-1}\{(\chi_{i+1} - \chi_i)/100\}$ such that for every $x \in \Lambda, n \in \mathbb{N}$ and $1 \leq j \leq \ell$ and each unit vector $u \in E_j(x)$, it holds that

$$C^{-1} \exp[n(\chi_j - \varepsilon)] \leq \|D_x f^n(u)\| \leq C \exp[n(\chi_j + \varepsilon)].$$

In particular, it is clear that $E_1 \preceq \cdots \preceq E_\ell$. We shall use the notion $\{\chi_j\}$-dominated when we want to stress the dependence on the numbers $\{\chi_j\}$.

Refining Katok's approximation theory in non-uniformly hyperbolic dynamical systems [18], Avila, Crovisier and Wilkinson [2] obtained the following approximation result.

THEOREM 2.4. [2] *Let f be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M, and let μ be an ergodic hyperbolic measure with $h_\mu(f) > 0$. Then for every $\varepsilon > 0$ and weak* neighbourhood $\mathcal{V}$ of μ in the space of f-invariant probability measures on M, there exists an f-invariant compact subset $\Lambda_\varepsilon \subset M$ such that:*

(a) *Λ_ε is ε-close to the support set of μ in the Hausdorff distance;*
(b) *$|h_{top}(f|_{\Lambda_\varepsilon}) - h_\mu(f)| \leq \varepsilon$;*
(c) *all the invariant probability measures supported on Λ_ε lie in $\mathcal{V}$;*
(d) *there is a $\{\chi_j(\mu)\}$-dominated splitting $TM = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell$ over Λ_ε, where $\chi_1(\mu) < \cdots < \chi_\ell(\mu)$ are distinct Lyapunov exponents of f with respect to the measure μ.*

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In the second statement, the original result does not show that $h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) \le h_\mu(f) + \varepsilon$. However, only a slight modification can give the upper bound of the topological entropy of f on the horseshoe.

3. *Proofs*

This section provides the detailed proofs of the main results presented in the previous section.

3.1. *Proof of Theorem A.* (i) Since μ is a hyperbolic ergodic SRB measure for a $C^{1+\alpha}$ diffeomorphism f and has a one-dimensional stable manifold, by [34, Lemma 15 and 25], one has

$$d_\mu^u(x) := \lim_{r\to 0} \frac{\log \mu_x^u(B^u(x,r))}{\log r} = u \quad \text{and} \quad d_\mu^s(x) := \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r} = \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)}$$

for μ-a.e. x. Barreira, Pesin and Schmeling [4] proved that $d_\mu(x) = d_\mu^u(x) + d_\mu^s(x)$ for μ-a.e. x. As a consequence, one has that

$$d_\mu(x) = u + \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)}$$

for μ-a.e. x. Hence, one has

$$\dim_H \mu = u + \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)}.$$

If $\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) < 0$, then one can show that

$$\dim_L \mu = u + \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)},$$

since μ is an SRB measure and has a one-dimensional stable manifold. Therefore, we have that

$$\dim_H \mu = \dim_L \mu.$$

If $\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) \ge 0$, it follows from the definition of Lyapunov dimension that $\dim_L \mu = m_0$. Since μ is an SRB measure for f and has a one-dimensional stable manifold, one has that

$$h_\mu(f) = \lambda_1(\mu) + \cdots + \lambda_{m_0-1}(\mu) \ge -\lambda_{m_0}(\mu).$$

This together with the fact that

$$1 \ge d_\mu^s(x) = \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)} \quad \mu\text{-a.e. } x$$

implies that $(h_\mu(f)/-\lambda_{m_0}(\mu)) = 1$. This yields that $\dim_H \mu = \dim_L \mu$.

By [34, Theorem B], there exists a sequence of hyperbolic sets Λ_n such that

$$\dim_H \Lambda_n \to \dim_L \mu \ (n \to \infty).$$

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(ii) Since μ is a hyperbolic ergodic SRB measure for a $C^{1+\alpha}$ diffeomorphism f, by [34, Lemma 15], one has that

$$d_\mu^u(x) = u \quad \mu\text{-a.e. } x.$$

Considering f^{-1} instead of f, since $h_\mu(f) = -\lambda_{u+1}(\mu) - \lambda_{u+2}(\mu) - \cdots - \lambda_{m_0}(\mu)$, by [21, Theorem A], we have that the measure μ has absolutely continuous conditional measures on stable manifolds of f. Using the same arguments as the proof of [34, Lemma 15], we have that

$$d_\mu^s(x) = s \quad \mu\text{-a.e. } x.$$

Hence, $d_\mu(x) = u + s = m_0$ for μ-a.e. x, which implies that $\dim_H \mu = m_0$. Since μ is an SRB measure for f and $h_\mu(f) = -\lambda_{u+1}(\mu) - \lambda_{u+2}(\mu) - \cdots - \lambda_{m_0}(\mu)$, one can conclude that

$$\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) = 0,$$

then $\dim_L \mu = m_0$ by the definition of Lyapunov dimension. This proves that

$$\dim_L \mu = \dim_H \mu.$$

Finally, for each $\varepsilon > 0$, there exists a hyperbolic set Λ_ε satisfying properties (a)–(d) in Theorem 2.4. Fix a positive integer $n \geq 1$. Let t_n be the unique root of Bowen's equation $P(f^{2^n}|\Lambda_\varepsilon, -\psi^t(\cdot, f^{2^n})) = 0$ and let μ_n^u be the unique equilibrium state for the topological pressure $P(f^{2^n}|\Lambda_\varepsilon, -\psi^{t_n}(\cdot, f^{2^n}))$. Similarly, let t_n' be the root of Bowen's equation $P(f^{2^n}|\Lambda_\varepsilon, \phi^t(\cdot, f^{2^n})) = 0$ and let μ_n^s be the unique equilibrium state for the topological pressure $P(f^{2^n}|\Lambda_\varepsilon, \phi^{t_n}(\cdot, f^{2^n}))$. As in the proof of [34, Theorem B], the following properties hold:

(e) $\lim_{\varepsilon\to 0} \lim_{n\to\infty} t_n = u$ and $\lim_{\varepsilon\to 0} \lim_{n\to\infty} t_n' = s$;

(f) there is a Markov partition $\mathcal{P} = \{P_1, P_2, \ldots, P_\ell\}$ of Λ_ε. For every $i \in \{1, 2, \ldots, \ell\}$, there is a family of conditional measures $\{\mu_{n,x}^u\}_{x\in P_i}$ ($\{\mu_{n,x}^s\}_{x\in P_i}$) of μ_n^u (μ_n^s) on the local unstable (stable) sets $W_{P_i}^u$ ($W_{P_i}^s$) such that for every $x \in P_i$, there is small $r_0 > 0$ such that for every $r \in (0, r_0)$,

$$r^{u+\varepsilon} \leq \mu_{n,x}^u(B^u(x, r)) \leq r^{t_n-\varepsilon}$$

and

$$r^{s+\varepsilon} \leq \mu_{n,x}^s(B^s(x, r)) \leq r^{t_n'-\varepsilon},$$

where $W_{P_i}^u(x, f) := W_{loc}^u(x, f) \cap P_i$ and $W_{P_i}^s(x, f) := W_{loc}^s(x, f) \cap P_i$ for every $x \in P_i$.

Define a measure $\hat{\mu}_n$ on P_i as follows:

$$\hat{\mu}_n(B(x, r)) = \mu_{n,x}^u(B^u(x, r)) \cdot \mu_{n,x}^s(B^s(x, r))$$

for every $x \in P_i$ and each sufficiently small $r > 0$. This yields that

$$t_n + t_n' - 2\varepsilon \leq \underline{d}_{\hat{\mu}_n}(x) \leq \overline{d}_{\hat{\mu}_n}(x) \leq m_0 + 2\varepsilon$$

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for every $x \in P_i$. By Proposition 2.1 and the fact that $\Lambda_\varepsilon = \bigcup_{i=1}^{\ell} P_i$, we have that

$$\lim_{\varepsilon \to 0} \dim_H \Lambda_\varepsilon = m_0 = \dim_L \mu.$$

This completes the proof of Theorem A.

3.2. *Proof of Theorem B.* Let Γ be the set of points which are regular in the sense of Oseledets [26] with respect to the measure μ. For every $x \in \Gamma$, denote its Lyapunov exponents by

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_u(\mu) > 0 > \lambda_{u+1}(\mu) \geq \cdots \geq \lambda_{m_0}(\mu).$$

To prove Theorem B, we need a coarse upper bound for the unstable and stable pointwise dimension $d^u_\mu(x)$, $d^s_\mu(x)$ of an ergodic f-invariant hyperbolic probability measure μ for almost every x. We now provide the following useful lemma, which estimates the Hausdorff measure of the image of a small ball along unstable/stable direction under f.

LEMMA 3.1. *Fix $t \in [0, u]$, then for any $b_0 > 2\sqrt{u}$ and $C_0 > 2^t u^{t/2}$, there is $\rho_0 > 0$ such that for all $x \in \Gamma$, if $B^u(x, \rho) \subset B(x, \rho_0) \cap W^u(x, f)$ for some $0 < \rho < \rho_0$, then we have*

$$\mathcal{H}^t_{b\rho}(B^u(x, \rho)) \leq C\mathcal{H}^t_\rho(f(B^u(x, \rho))),$$

where $b = b_0 \exp\{-\log \alpha_{u-[t]}(x, f)\}$ and $C = C_0 \exp\{-\psi^t(x, f)\}$.

Proof. For simplicity, we just prove the lemma on the assumption that M is the Euclid space $\mathbb{R}^{m_0}$. For the general case, one can use local charts to prove it.

Given a small positive number ε with $e^\varepsilon/(1-\varepsilon) < 2$, since $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism on M, there exists $\rho_0 > 0$ such that for every $y, z \in B(x, \rho_0) \cap W^u(x, f)$, the following properties hold:

(a) $\|y - z - (D_y f)^{-1}(f(y) - f(z))\| \leq \varepsilon \|y - z\|$;

(b) $|\log \alpha_i(y, f) - \log \alpha_i(z, f)| \leq \varepsilon$ for $i = 1, 2, \ldots, u$.

See [17, Lemma 4] for the detailed proof of the above properties. Fix $0 < \rho < \rho_0$. Let $A := B^u(x, \rho)$ and $a = \mathcal{H}^t_\rho(f(A))$. Assume that a is finite, otherwise the conclusion is clear. For every $\eta > 0$, there are points $\{z_j\} \subset f(B(x, \rho_0) \cap W^u(x, f))$ such that

$$f(A) \subset \bigcup_j B^u(z_j, r_j)$$

with $r_j \leq \rho$ for each j and

$$\sum_j r_j^t < a + \eta.$$

Let $B'_j = \{y \in A : f(y) \in B^u(z_j, r_j)\}$, then $A \subset \bigcup_j B'_j$. By property (a), we conclude that B'_j is contained in an ellipse with principal axes

$$\frac{1}{1-\varepsilon} r_j \cdot \alpha_1(y_j, f)^{-1}, \frac{1}{1-\varepsilon} r_j \cdot \alpha_2(y_j, f)^{-1}, \ldots, \frac{1}{1-\varepsilon} r_j \cdot \alpha_u(y_j, f)^{-1},$$

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where $y_j \in B^u(x, \rho)$ and $f(y_j) = z_j$. This together with property (b) yield that B'_j is contained in an ellipse with principal axes

$$\frac{e^\varepsilon}{1-\varepsilon} r_j \cdot \alpha_1(x, f)^{-1}, \frac{e^\varepsilon}{1-\varepsilon} r_j \cdot \alpha_2(x, f)^{-1}, \ldots, \frac{e^\varepsilon}{1-\varepsilon} r_j \cdot \alpha_u(x, f)^{-1}.$$

Hence, B'_j is covered by

$$\frac{\exp\{-\sum_{j=u-[t]+1}^{u} \log \alpha_j(x, f)\}}{\exp\{-[t] \log \alpha_{u-[t]}(x, f)\}}$$

balls with radius $(e^\varepsilon/(1-\varepsilon))\sqrt{u} r_j \cdot \exp\{-\log \alpha_{u-[t]}(x, f)\}$. In fact, the radius

$$\begin{aligned} &\frac{e^\varepsilon}{1-\varepsilon} \sqrt{u} r_j \cdot \exp\{-\log \alpha_{u-[t]}(x, f)\} \\ &\quad \le 2\sqrt{u} \exp\{-\log \alpha_{u-[t]}(x, f)\} \cdot \rho \\ &\quad \le b\rho. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{H}^t_{b\rho}(B'_j) &\le \exp\left\{ -\sum_{j=u-[t]+1}^{u} \log \alpha_j(x, f) + [t] \log \alpha_{u-[t]}(x, f) \right\} \\ &\quad \cdot \left(\frac{e^\varepsilon}{1-\varepsilon} \sqrt{u} \right)^t r_j^t \cdot \exp\{-t \log \alpha_{u-[t]}(x, f)\} \\ &\le (2\sqrt{u})^t \cdot \exp\{-\psi^t(x, f)\} \cdot r_j^t. \end{aligned}$$

Summing up over all j, we have that

$$\begin{aligned} \mathcal{H}^t_{b\rho}(A) &\le \sum_j \mathcal{H}^t_{b\rho}(B'_j) \\ &\le 2^t (\sqrt{u})^t \exp\{-\psi^t(x, f)\} \cdot \sum_j r_j^t \\ &\le 2^t (\sqrt{u})^t \exp\{-\psi^t(x, f)\} \cdot (a + \eta). \end{aligned}$$

The choice of C_0 and the arbitrariness of $\eta > 0$ implies the desired result. □

The following result relates the zero of measure-theoretic pressure with the upper bound of the unstable pointwise dimension of μ.

LEMMA 3.2. *For μ-a.e. x, $d^u_\mu(x) \le t^*_{u,1}$, where $t^*_{u,1}$ is the unique solution of the equation $P_\mu(f, -\psi^t(\cdot, f)) = 0$.*

Proof. Fix a small number $\varepsilon > 0$ such that $-\lambda_u(\mu) + 2\varepsilon < 0$ and choose $t > t^*_{u,1}$ such that

$$h_\mu(f) - \int \psi^t(x, f)\, d\mu = -3\varepsilon.$$

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CLAIM. *There exists an integer N_1 (depending only on ε) such that, for μ-a.e. x and every $N \geq N_1$, the Birkhoff averages*

$$\frac{1}{kN}\sum_{j=0}^{k-1}\log \alpha_u(f^{jN}x,\, f^N)$$

converge towards a number bigger than $\lambda_u(\mu)-\varepsilon$, as k goes to $+\infty$.

Proof of the Claim. We give the proof of the Claim by modifying slightly the arguments in the proof of [1, Lemma 8.4].

Since $\lim_{n\to\infty}(1/n)\log \alpha_u(x,\, f^n) = \lim_{n\to\infty}(1/n)\int \log \alpha_u(x,\, f^n)\, d\mu = \lambda_u(\mu)$ for μ-a.e. x, there exists a positive integer L such that

$$\int \log \alpha_u(x,\, f^L)\, d\mu \geq (\lambda_u(\mu)-\varepsilon/2)L. \tag{11}$$

The measure μ may be not ergodic for f^L, one can decompose it as

$$\mu = \frac{1}{m}(\mu_1+\mu_2+\cdots+\mu_m),$$

where $m \in \mathbb{N}^+$ divides L and each μ_i is an ergodic f^L-invariant measure such that $f_*\mu_i = \mu_{i+1}$ for each $i\,(\mathrm{mod}\ m)$. Let $A_1 \cup A_2 \cup \cdots \cup A_m$ be a measurable partition of (M,μ) such that $f(A_i) = A_{i+1}$ for each $i\,(\mathrm{mod}\ m)$ and $\mu_i(A_i) = 1$. By equation (11), there exists $j_0 \in \{1,2,\ldots,m\}$ such that

$$\int \log \alpha_u(x,\, f^L)\, d\mu_{j_0} \geq (\lambda_u(\mu)-\varepsilon/2)L.$$

For every $N \geq 1$ and μ-a.e. x, one decomposes the orbit $\{f^i(x)\}_{i=0}^{N-1}$ as $(x,\ldots,f^{j-1}(x))$, $(f^j(x),\ldots,f^{j+(r-1)L-1}(x))$ and $(f^{j+(r-1)L}(x),\ldots,f^{N-1}(x))$, where $j<L$, $j+rL \geq N$ and the points $\{f^{j+sL}(x)\}_{s=0}^{r}$ belong to A_{j_0}. Using the super-additivity of $\{\log \alpha_u(x,\, f^n)\}_{n\geq 1}$, we have that

$$\begin{aligned}\log \alpha_u(x,\, f^N) \geq \log \alpha_u(x,\, f^j) &+ \sum_{s=0}^{r-2}\log \alpha_u(f^{j+sL}x,\, f^L)\\ &+ \log \alpha_u(f^{j+(r-1)L}x,\, f^{N-j-L(r-1)}).\end{aligned}$$

Hence, one has

$$\log \alpha_u(x,\, f^N) \geq 2C_f + \sum_{s=0}^{r-2}\log \alpha_u(f^{j+sL}x,\, f^L),$$

where $C_f = \max_{0\leq i<L}\max_{x\in M}|\log \alpha_u(x,\, f^i)|$ with the convention that $|\log \alpha_u(x,\, f^0)| = 0$. Since

$$\lim_{k\to+\infty}\frac{1}{kL}\sum_{\ell=0}^{k-1}\log \alpha_u(f^{j+\ell L}x,\, f^L) = \frac{1}{L}\int \log \alpha_u(x,\, f^L)\, d\mu_{j_0} \geq \lambda_u(\mu)-\varepsilon/2$$

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and

$$\lim_{k\to+\infty}\frac{1}{kN}\sum_{j=0}^{k-1}\log\alpha_u(f^{jN}x,\,f^N)\geq\frac{2C_f}{N}+\lim_{k\to+\infty}\frac{1}{kL}\sum_{\ell=0}^{k-1}\log\alpha_u(f^{j+\ell L}x,\,f^L),$$

and there exists an integer N_1 (depending on ε) so that $|2C_f/N|<\varepsilon/2$ for every $N>N_1$, for μ-a.e. x and every $N>N_1$, we have that

$$\lim_{k\to+\infty}\frac{1}{kN}\sum_{j=0}^{k-1}\log\alpha_u(f^{jN}x,\,f^N)>\lambda_u(\mu)-\varepsilon.$$ □

Take $b_0>2\sqrt{u}$ and $C_0>2^t u^{t/2}$, choose $N>N_1$ large enough such that

$$C_0e^{-N\varepsilon}<1\quad\text{and}\quad e^{[\lambda_u(\mu)-2\varepsilon]N}>b_0. \tag{12}$$

By the above Claim and Birkhoff ergodic theorem, for μ-a.e. $x\in M$, we have that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\alpha_{u-[t]}(f^{jN}x,\,f^N)&\geq\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\alpha_u(f^{jN}x,\,f^N)\\&\geq(\lambda_u(\mu)-\varepsilon)N\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{1}{nN}\sum_{j=0}^{nN-1}\psi^t(f^jx,\,f)=\int\psi^t(x,\,f)\,d\mu.$$

Let ρ_0 be as in Lemma 3.1. Fix $\delta\in(0,\,\rho_0)$. Ledrappier and Young [22] proved that

$$\begin{aligned}\limsup_{n\to\infty}\frac{-\log\mu_x^u(B^u(x,n,\delta/2))}{n}&\leq\lim_{\delta\to0}\limsup_{n\to\infty}\frac{-\log\mu_x^u(B^u(x,n,\delta/2))}{n}\\&=h_\mu(f)\ \mu\text{-a.e. }x,\end{aligned}$$

where $B^u(x,n,\delta/2):=\{y\in W^u(x,\,f):d^u(f^jx,\,f^jy)<\delta/2\text{ for }0\leq j<n\}$. Hence, one can find sets $A_n\subset\Gamma$ with $\mu(A_n)\to1$ $(n\to\infty)$, for every $x\in A_n$ where the following properties hold:

(a) $\exp[-nN(h_\mu(f)+\varepsilon)]\leq\mu_x^u(B^u(x,nN,\delta/2))$;

(b) $nN(-\int\psi^t(x,\,f)-\varepsilon)\leq-\sum_{j=0}^{nN-1}\psi^t(f^jx,\,f)\leq nN(-\int\psi^t(x,\,f)+\varepsilon)$;

(c) $\sum_{j=0}^{n-1}\log\alpha_{u-[t]}(f^{jN}x,\,f^N)\geq nN(\lambda_u(\mu)-2\varepsilon)$.

Take a point $x\in A_n$. Let E be a maximal $(nN,\,\delta)$-separated subset of $A_n\cap\xi^u(x)$, then

$$A_n\cap\xi^u(x)\subset\bigcup_{x_j\in E}B^u(x_j,nN,\delta).$$

Furthermore, by property (a), the number of balls $B^u(x_j,nN,\delta/2)$ is less than or equal to $\exp\{nN[h_\mu(f)+\varepsilon]\}$. Let

$$b_k(x)=(b_0)^k\exp\left[-\sum_{j=n-k}^{n-1}\log\alpha_{u-[t]}(f^{jN}x,\,f^N)\right]$$

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for $k = 1, 2, \ldots, n$ and $\beta_n = \{b_0 \exp[(-\lambda_u(\mu) + 2\varepsilon)N]\}^n \cdot \rho$, where $0 < \rho < \rho_0$. By property (c), we have

$$\begin{aligned} b_n(x)\rho &= (b_0)^n \exp\left[-\sum_{j=0}^{n-1} \log \alpha_{u-[t]}(f^{jN}x, f^N) \right] \cdot \rho \\ &\le (b_0)^n \exp[nN(-\lambda_u(\mu) + 2\varepsilon)] \cdot \rho \\ &= [b_0 e^{(-\lambda_u(\mu)+2\varepsilon)N}]^n \cdot \rho \\ &= \beta_n. \end{aligned}$$

For each $x_j \in E$, using Lemma 3.1 n times, we conclude that

$$\begin{aligned} \mathcal{H}^t_{\beta_n}(B^u(x_j, nN, \delta)) &\le \mathcal{H}^t_{b_n(x_j)\rho}(B^u(x_j, nN, \delta)) \\ &\le C_0 \exp\{-\psi^t(x_j, f^N)\} \cdot \mathcal{H}^t_{b_{n-1}(x_j)\rho}(f^N(B^u(x_j, nN, \delta))) \\ &\le C_0 \exp\{-\psi^t(x_j, f^N)\} \cdot \mathcal{H}^t_{b_{n-1}(x_j)\rho}(B^u(f^N(x_j), (n-1)N, \delta)) \\ &\le (C_0)^2 \exp\{-\psi^t(x_j, f^N)\} \cdot \exp\{-\psi^t(f^N(x_j), f^N)\} \\ &\quad \cdot \mathcal{H}^t_{b_{n-2}(x_j)\rho}(B^u(f^{2N}(x_j), (n-2)N, \delta)) \\ &\le \cdots \\ &\le (C_0)^n \exp\left\{ -\sum_{j=0}^{n-1} \psi^t(f^{jN}x_j, f^N) \right\} \cdot \mathcal{H}^t_{\rho}(B^u(f^{nN}x_j, \delta)) \\ &\le (C_0)^n C_1 \cdot \exp\left\{ -\sum_{j=0}^{n-1} \psi^t(f^{jN}x_j, f^N) \right\}, \end{aligned}$$

where $C_1 = \sup_{y \in M} \mathcal{H}^t_\rho(B(y, \delta))$. By property (b) and the sub-additivity of $\{-\psi^t(\cdot, f^n)\}_{n \ge 1}$, we have that

$$\begin{aligned} \mathcal{H}^t_{\beta_n}(A_n \cap \xi^u(x)) &\le \sum_{x_j \in E} \mathcal{H}^t_{\beta_n}(B^u(x_j, nN, \delta)) \\ &\le \sum_{x_j \in E} (C_0)^n C_1 \cdot \exp\left\{ -\sum_{i=0}^{n-1} \psi^t(f^{jN}x_j, f^N) \right\} \\ &\le \sum_{x_j \in E} (C_0)^n C_1 \cdot \exp\left\{ -\sum_{i=0}^{nN-1} \psi^t(f^i x_j, f) \right\} \\ &\le (C_0)^n C_1 \cdot \exp[nN(h_\mu(f) + \varepsilon)] \cdot \exp\left[nN\left(-\int \psi^t(x, f)\, d\mu + \varepsilon \right) \right] \\ &= (C_0)^n C_1 \cdot \exp\left[nN\left(h_\mu(f) - \int \psi^t(x, f)\, d\mu + 2\varepsilon \right) \right] \\ &= (C_0)^n C_1 \cdot e^{-nN\varepsilon} \\ &= (C_0 e^{-N\varepsilon})^n C_1. \end{aligned}$$

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Since N satisfies $C_0 e^{-N\varepsilon} < 1$, we have that

$$\lim_{n\to\infty} \mathcal{H}^t_{\beta_n}(A_n \cap \xi^u(x)) = 0.$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \mu^u_x(A_n \cap \xi^u(x)) = 1$ for μ-a.e. x, by [17, Lemma 6], we obtain that

$$\dim_H \mu^u_x \le t$$

for μ-a.e. x. Combining with equation (5) and the choice of t yield that $d^u_\mu(x) \le t^*_{u,1}$ for μ-a.e. x. □

Now we are ready to present the proof of Theorem B.

Proof of Theorem B. For each $n > 1$, the measure μ is f-invariant ergodic, but it may be not ergodic for f^n although μ is still f^n-invariant. In either case, one can find an f^n-invariant ergodic probability measure ν such that

$$\mu = \frac{1}{m}[\nu + f_*\nu + \cdots + f_*^{m-1}\nu],$$

where $m \in \mathbb{N} \setminus \{0\}$ divides n. Let

$$\widetilde{P}_\mu(f^n, -\psi^n(\cdot, f^n)) := h_\mu(f^n) - \int \psi^n(x, f^n)\, d\mu,$$

then one can show that

$$\begin{aligned}\widetilde{P}_\mu(f^n, -\psi^n(\cdot, f^n)) &= \frac{1}{m}\sum_{i=0}^{m-1}\left(h_{f_*^i\nu}(f^n) - \int \psi^n(x, f^n)\, df_*^i\nu\right)\\ &= \frac{1}{m}\sum_{i=0}^{m-1} P_{f_*^i\nu}(f^n, -\psi^n(\cdot, f^n)).\end{aligned}$$

Hence, there exists $j_0 \in \{0, 1, \ldots, m-1\}$ such that

$$\widetilde{P}_\mu(f^n, -\psi^t(\cdot, f^n)) \ge P_{f_*^{j_0}\nu}(f^n, -\psi^t(\cdot, f^n)).$$

Since $f_*^{j_0}\nu$ is hyperbolic and f^n-invariant ergodic, by Lemma 3.2, there is a set $\tilde{A}$ with $\nu \circ f^{-j_0}(\tilde{A}) = 1$ such that for each $x \in \tilde{A}$,

$$d^u_{f_*^{j_0}\nu}(x) \le t^*_{u,n},$$

where $t^*_{u,n}$ is the unique root of the equation $P_\mu(f^n, -\psi^t(\cdot, f^n)) = 0$. Note that $d^u_\mu(x)$, $d^u_{f_*^{j_0}\nu}(x)$ are constants almost everywhere (see [22]) and $d^u_\mu(x) \le d^u_{f_*^{j_0}\nu}(x) \le t^*_{u,n}$ for each $x \in \tilde{A}$ with $\mu(\tilde{A}) \ge 1/m$. Consequently, we have that

$$d^u_\mu(x) \le t^*_{u,n}$$

for μ-a.e. x.

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By the sub-additive of $\{-\psi^t(\cdot, f^n)\}_{n\geq 1}$, we obtain

$$\frac{1}{2^{k+1}}\left[h_\mu(f^{2^{k+1}}) - \int \psi^t(x, f^{2^{k+1}})\, d\mu\right] \leq \frac{1}{2^k}\left[h_\mu(f^{2^k}) - \int \psi^t(x, f^{2^k})\, d\mu\right].$$

Hence,

$$\frac{1}{2^{k+1}}\widetilde{P}_\mu(f^{2^{k+1}}, -\psi^t(\cdot, f^{2^{k+1}})) \leq \frac{1}{2^k}\widetilde{P}_\mu(f^{2^k}, -\psi^t(\cdot, f^{2^k})).$$

This yields that $t^*_{u,2^{k+1}} \leq t^*_{u,2^k}$ for every $k \geq 1$. Let $t^*_u := \lim_{k\to\infty} t^*_{u,2^k}$, then one has that

$$d^u_\mu(x) \leq t^*_u \quad \mu\text{-a.e. } x.$$

Since $P_\mu(f, \{-\psi^t(\cdot, f^n)\})$ is continuous and strictly decreasing with respect to t, there exists at most one solution of the equation. To complete the proof of Theorem B, it suffices to show that $P_\mu(f, \{-\psi^{t^*}(\cdot, f^n)\}) = 0$.

Since $t^*_{u,2^k} \geq t^*_u$ for every $k \geq 1$, by Theorem 2.2, one has that

$$0 \leq \lim_{k\to\infty} \frac{1}{2^k}\widetilde{P}_\mu(f^{2^k}, -\psi^{t^*_u}(\cdot, f^{2^k})) = P_\mu(f, \{-\psi^{t^*_u}(\cdot, f^n)\}).$$

However, for each small number $\varepsilon > 0$, there exists K so that $t^*_{u,2^k} \leq t^*_u + \varepsilon$ for every $k \geq K$. Hence, we have that

$$\begin{aligned} P_\mu(f, \{-\psi^{t^*_u+\varepsilon}(\cdot, f^n)\}) &= h_\mu(f) - \lim_{n\to\infty} \frac{1}{n}\int \psi^{t^*_u+\varepsilon}(x, f^n)\, d\mu \\ &= \lim_{k\to\infty} \frac{1}{2^k}\widetilde{P}_\mu(f^{2^k}, -\psi^{t^*_u+\varepsilon}(\cdot, f^{2^k})) \leq 0. \end{aligned}$$

The previous arguments imply that $P_\mu(f, \{-\psi^{t^*}(\cdot, f^n)\}) = 0$. One can prove in a similar fashion that $d^s_\mu(x) \leq t^*_s$ for μ-a.e. x. This completes the proof of Theorem B. □

3.3. *Proof of Theorem C.* For each $\varepsilon > 0$, there exists a hyperbolic set Λ_ε satisfying properties (a)–(d) in Theorem 2.4. The following lemma shows that the zero of the super-additive topological pressure of $\Phi_f(t)$ provides a lower bound of the Carathéodory singular dimension of the hyperbolic set on the local unstable leaf with respect to the super-additive singular valued potential $\Phi_f(t)$.

LEMMA 3.3. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and let $\Lambda \subset M$ be a hyperbolic set. Assume that $f|_\Lambda$ is topologically transitive, then for every $x \in \Lambda$,*

$$\dim_C^{\Phi_f}(\Lambda \cap W^u_{\mathrm{loc}}(x, f)) \geq t_*,$$

where t_* is the unique root of the equation $P(f|_\Lambda, \Phi_f(t)) = 0$.

Proof. For every $x \in \Lambda$, we denote $Z = \Lambda \cap W^u_{\mathrm{loc}}(x, f)$ and $P(t) = P(f|_\Lambda, \Phi_f(t))$. Since the function $P(t)$ is strictly decreasing in t, then for each $t < t_*$, we have that

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$P(t) > 0$. Fix such a number t and take $\varepsilon > 0$ with $P(t) - \varepsilon > 0$. By Proposition 2.2, one has that

$$P(t) = \lim_{n\to\infty} \frac{1}{n} P(f^n|_\Lambda, -\phi^t(\cdot, f^n)),$$

then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, we obtain

$$P(f^n|_\Lambda, -\phi^t(\cdot, f^n)) > n(P(t) - \varepsilon) > 0.$$

Fix an integer $L \geq N_1$, by [11, Proposition 5.4], one has that

$$P(f^L|_\Lambda, -\phi^t(\cdot, f^L)) = P_Z(f^L|_\Lambda, -\phi^t(\cdot, f^L)).$$

Hence, there is $\delta_1 > 0$ such that

$$P_Z(f^L|_\Lambda, -\phi^t(\cdot, f^L), \delta) > (P(t) - \varepsilon)L$$

for every $0 < \delta < \delta_1$. Consequently, fixing such a $\delta > 0$, one has that

$$m(Z, -\phi^t(\cdot, f^L), (P(t) - \varepsilon)L, \delta) = +\infty.$$

Hence, for each $K > 0$, there exists $S \in \mathbb{N}$ such that for each $N \geq S$, we have that

$$\begin{aligned} K &\leq \inf \sum_i \exp[-(P(t) - \varepsilon)Lm_i - S_{m_i}\phi^t(x_i, f^L)] \\ &\leq e^{-NL(P(t)-\varepsilon)} \inf \sum_i \exp[-S_{m_i}\phi^t(x_i, f^L)], \end{aligned} \tag{13}$$

where the infimum is taken over all collections $\{B^u_{m_i}(x_i, \delta, f^L)\}$ with $x_i \in \Lambda$, $m_i \geq N$ which cover Z, $-S_{m_i}\phi^t(x_i, f^L) = -\phi^t(x_i, f^L) - \phi^t(f^L x_i, f^L) - \cdots - \phi^t(f^{(m_i-1)L}x_i, f^L)$ and

$$B^u_{m_i}(x_i, \delta, f^L) := \left\{ y \in W^u(x_i, f) : \max_{0\leq j<m_i} d^u(f^{jL}(y), f^{jL}(x_i)) < \delta \right\}.$$

Fixing such an N and taking an integer $R \geq NL$, let the collection of balls $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda$, $n_i \geq R$ be a cover of Z. One can write $n_i = m_i L + s_i$ with $0 \leq s_i < L$ and $m_i \geq N$ for each i. Since $B^u_{n_i}(x_i, \delta) \subset B^u_{m_i}(x_i, \delta, f^L)$ for each i, the collection of balls $B^u_{m_i}(x_i, \delta, f^L)$ is also a cover of Z with $x_i \in \Lambda$, $m_i \geq N$. By the super-additivity of $\{-\phi^t(\cdot, f^n)\}_{n\geq 1}$, one has

$$\begin{aligned} \sum_i \exp[-\phi^t(x_i, f^{n_i})] &\geq \sum_i \exp[-S_{m_i}\phi^t(x_i, f^L) - \phi^t(f^{m_i L}y, f^{s_i})] \\ &\geq C \sum_i \exp[-S_{m_i}\phi^t(x_i, f^L)], \end{aligned}$$

where $C = \min_{0\leq s<L} \min_{x\in M} \exp[-\phi^t(x, f^s)]$. This together with equation (13) yield that

$$\sum_i \exp[-\phi^t(x_i, f^{n_i})] \geq CKe^{NL(P(t)-\varepsilon)}.$$

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Since the cover of Z is taken arbitrarily, one can conclude that

$$\inf \sum_i \exp[-\phi^t(x_i, f^{n_i})] \geq CKe^{NL(P(t)-\varepsilon)},$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda$, $n_i \geq NL$ which cover Z. Letting $N \to \infty$, we obtain

$$m(Z, \Phi_f(t), \delta) = +\infty$$

for every $t < t_*$. This implies that

$$\dim_C^{\Phi_f} Z \geq t_*. \qquad \square$$

Proof of Theorem C(i). By Lemma 3.3, for every $x \in \Lambda_\varepsilon$, we obtain

$$\dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W^u_{\mathrm{loc}}(x, f)) \geq t_{\varepsilon *},$$

where $t_{\varepsilon *}$ is the unique root of the equation $P(f|_{\Lambda_\varepsilon}, \Phi_f(t)) = 0$. By the variational principle of topological entropy, take $\nu \in \mathcal{M}_{\mathrm{inv}}(f|_{\Lambda_\varepsilon})$ such that $h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) = h_\nu(f|_{\Lambda_\varepsilon})$. By properties (b) and (d) in Theorem 2.4, it holds that

$$\begin{aligned}
0 &= P(f|_{\Lambda_\varepsilon}, \Phi_f(t_{\varepsilon *})) \\
&= \sup\left\{h_\nu(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]} \lambda_i(\nu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\nu) : \ \nu \in \mathcal{M}_{\mathrm{inv}}(f|_{\Lambda_\varepsilon})\right\} \\
&\geq h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]} \lambda_i(\nu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\nu) \\
&\geq h_\mu(f) - \sum_{i=1}^{[t_{\varepsilon *}]} \lambda_i(\mu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\mu) - (u+1)\varepsilon, \qquad (14)
\end{aligned}$$

where $\lambda_1(\nu) \geq \lambda_2(\nu) \geq \cdots \geq \lambda_{m_0}(\nu)$ are the Lyapunov exponents of ν. However, let $\tau \in \mathcal{M}_{\mathrm{inv}}(f|_{\Lambda_\varepsilon})$ be an equilibrium state of $P(f|_{\Lambda_\varepsilon}, \Phi_f(t_{\varepsilon *}))$, then one has that

$$\begin{aligned}
0 &= P(f|_{\Lambda_\varepsilon}, \Phi_f(t_{\varepsilon *})) \\
&= h_\tau(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]} \lambda_i(\tau) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\tau) \\
&\leq h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]} \lambda_i(\tau) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\tau) \\
&\leq h_\mu(f) - \sum_{i=1}^{[t_{\varepsilon *}]} \lambda_i(\mu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\mu) + (u+1)\varepsilon.
\end{aligned}$$

This together with equation (14) yield that

$$-(u+1)\varepsilon \leq P_\mu(f, \Phi_f(t_{\varepsilon *})) \leq (u+1)\varepsilon.$$

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Hence, we have that

$$\lim_{\varepsilon\to 0} P_\mu(f, \Phi_f(t_{\varepsilon *})) = 0.$$

This implies that $\lim_{\varepsilon\to 0} t_{\varepsilon *} = t_{u*}$, where t_{u*} is the unique root of $P_\mu(f, \Phi_f(t)) = 0$. Consequently, we have that

$$\liminf_{\varepsilon\to 0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W^u_{\mathrm{loc}}(x, f)) \geq t_{u*}. \tag{15}$$

□

As a counterpart of Lemma 3.3, we have the following result.

LEMMA 3.4. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and let $\Lambda \subset M$ be a hyperbolic set. Assume that $f|_\Lambda$ is topologically transitive. Then for every $x \in \Lambda$,*

$$\dim_C^{\Psi_f}(\Lambda \cap W^u_{\mathrm{loc}}(x, f)) \leq t^*,$$

where t^* is the unique root of the equation $P(f|_\Lambda, \Psi_f(t)) = 0$.

Proof. Denote $P(t) = P(f|_\Lambda, \Psi_f(t))$. For each $t > t_*$,

$$0 > P(t) = \lim_{n\to\infty} \frac{1}{n} P(f^n, -\psi^t(\cdot, f^n)).$$

Fix such a number t and take $\varepsilon > 0$ with $P(t) + \varepsilon < 0$. Then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, we obtain

$$P(f^n, -\psi^t(\cdot, f^n)) < n(P(t) + \varepsilon) < 0.$$

Fix an integer $L \geq N_1$ such that

$$P(f^L, -\psi^t(\cdot, f^L)) < L(P(t) + \varepsilon) < 0.$$

For each $x \in \Lambda$, set $Z = \Lambda \cap W^u_{\mathrm{loc}}(x, f)$. By [11, Proposition 5.4], one has that

$$P(f^L, -\psi^t(\cdot, f^L)) = P_Z(f^L, -\psi^t(\cdot, f^L)).$$

Thus, there is $\delta_1 > 0$ such that for every $0 < \delta < \delta_1$, one has

$$P_Z(f^L, -\psi^t(\cdot, f^L), \delta) < (P(t) + \varepsilon)L.$$

Hence, one has that

$$m(Z, -\psi^t(\cdot, f^L), (P(t) + \varepsilon)L, \delta) = 0.$$

For each $\xi > 0$, there exists $N \in \mathbb{N}$ and a cover $\{B^u_{n_i}(x_i, \delta, f^L)\}$ of Z with $x_i \in \Lambda$, $n_i \geq N$ such that

$$\begin{aligned}\xi &\geq \sum_i \exp\Big[-(P(t) + \varepsilon)Ln_i + \sup_{y\in B^u_{n_i}(x_i,\delta,f^L)} -S_{n_i}\psi^t(y, f^L)\Big].\\ &\geq e^{-NL(P(t)+\varepsilon)} \sum_i \exp\Big[\sup_{y\in B^u_{n_i}(x_i,\delta,f^L)} -S_{n_i}\psi^t(y, f^L)\Big].\end{aligned}$$

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Note that $d^u(f^L x, f^L y) < \delta$ implies $d^u(f^i x, f^i y) < \delta$ for $i = 0, 1, \ldots, L-1$, since f is expanding along the unstable manifold. This implies that $B^u_{(n_i-1)L+1}(x_i, \delta) = B^u_{n_i}(x_i, \delta, f^L)$ for every i. Since

$$\begin{aligned} -S_{n_i}\psi^t(y, f^L) &= -\psi^t(y, f^L) - \psi^t(f^L y, f^L) - \cdots - \psi^t(f^{(n_i-1)L} y, f^L) \\ &\geq -\psi^t(y, f^{(n_i-1)L}) + C_1 \\ &= -\psi^t(y, f^{(n_i-1)L}) - \psi^t(f^{(n_i-1)L} y, f) + \psi^t(f^{(n_i-1)L} y, f) + C_1 \\ &\geq -\psi^t(y, f^{(n_i-1)L+1}) + C_1 + C_2, \end{aligned}$$

where $C_1 = \min_{x\in M}\{-\psi^t(x, f^L)\}$ and $C_2 = \min_{x\in M} \psi^t(x, f)$, we have that

$$\begin{aligned} \xi &\geq e^{-NL(P(t)+\varepsilon)} e^{C_1+C_2} \sum_i \exp\Big[\sup_{y\in B^u_{(n_i-1)L+1}(x_i,\delta)} -\psi^t(y, f^{(n_i-1)L+1})\Big] \\ &\geq e^{-NL(P(t)+\varepsilon)} e^{C_1+C_2} \inf \sum_i \exp\Big[\sup_{y\in B^u_{m_i}(x_i,\delta)} -\psi^t(y, f^{m_i})\Big] \end{aligned}$$

and

$$\inf \sum_i \exp\Big[\sup_{y\in B^u_{m_i}(x_i,\delta)} -\psi^t(y, f^{m_i})\Big] \leq \xi e^{NL(P(t)+\varepsilon)} e^{-C_1-C_2},$$

where the infimum is taken over all collections $\{B^u_{m_i}(x_i, \delta)\}$ with $x_i \in \Lambda$, $m_i \geq (N-1)L$ which cover Z. Letting $N \to \infty$, we obtain

$$m(Z, \Psi_f(t), \delta) = 0$$

for every $t > t_*$. This yields that

$$\dim_C^{\Psi_f} Z \leq t_*. \qquad \square$$

Remark 3.1. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact Riemannian manifold M and $\Lambda \subset M$ be a hyperbolic set. Assume that $f|_\Lambda$ is topologically transitive. Then for every $x \in \Lambda$,

$$\dim_C^{\Phi_f}(\Lambda \cap W^u_{\mathrm{loc}}(x, f)) = t_u^{\Phi_f}, \quad \dim_C^{\Psi_f}(\Lambda \cap W^u_{\mathrm{loc}}(x, f)) = t_u^{\Psi_f},$$

where $t_u^{\Phi_f}, t_u^{\Psi_f}$ are the unique roots of the equations

$$P_{\Lambda\cap W^u(x,f)}(f, \Phi_f(t)) = 0, \quad P_{\Lambda\cap W^u(x,f)}(f, \Psi_f(t)) = 0,$$

respectively. The proof is a slight modification of Lemmas 3.3 and 3.4. See [9] for more details about the Carathéodory singular dimension of each subset of a repeller. However, we do not know whether $P_{\Lambda\cap W^u(x,f)}(f, \Phi_f(t)) = P_\Lambda(f, \Phi_f(t))$ and $P_{\Lambda\cap W^u(x,f)}(f, \Psi_f(t)) = P_\Lambda(f, \Psi_f(t))$ hold.

Proof of Theorem C(ii). By Lemma 3.4, we have that

$$\dim_C^{\Psi_f}(\Lambda_\varepsilon \cap W^u_{\mathrm{loc}}(x, f)) \leq t^*_\varepsilon,$$

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where t_ε^* is the unique root of the equation $P(f|_{\Lambda_\varepsilon}, \Psi_f(t)) = 0$. Take $\nu \in \mathcal{M}_{\text{inv}}(f|_{\Lambda_\varepsilon})$ such that $h_\nu(f|_{\Lambda_\varepsilon}) = h_{\text{top}}(f|_{\Lambda_\varepsilon})$, by properties (b) and (d) in Theorem 2.4, it holds that

$$\begin{aligned}
0 &= P(f|_{\Lambda_\varepsilon}, \Psi_f(t_\varepsilon^*)) \\
&= \sup\left\{h_\nu(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\nu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{u-[t_\varepsilon^*]}(\nu) : \quad \nu \in \mathcal{M}_{\text{inv}}(f|_{\Lambda_\varepsilon})\right\} \\
&\geq h_{\text{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\nu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{u-[t_\varepsilon^*]}(\nu) \\
&\geq h_\mu(f) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\mu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{u-[t_\varepsilon^*]}(\mu) - (u+1)\varepsilon, \qquad (16)
\end{aligned}$$

where $\lambda_1(\nu) \geq \lambda_2(\nu) \geq \cdots \geq \lambda_{m_0}(\nu)$ are the Lyapunov exponents of ν. Similarly, let $\tau \in \mathcal{M}_{\text{inv}}(f|_{\Lambda_\varepsilon})$ be an equilibrium state of $P(f|_{\Lambda_\varepsilon}, \Psi_f(t_\varepsilon^*))$, then one has that

$$\begin{aligned}
0 &= P(f|_{\Lambda_\varepsilon}, \Psi_f(t_\varepsilon^*)) \\
&= h_\tau(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\tau) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{[t_\varepsilon^*]+1}(\tau) \\
&\leq h_{\text{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\tau) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{[t_\varepsilon^*]+1}(\tau) \\
&\leq h_\mu(f) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\mu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{[t_\varepsilon^*]+1}(\mu) + (u+1)\varepsilon.
\end{aligned}$$

This together with equation (16) yield that

$$-(u+1)\varepsilon \leq P_\mu(f, \Psi_f(t_\varepsilon^*)) \leq (u+1)\varepsilon.$$

Hence, we have that

$$\lim_{\varepsilon\to 0} P_\mu(f, \Psi_f(t_\varepsilon^*)) = 0.$$

This implies that $\lim_{\varepsilon\to 0} t_\varepsilon^* = t_u^*$, where t_u^* is the unique root of $P_\mu(f, \Psi_f(t)) = 0$. Consequently, we have that

$$\limsup_{\varepsilon\to 0} \dim_C^{\Psi_f}(\Lambda_\varepsilon \cap W_{\text{loc}}^u(x, f)) \leq t_u^*. \qquad \square$$

Finally, to complete the proof of Theorem C, assume that μ is an SRB measure from now on, then $h_\mu(f) = \lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_u(\mu)$. Thus, $P_\mu(f, \Phi_f(u)) = 0$. Since the Carathéodory singular dimension with respect to Ψ_f is always less than u, by property (i) of Theorem C, we have that

$$\lim_{\varepsilon\to 0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W_{\text{loc}}^u(x, f)) = u.$$

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If $\lambda_1(\mu)+\lambda_2(\mu)+\cdots+\lambda_{m_0}(\mu)\geq 0$, then $P_\mu(f, \Xi_f(m_0-u))\geq 0$ since μ is an SRB measure for f. Consider f^{-1}, by Margulis–Ruelle inequality, we have that

$$h_\mu(f)=h_\mu(f^{-1})\leq -\lambda_{u+1}(\mu)-\cdots-\lambda_{m_0}(\mu),$$

which implies that $P_\mu(f, \Xi_f(m_0-u))\leq 0$. Hence, we have that

$$P_\mu(f, \Xi_f(m_0-u))=0.$$

Thus, we have that $t_s^*=m_0-u$. By the definition of Lyapunov dimension, we have that $\dim_L\mu=m_0=u+t_s^*$.

Now, we assume that $\lambda_1(\mu)+\lambda_2(\mu)+\cdots+\lambda_{m_0}(\mu)<0$ and let ℓ be the largest integer such that $\lambda_1(\mu)+\lambda_2(\mu)+\cdots+\lambda_\ell(\mu)\geq 0$. By a standard computation, one can show that

$$t_s^*=\ell-u-\frac{h_\mu(f)+\lambda_{u+1}(\mu)+\cdots+\lambda_\ell(\mu)}{\lambda_{\ell+1}(\mu)}.$$

Combining with

$$\dim_L\mu=\ell+\frac{h_\mu(f)+\lambda_{u+1}(\mu)+\cdots+\lambda_\ell(\mu)}{|\lambda_{\ell+1}(\mu)|},$$

one has

$$\dim_L\mu=u+t_s^*.$$

This completes the proof of Theorem C.

Acknowledgements. This work is partially supported by The National Key Research and Development Program of China (2022YFA1005802). Y.C. is partially supported by NSFC (12371194), Y.Z. is partially supported by NSFC (12271386) and Qinglan project of Jiangsu Province.

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