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# *Technical Note*
# *Decidability of the Clark's completion semantics for monadic programs and queries*

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**Abstract**

There are many different semantics for general logic programs (i.e. programs that use negation in the bodies of clauses). Most of these semantics are Turing complete (in a sense that can be made precise), implying that they are undecidable. To obtain decidability one needs to put additional restrictions on programs and queries. In logic programming it is natural to put restrictions on the underlying first-order language. In this note, we show the decidability of the Clark's completion semantics for monadic general programs and queries.

*KEYWORDS*: monadic programs, general programs, Clark's completion, decidability

## 1 Introduction

*Definite* monadic programs have been studied by Matos (1997) and Matsushita and Runciman (2001). Both of these studies independently conclude that the least Herbrand model of a monadic program is a regular set. Matos (1997) further notes that as a consequence it is decidable whether a query follows from a monadic program. However, if one is only interested in the decidability, then it simply follows from the fact that monadic first-order logic without equality is decidable, see Gurevich (1966).

If we move from definite programs to general programs with the Clark's completion semantics, the decidability in monadic languages does not come that cheap – monadic logic with equality is undecidable. More precisely the satisfiability of formulas using equality and a single monadic functional symbol is decidable, but it becomes undecidable if formulas are allowed to use two monadic functional symbols, see Gurevich (1976). The central result of this note is that the satisfiability in monadic languages becomes decidable if we consider only models that satisfy the Clark's equational theory. As a consequence, we obtain a decidable interpreter for monadic general programs and queries. Our proof is based on the decidability of Rabin's monadic second-order logic of successor functions (Rabin 1969).

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The rest of the paper is organised as follows. In the next section, we briefly go over the preliminaries. Section 3 is devoted to detailed analysis of models of monadic Clark's equational theory. Next, we state and prove the main result. The final section contains some concluding remarks.

The results were obtained when the author was at Yerevan State University.

## 2 Preliminaries

In this section, we recall basic definitions and results concerning logic programs, the Clark's completion and Rabin's monadic second-order logic of successor functions.

Consider a first-order language $\mathscr{L}$. Variables are usually denoted by $x, y, z$, constant symbols by $a, b, c$, functional symbols by $f, g, h$, predicate symbols by $p, q, r$, terms by $s, t$, atomic formulas (or atoms) by $A, B, C$ and formulas by $F, G$ (all possibly subscripted or superscripted). A *literal* is an atom (positive literal) or a negation of an atom (negative literal). A *program clause* is a formula of the form

$$\forall(L_1 \wedge \cdots \wedge L_m \rightarrow A),$$

where $m \geqslant 0$, $L_1, \ldots, L_m$ are literals and $A$ is an atom. We abbreviate the above clause to

$$A \leftarrow L_1, \ldots, L_m.$$

The atom $A$ is called the *head* and $L_1, \ldots, L_m$ the *body* of the clause. A *query* is a formula of the form

$$\exists(S_1 \wedge \cdots \wedge S_k),$$

where $k > 0$ and $S_1, \ldots, S_k$ are literals. A *(general) program* is a finite set of program clauses. A logic program or a query is called *definite* if all its literals are positive. A substitution $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$ is a finite set of pairs, where $x_i$ is a variable and $t_i$ is a term. If $F$ is a formula, then $F\theta$ denotes the formula obtained from $F$ by substituting all free occurrences of $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$ respectively.

*Structures* (or *interpretations*) are usually denoted by $\mathscr{A}, \mathscr{B}, \mathscr{M}, \mathscr{N}$. A structure $\mathscr{M}$ consists of a nonempty set $M$ and interpretations of symbols in $\mathscr{L}$. That is an element $c^{\mathscr{M}} \in M$ for each constant symbol $c$ of $\mathscr{L}$, an $n$-ary function $f^{\mathscr{M}} : M^n \rightarrow M$ for each $n$-ary functional symbol $f$ of $\mathscr{L}$ and an $n$-ary relation $p^{\mathscr{M}} \subseteq M^n$ for each $n$-ary predicate symbol of $\mathscr{L}$. We extend this notation to terms of $\mathscr{L}$. That is if $t(x_1, \ldots, x_n)$ is a term, then $t^{\mathscr{M}}$ denotes the $n$-ary function that is the interpretation of $t$ in $\mathscr{M}$.

In this paper, we study the most widely accepted semantics of general programs – the Clark's completion semantics from Clark (1978). To a program $P$ we assoc iate another set of formulas $c_{\mathscr{L}}(P)$ as follows. First we rewrite each clause

$$p(t_1, \ldots, t_n) \leftarrow L_1, \ldots, L_m$$

in the general form

$$p(x_1, \ldots, x_n) \leftarrow \exists y_1, \ldots, y_k(x_1 = t_1 \wedge \cdots \wedge x_n = t_n \wedge L_1 \wedge \cdots \wedge L_m),$$

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where $x_1, \dots, x_n$ are new variables and $y_1, \dots, y_k$ are the variables of the original clause. If

$$p(x_1, \dots, x_n) \leftarrow E_1$$
$$\vdots$$
$$p(x_1, \dots, x_n) \leftarrow E_k$$

are all the general forms of clauses with $p$ in the head, then the *definition* of $p$ is the formula

$$\forall(p(x_1, \dots, x_n) \leftrightarrow E_1 \vee \dots \vee E_k).$$

The empty disjunction (i.e. if $k = 0$) is understood as a logical falsehood. It is assumed that $=$ is a new binary predicate symbol. The set of definitions of all predicate symbols of the language is then denoted by $c_{\mathscr{L}}(P)$. The *completion* of $P$, denoted by $comp_{\mathscr{L}}(P)$, is the union of $c_{\mathscr{L}}(P)$ together with the following equality and freeness axioms referred as $CET_{\mathscr{L}}$ (Clark's equational theory).

Equality axioms:

- $\forall(x = x)$;
- $\forall(x = y \rightarrow y = x)$;
- $\forall(x = y \wedge y = z \rightarrow x = z)$;
- $\forall((x_1 = y_1 \wedge \dots \wedge x_n = y_n) \rightarrow (p(x_1, \dots, x_n) \leftrightarrow p(y_1, \dots, y_n)))$, for each predicate symbol $p$ of $\mathscr{L}$;
- $\forall((x_1 = y_1 \wedge \dots \wedge x_n = y_n) \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n))$, for each functional symbol $f$ of $\mathscr{L}$.

Freeness axioms:

- $\forall(f(x_1, \dots, x_n) \neq g(y_1, \dots, y_m))$, for each pair of distinct functional symbols $f$ and $g$ of $\mathscr{L}$ (here constants are treated as nullary functional symbols);
- $\forall(f(x_1, \dots, x_n) = f(y_1, \dots y_n) \rightarrow (x_1 = y_1 \wedge \dots \wedge x_n = y_n))$, for each functional symbol $f$ of $\mathscr{L}$;
- $\forall(t(x) \neq x)$, for each term $t(x)$ of $\mathscr{L}$, where $x$ is a proper subterm.

As the notation indicates $comp_{\mathscr{L}}(P)$ depends not only on $P$ but also on the underlying first-order language $\mathscr{L}$. This dependence is discussed in details in Shepherdson (2002). We agree to drop the language subscript, whenever it is clear to which language we refer.

According to Clark (1978), a logic programming system should derive consequences of $comp(P)$ rather than $P$ itself. So given a general program $P$ and a query $Q$, the interpreter should be able to answer the following questions:

- whether $comp(P) \models Q$;
- whether $comp(P) \models \neg Q$.

Remarkably this semantics is compatible with the widely accepted semantics of definite logic programs. That is, for a definite program $P$ and a definite query $Q$, we have $comp(P) \models Q$ if and only if $P \models Q$ (see Lloyd (1984) for the proof). This, however, implies that a decidable interpreter for general programs does not exist, since it could be used to decide whether a definite query is a consequence of a

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definite program. Curiously, however, if the language has no predicate symbols, then $CET$ itself is decidable, see e.g. Kunen (1987).

Next, we introduce the monadic second-order logic of successor functions ($SnS$) adopted from Rabin (1969). Briefly, $SnS$ is the monadic second-order theory of $\{1,\ldots,n\}^*$ (finite words on $\{1,\ldots,n\}$) with $n$ functional symbols for functions $x \mapsto xi$ ($i = 1,\ldots,n$). A more precise definition follows.

The alphabet of $SnS$ consists of a countable set of object variables (usually denoted by $x, y, z$, possibly subscripted or superscripted), a countable set of monadic predicate variables (usually denoted by $X, Y, Z$, possibly subscripted or superscripted), a single constant symbol $\Lambda$ and $n$ unary functional symbols $r_1,\ldots,r_n$, usual logical connectives, quantifiers and punctuation symbols. Terms of $SnS$ are the usual first-order terms constructed from object variables, $\Lambda$ and $r_1,\ldots,r_n$. Formulas of $SnS$ are defined as follows:

- if $t, s$ are terms and $X$ is a predicate variable, then $t = s$ and $X(t)$ are (atomic) formulas ($X(t)$ is also written as $t \in X$);
- if $F, G$ are formulas, $x$ is an object variable and $X$ is a predicate variable, then $(\neg F)$, $(F \wedge G)$, $(F \vee G)$, $(F \to G)$, $(F \leftrightarrow G)$, $(\exists x F)$, $(\exists X F)$, $(\forall x F)$, $(\forall X F)$ are formulas.

The semantics of $SnS$ formulas is defined with respect to the term interpretation and the usual second-order semantics. That is, consider the structure $\mathscr{U}_{SnS}$ whose domain is the set of ground terms. The constant symbol $\Lambda$ is interpreted by the ground term $\Lambda$ and each functional symbol $r$ is interpreted by the function $t \mapsto r(t)$. Given an $SnS$ sentence $F$, the relation $\mathscr{U}_{SnS} \models F$ is defined as in the standard second-order semantics. That is object quantifiers range over the domain, and predicate quantifiers range over all subsets of the domain. We usually surpass $\mathscr{U}_{SnS}$ from the notation and say that a sentence $F$ is true if $\mathscr{U}_{SnS} \models F$. The decidability of $SnS$ is crucial for our purposes.

*Theorem 1* (*see Rabin (1969)*)
There is an algorithm for deciding if a given $SnS$ formula is true.

From now on we fix a finite monadic language $\mathscr{L}$.

## 3 Models of monadic Clark's equational theory

It is well known that in the study of theories that contain equality axioms, one can restrict attention to structures where $=$ is interpreted as the equality in the domain. So without loss of generality, we will assume that in all structures $=$ is interpreted as the equality. This further ensures that equality axioms of $CET$ hold. So $CET$ is reduced to freeness axioms only.

Let $\mathscr{M}$ be a structure with domain $M$. An element $a \in M$ is said to *proceed* $b \in M$ if there is a term $t(x)$ containing the variable $x$ such that $t^{\mathscr{M}}(a) = b$. A *root* element is an element that does not have predecessors, apart from itself. Let $\mathscr{L}_0$ denote the language consisting of only the functional symbols of $\mathscr{L}$.

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*Proposition 1*
A structure is a model of $CET_{\mathscr{L}}$ if and only if it is a model of $CET_{\mathscr{L}_0}$ and each constant symbol is interpreted as a distinct root element.

*Proof*
Let $\mathscr{A}$ be a model of $CET_{\mathscr{L}}$. Then $\mathscr{A}$ is a model of $CET_{\mathscr{L}_0}$ since $CET_{\mathscr{L}_0} \subseteq CET_{\mathscr{L}}$. Further,

$$\forall x(f(x) \neq a)$$

is in $CET_{\mathscr{L}}$ for each functional symbol $f$ and each constant symbol $a$. So each constant symbol is interpreted as a root element. Since

$$a \neq b$$

is in $CET_{\mathscr{L}}$ for distinct constant symbols $a$ and $b$, each constant symbol is interpreted as a distinct element.

Conversely let $\mathscr{A}$ be a model of $CET_{\mathscr{L}_0}$ where each constant symbol is interpreted by a distinct root element. Note that $CET_{\mathscr{L}}$ is obtained from $CET_{\mathscr{L}_0}$ by adding axioms

$$\forall x(f(x) \neq a)$$

for each functional symbol $f$ and each constant symbol $a$ and

$$a \neq b$$

for each distinct constant symbols $a$ and $b$. All these axioms hold in $\mathscr{A}$, so it is a model of $CET_{\mathscr{L}}$. □

With this characterisation in mind, let us study structures in $\mathscr{L}_0$. Let $\{\mathscr{M}_i : i \in I\}$ be a set of structures in $\mathscr{L}_0$. We can define their disjoint union $\coprod_{i\in I} \mathscr{M}_i$ as the structure whose domain is the disjoint union of domains of $\mathscr{M}_i$ and each functional symbol $f$ is interpreted as $f^{\mathscr{M}_i}$ in the domain of $\mathscr{M}_i$.

*Proposition 2*
The structure $\coprod_{i\in I} \mathscr{M}_i$ is a model of $CET_{\mathscr{L}_0}$ if and only if each $\mathscr{M}_i$ is.

*Proof*
The structure $\mathscr{M}_i$ is a substructure of $\coprod_{i\in I} \mathscr{M}_i$. So, if the latter is a model of $CET_{\mathscr{L}_0}$, then so is the former since $CET_{\mathscr{L}_0}$ is a universal theory.

Conversely assume that $\mathscr{M}_i \models CET_{\mathscr{L}_0}$ for each $i \in I$. Let $f$ and $g$ be distinct functional symbols. If $x$ and $y$ belong to the domains of different structures than $f(x) \neq g(y)$ holds in $\coprod_{i\in I} \mathscr{M}_i$. If $x$ and $y$ belong to the domain of $\mathscr{M}_i$, then $f(x) \neq g(y)$ holds in $\mathscr{M}_i$ and hence in $\coprod_{i\in I} \mathscr{M}_i$. So

$$\coprod_{i\in I} \mathscr{M}_i \models \forall(f(x) \neq g(y)).$$

Other axioms of $CET_{\mathscr{L}_0}$ are checked similarly. □

Now, let $\mathscr{M}$ be a model of $CET_{\mathscr{L}_0}$. Two elements $a, b \in M$ are called *connected* (in symbols $a \sim b$) if they have a common predecessor.

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*Proposition 3*
The relation $\sim$ is an equivalence relation.

*Proof*
It is easy to see that $\sim$ is reflexive and symmetric. For transitivity let $a_1 \sim a_2$ and $a_2 \sim a_3$. Let $b$ be the common predecessor of $a_1$ and $a_2$ and $c$ be the common predecessor of $a_2$ and $a_3$. Then there exist terms $t(x) = h_1(\ldots h_k(x)\ldots)$ and $s(x) = g_1(\ldots g_l(x)\ldots)$ such that $t^{\mathscr{M}}(b) = a_2 = s^{\mathscr{M}}(c)$. Without loss of generality assume that $l \leqslant k$. Then by freeness axioms $h_1 = g_1, \ldots, h_l = g_l$ and $c = h_{l+1}^{\mathscr{M}}(\ldots h_k^{\mathscr{M}}(b)\ldots)$. Thus $b$ proceeds $c$ and hence $a_3$ and so $a_1$ and $a_3$ are connected. □

Thus $\sim$ partitions $M$ into equivalence classes. Each class is closed under the interpretations of the functional symbols and hence generates a substructure. We will refer to these substructures as the *components* of $\mathscr{M}$. Thus $\mathscr{M}$ is isomorphic to the disjoint union of all of its components. Our goal is to characterise each component.

There can be at most one root element in each component. We will refer to components containing root elements as *root components*. If $\mathscr{A}$ is a root component with domain $A$ and root element $a$, then $A = \{t^{\mathscr{A}}(a) : t(x) \text{ is a term containing } x\}$. Further, by freeness axioms, elements $t^{\mathscr{A}}(a)$ are all different for different terms $t(x)$. Thus the substructure generated by $A$ is isomorphic to the term structure of $\langle c, f_1, \ldots, f_n\rangle$, where $c$ is some constant symbol and $f_1, \ldots, f_n$ are the functional symbols of $\mathscr{L}_0$. We will call this structure the *root structure*. Thus all root components are isomorphic to the root structure (and hence are isomorphic to each other).

Now let us study components that do not contain a root element – *non-root components*. Let $\mathscr{A}$ be a non-root component with domain $A$. Pick arbitrary $a_0 \in A$. Then there are $a_1 \in A$ and a functional symbol $h_1$ such that $a_0 = h_1^{\mathscr{A}}(a_1)$. Similarly there are $a_2 \in A$ and a functional symbol $h_2$ such that $a_1 = h_2^{\mathscr{A}}(a_2)$. Continuing this way we will get an infinite sequence $a_0, a_1, a_2, \ldots$ of elements of $A$ and an infinite sequence of functional symbols $h_1, h_2, \ldots$. By freeness axioms all $a_j$ are different. Observe that $\{a_0, a_1, \ldots\}$ is the set of predecessors of $a_0$. For an arbitrary $b \in A$, elements $b$ and $a_0$ should have a common predecessor. Let $a_j$ be the one with the minimal index and let $b = g_1^{\mathscr{A}}(\ldots g_k^{\mathscr{A}}(a_j)\ldots)$. In case that $j > 0$ and $k > 0$ we would further have $g_k \neq h_j$. Let $c_0, c_1, \ldots$ be new constant symbols, $f_1, \ldots, f_n$ be the functional symbols of $\mathscr{L}_0$ and consider the set of ground terms over $\langle c_0, c_1, \ldots, f_1, \ldots, f_n\rangle$ that do not contain $h_j(c_j)$ as a subterm for $j = 1, 2, \ldots$. Define the interpretations of functional symbols as

$$f(t) = \begin{cases} c_{j-1} & \text{if } t = c_j \text{ and } f = h_j \text{ for some } j = 1, 2, \ldots \\ f(t) & \text{otherwise} \end{cases}$$

From the above discussion it follows that this structure is isomorphic to $\mathscr{A}$. We will refer to such structures as *non-root structures*. The sequence $h_1, h_2, \ldots$ is called the signature of the structure. Note that non-root structures with different signatures may be isomorphic. To sum up, we obtain the following:

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*Theorem 2*
A structure is a model of $CET$ if and only if it is a disjoint union of root structures and non-root structures and each constant symbol is interpreted as a root element.

## 4 Decidability for monadic programs and queries

In this section, we construct an algorithm to decide whether a monadic query (or its negation) is a consequence of $comp(P)$ for a monadic program $P$. In fact we show slightly more: given an arbitrary monadic formula $F$ (probably using equality) it is decidable whether $\{F\} \cup CET$ is consistent or not. Let $c_1, \dots, c_k$ be the constant, $f_1, \dots, f_n$ – the functional and $p_1, \dots, p_m$ – the predicate symbols of $\mathscr{L}$. We will construct a formula of $S(2n+1)S$ that would be true if and only if $\{F\} \cup CET$ is consistent. For convenience we will refer to the functional symbols of $S(2n+1)S$ as $f_0, f_1, \dots, f_n, f_1^{-1}, \dots, f_n^{-1}$. This is a bit confusing since for positive $j$, we also use $f_i^j(t)$ to denote the term $f_i(\dots f_i(t)\dots)$, where $f_i$ is repeated $j$ times. However, the notation $f_i^{-1}$ indicates exactly how we are going to use that functional symbol.

Let $D$ be a subset of $U_{S(2n+1)S}$ such that

- $f_0^j(\Lambda) \in D$ for $j = 1, \dots, k$;
- for every $x \in D$ and every $i = 1, \dots, n$ either $f_i(x) \in D$ or $x = f_i^{-1}(y)$ for some $y \in D$, but not both.

Let $P_1, \dots, P_m$ be subsets of $D$. The tuple $\langle D, P_1, \dots, P_m \rangle$ defines an interpretation of $\mathscr{L}$ in the following way:

- the domain of the interpretation is $D$;
- the constant symbol $c_j$ is interpreted as $f_0^j(\Lambda)$;
- the functional symbol $f_i$ is interpreted as the function
$$x \mapsto \begin{cases} f_i(x) & \text{if } f_i(x) \in D \\ y & \text{if } x = f_i^{-1}(y); \end{cases}$$
- the predicate symbol $p_l$ is interpreted as the set $P_l$.

We want to find and express in $S(2n+1)S$ sufficient conditions on $D$, such that structures defined by $\langle D, P_1, \dots, P_m \rangle$ enumerate countable models of $CET$ and only those.

Let $domain(X)$ denote the following $S(2n+1)S$ formula:

$$\begin{aligned}
&\textstyle\bigwedge_{j=1,\dots,k} f_0^j(\Lambda) \in X \\
\wedge\ &\forall x (x \in X \to \textstyle\bigwedge_{i=1,\dots,n} (f_i(x) \in X \veebar \exists y \in X\ x = f_i^{-1}(y))) \\
\wedge\ &\textstyle\bigwedge_{\substack{j=1,\dots,k \\ i=1,\dots,n}} f_i^{-1}(f_0^j(\Lambda)) \notin X \\
\wedge\ &\textstyle\bigwedge_{i=1,\dots,n} \forall x (x \in X \wedge f_i(x) \in X \to \bigwedge_{i'=1,\dots,n} f_{i'}^{-1}(f_i(x)) \notin X) \\
\wedge\ &\forall x (\neg \textstyle\bigvee_{\substack{i=1,\dots,n-1 \\ i'=i+1,\dots,n}} (f_i^{-1}(x) \in X \wedge f_{i'}^{-1}(x) \in X)),
\end{aligned}$$

where $\veebar$ stands for the exclusive or.

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*Proposition 4*
If *domain*$(D)$ holds for $D \subseteq U_{S(2n+1)S}$ and $P_1, \ldots, P_m \subseteq D$ then the interpretation defined by $\langle D, P_1, \ldots, P_m \rangle$ is a model of $CET$.

*Proof*
Let $D$ be a subset of $U_{S(2n+1)S}$ such that *domain*$(D)$ holds and $P_1, \ldots, P_m \subseteq D$. The first two clauses of the definition of *domain* ensure that $\langle D, P_1, \ldots, P_m \rangle$ defines a structure $\mathscr{D}$. Let us show that it satisfies $CET$. By the third clause $f_0^j(\Lambda)$ is a root element for $j = 1, \ldots, k$. So we need to only check the axioms of $CET_{\mathscr{L}_0}$.

Let $a, b \in D$ and assume that $f_i^{\mathscr{D}}(a) = f_{i'}^{\mathscr{D}}(b)$. Consider two cases.

- If $f_i(a) \in D$, then $f_{i'}^{\mathscr{D}}(b) = f_i^{\mathscr{D}}(a) = f_i(a)$. But then by the fourth clause $f_{i'}^{-1}(f_i(a)) \notin D$ and so $b \neq f_{i'}^{-1}(f_i(a))$. It follows that $i = i'$ and $a = b$.
- If $f_i(a) \notin D$, then $a = f_i^{-1}(c)$ for some $c \in D$. Thus, we have $f_{i'}^{\mathscr{D}}(b) = c$. Then $f_{i'}(b) \notin D$, since otherwise $a = f_i^{-1}(f_{i'}(b)) \in D$ contrary to the fourth clause. But then $f_{i'}^{-1}(c) = b \in D$. By the fifth clause $i = i'$ and $a = b$.

Thus in both cases we have $i = i'$ and $a = b$. Therefore

$$\mathscr{D} \models \forall(f_i(x) \neq f_{i'}(y)),$$

for $i \neq i'$ and

$$\mathscr{D} \models \forall(f_i(x) = f_i(y) \rightarrow x = y).$$

To show that the third axiom scheme of $CET_{\mathscr{L}_0}$ holds assume that $a \in D$ and $f_{n_1}^{\mathscr{D}}(\ldots f_{n_i}^{\mathscr{D}}(a)\ldots) = a$. Denote $b = f_{n_2}^{\mathscr{D}}(\ldots f_{n_i}^{\mathscr{D}}(a)\ldots)$, so that $f_{n_1}^{\mathscr{D}}(b) = a$. Again consider two cases.

- Assume $a = f_{n_1}(b)$. But then for no $a' \in D$, $a = f_{n_i}^{-1}(a')$. So $f_{n_i}(a) \in D$. Similarly $f_{n_{i-1}}(f_{n_i}(a)) \in D$ and continuing this way we will get that $f_{n_1}(\ldots f_{n_i}(a)\ldots) \in D$. So $f_{n_1}^{\mathscr{D}}(\ldots f_{n_i}^{\mathscr{D}}(a)\ldots) = f_{n_1}(\ldots f_{n_i}(a)\ldots) \neq a$, which contradicts our assumption.
- Otherwise $b = f_{n_1}^{-1}(a)$. But we have $b = f_{n_2}^{\mathscr{D}}(\ldots f_{n_i}^{\mathscr{D}}(a),)$. So $f_{n_3}^{\mathscr{D}}(\ldots f_{n_i}^{\mathscr{D}}(a)\ldots) = f_{n_2}^{-1}(b) = f_{n_2}^{-1}(f_{n_1}^{-1}(a))$. Continuing this way we will get $a = f_{n_i}^{-1}(\ldots f_{n_1}^{-1}(a)\ldots)$ which is not possible.

Thus in both cases we obtain a contradiction, which proves that the third axiom scheme of $CET_{\mathscr{L}_0}$ holds. □

The last proposition ensures that whenever *domain*$(D)$ holds, $D$ defines a model of $CET$. We also need each countable model of $CET$ to have such a representation.

*Proposition 5*
For every countable model $\mathscr{D}$ of $CET$, there are $D \subseteq U_{S(2n+1)S}$ and $P_1, \ldots, P_m \subseteq D$ such that *domain*$(D)$ holds and the structure defined by $\langle D, P_1, \ldots, P_m \rangle$ is isomorphic to $\mathscr{D}$.

*Proof*
Let $\mathscr{D}$ be a countable model of $CET$. Let $\mathscr{D}$ be obtained by interpreting $c_1, \ldots, c_k$ as root elements in $\coprod_{j \in J} \mathscr{D}_j$, where $\mathscr{D}_j$ is either the root structure or some non-root structure. Since $\mathscr{D}$ is countable, $J$ is at most countable. So without loss of generality, we can assume that $J \subseteq \mathbb{N}$. We can also assume that $\{1, \ldots, k\} \subseteq J$ and that $c_1, \ldots, c_k$

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are interpreted as the root elements of $\mathscr{D}_1, \dots, \mathscr{D}_k$. Since $P_1, \dots, P_m$ can be chosen arbitrarily, it is enough to find $D \subseteq U_{S(2n+1)S}$ such that *domain*$(D)$ holds and the structure generated by $D$ is $\mathscr{L}_0$-isomorphic to $\coprod_{j\in J} \mathscr{D}_j$ (since *domain*$(D)$ holds $f_0^j(\Lambda)$ is a root element for $j = 1, \dots, k$).

A $S(2n+1)S$ term $t(x)$ is called a *main* term if it contains $x$ and does not contain $f_0$. For $a \in U_{S(2n+1)S}$, define the subtree rooted in $a$ as the set $T(a) = \{t(a) : t(x) \text{ is a main term}\}$. We will represent $D$ as a union $D = \bigcup_{j\in J} D_j$, where $D_j \subseteq T(f_0^j(\Lambda))$. If $\mathscr{D}_j$ is a root structure, the choice of $D_j$ is straightforward: $D_j = \{t(f_0^j(\Lambda)) : t(x) \text{ is a term over } f_1, \dots, f_n \text{ and } x\}$. Clearly $D_j$ generates a root structure with $f_0^j(\Lambda)$ as the root element.

Now let $\mathscr{D}_j$ be a non-root structure over $\langle d_0, d_1, \dots, f_1, \dots, f_n\rangle$ and $f_{n_1}, f_{n_2}, \dots$ be its signature. Thus the domain of $\mathscr{D}_j$ consists of ground terms not containing $f_{n_i}(d_i)$ as subterms for $i = 1, 2, \dots$ and $f_{i'}$ is interpreted as $t \mapsto f_{i'}(t)$ with the exception that $f_{n_i}(d_i) = d_{i-1}$. To form $D_j$ we pick the element $f_{m_1}(\dots f_{m_l}(f_{n_i}^{-1}(\dots f_{n_1}^{-1}(f_0^j(\Lambda))\dots))\dots)$ for the element $f_{m_1}(\dots f_{m_l}(d_i)\dots)$ of $\mathscr{D}_j$. Denote the structure generated by $D_j$ as $\mathscr{D}'_j$. Note that $f_{n_i}^{\mathscr{D}'_j}(f_{n_i}^{-1}(\dots f_{n_1}^{-1}(f_0^j(\Lambda))\dots)) = f_{n_{i-1}}^{-1}(\dots f_{n_1}^{-1}(f_0^j(\Lambda))\dots)$ (which corresponds to $f_{n_j}(d_j) = d_{j-1}$) and $f_i^{\mathscr{D}'_j}(t) = f_i(t)$ otherwise. This shows that $\mathscr{D}_j$ and $\mathscr{D}'_j$ are indeed isomorphic.

It is routine to check that $D$ satisfies *domain*$(X)$. □

Last two propositions enable us to quantify over all countable models of $CET$. So to decide whether a formula $F$ has a model satisfying $CET$ we need to find an $S(2n+1)S$ formula to define the models of $F$. The formula $F$ is called *simple* if every functional symbol $f$ occurs in a subformula of the form $y = f(x)$.

*Proposition 6*

For a simple closed formula $F$, there is an $S(2n+1)S$ formula $Mod_F(X, Y_1, \dots, Y_m)$ such that for every $D \subseteq U_{S(2n+1)S}$ satisfying *domain*$(X)$ and every $P_1, \dots, P_m \subseteq D$ the following holds: $Mod_F(D, P_1, \dots, P_m)$ holds if and only if the structure defined by $\langle D, P_1, \dots, P_m\rangle$ is a model of $F$.

*Proof*

Without loss of generality, assume that $F$ uses only the connectives $\vee$ and $\neg$ and the quantifier $\exists$. To obtain $Mod_F(X, Y_1, \dots, Y_m)$ we do the following

- replace each subformula $\exists x G$, with $\exists x(x \in X \wedge G)$;
- replace each predicate symbol $p_l$ with a predicate variable $Y_l$;
- replace each constant symbol $c_j$ by the term $f_0^j(\Lambda)$;
- replace each subformula of the form $y = f(x)$ with $y = f(x) \vee x = f^{-1}(y)$.

Now let $D \subseteq U_{S(2n+1)S}$ be such that *domain*$(D)$ holds and $P_1, \dots, P_m \subseteq D$. Denote by $\mathscr{D}$ the structure defined by $\langle D, P_1, \dots, P_m\rangle$. Consider an arbitrary simple $\mathscr{L}$-formula $G$ (possibly with free variables). Let $\phi$ be an assignment of its free variables (with respect to $\mathscr{D}$). Since the range of $\phi$ is in $U_{S(2n+1)S}$ (the set of $S(2n+1)S$ ground terms), it defines an $S(2n+1)S$ substitution. We show by induction on the construction of $G$ that $\mathscr{D} \models_\phi G$ if and only if $Mod_G(D, P_1, \dots, P_m)\phi$ holds.

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- If $G$ is $p_l(t)$, then $t$ does not contain functional symbols. If $t$ is a variable $x$, then $\mathscr{D} \models_\phi p_l(x) \iff \phi(x) \in P_l$ by definition of $\mathscr{D}$. Otherwise $t$ is a constant symbol $c_j$ and then $\mathscr{D} \models_\phi p_l(c_j) \iff f_0^j(\Lambda) \in P_l$ again by definition of $\mathscr{D}$.
- If $G$ is $t_1 = t_2$, then consider two cases. If $t_1$ and $t_2$ do not contain functional symbols, then $\mathscr{D} \models_\phi t_1 = t_2 \iff Mod_G(D, P_1, \ldots, P_m)\phi$ can be shown similar to the previous case. Otherwise $G$ is of the form $y = f_i(x)$. In this case, we have $\mathscr{D} \models_\phi y = f_i(x) \iff \phi(y) = f_i^{\mathscr{D}}(\phi(x)) \iff \phi(y) = f_i(\phi(x)) \vee \phi(x) = f_i^{-1}(\phi(y))$.
- The cases $G = G_1 \vee G_2$ and $G = \neg G_1$ are completely straightforward.
- If $G$ is $\exists x G_1$, then $\mathscr{D} \models_\phi \exists x G_1 \iff$ for some $t \in D$, $\mathscr{D} \models_{\phi[x \to t]} G_1 \iff$ for some $t \in D$, we have $Mod_{G_1}(D, P_1, \ldots, P_m)\phi[x \mapsto t] \iff \exists x(x \in D \wedge Mod_{G_1}(D, P_1, \ldots, P_m)\phi \setminus x)) \iff Mod_G(D, P_1, \ldots, P_m)\phi$. Here $\phi[x \mapsto t]$ and $\phi \setminus x$ are substitutions that differ from $\phi$ only in assignment of $x$. The substitution $\phi[x \mapsto t]$ assigns $t$ to $x$ and $\phi \setminus x$ does not assign anything to $x$.

Now since $F$ does not contain free variables, $\mathscr{D} \models F \iff Mod_F(D, P_1, \ldots, P_m)$. □

It remains to glue all the pieces together.

*Theorem 3*
There is an algorithm that takes a finite monadic language $\mathscr{L}$ and a formula $F$ in $\mathscr{L}$ (possibly using equality) and decides whether $\{F\} \cup CET_{\mathscr{L}}$ is satisfiable.

*Proof*
First we transform $F$ into $F'$ by repeatedly replacing each atomic subformula $A(f(t))$ not of the form $y = f(x)$ by $\exists x, y(x = t \wedge y = f(x) \wedge A(y))$ until there is none left. Clearly $F'$ is simple and is logically equivalent to $F$. Then we form $Mod_{F'}$. By Propositions 4 and 5, the formula $domain(X)$ enumerates all countable models of $CET_{\mathscr{L}}$ and by Proposition 6, the formula $Mod_{F'}(X, Y_1, \ldots, Y_m)$ defines the truth of $F'$ in $\langle X, Y_1, \ldots, Y_m \rangle$. Hence $\{F\} \cup CET_{\mathscr{L}}$ is satisfiable if and only if the $S(2n+1)S$ formula

$$\exists X, Y_1, \ldots, Y_m(domain(X) \wedge Y_1 \subseteq X \wedge \ldots Y_m \subseteq Y_m \wedge Mod_{F'}(X, Y_1, \ldots, Y_1))$$

is true, which is decidable. □

*Corollary 1*
There is an algorithm that given a finite monadic language $\mathscr{L}$, a program $P$ and a query $Q$ decides the following questions:

- whether $comp_{\mathscr{L}}(P) \models Q$;
- whether $comp_{\mathscr{L}}(P) \models \neg Q$.

## 5 Conclusion

The precise computational complexity of the decision procedure for the Clark's completion semantics remains to be determined. The decision procedure for $SnS$ is primitive recursive, but not elementary recursive, (i.e. its complexity cannot be bound by a tower of exponentials of a fixed length) see Meyer (1975). This makes

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the proposed algorithm for deciding the Clark's completion semantics prohibitive for practical applications. For comparison exponential algorithms are known for deciding the satisfiability of a monadic first-order formula without equality and the satisfiability of a monadic first-order formula with equality but without functional symbols (Börger *et al*. 1997).

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