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Transport in partially degenerate, magnetized plasmas. Part 1. Collision operators

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The quantum Boltzmann collision operator is expanded to yield a degenerate form of the Fokker–Planck collision operator. This is analysed using Rosenbluth potentials to give a degenerate analogue of the Shkarofsky operator. The distribution function is then expanded about an equilibrium Fermi–Dirac distribution function using a tensor perturbation formulation to give a zeroth-order and a first-order collision operator. These equations are shown to satisfy the relevant conservation equations. It is shown that the distribution function relaxes to a Fermi–Dirac form through electron–electron collisions.

1. Introduction

Understanding plasma transport is essential for the correct modelling of plasma experiments. The most successful transport coefficients are those of Braginskii (1958), and include magnetic field effects and electron–electron collisions. However, with today's interest in dense plasmas, the Braginskii coefficients become invalid as electron degeneracy begins to become important.

The problem of degenerate electron conduction in liquid metals was studied by Ziman (1961). Although these coefficients are valid for liquid metals, electron–electron collisions are not included in the model, and hence they are not suitable for low-density fusion plasmas. Hubbard and Lampe (1969) used the Lennard–Balescu equation to include electron–electron collisions in a weakly coupled plasma, but did not include a magnetic field. Lee and More (1984) used a relaxation model to derive a full set of transport coefficients, but again the neglect of electron–electron collisions makes them unsuitable for fusion plasmas, i.e. hot, low-Z plasmas. An early calculation of thermal conductivity in the absence of magnetic field and electron–electron collisions was given by Mestel (1950).

The aim of this paper is to develop a partially degenerate kinetic equation equivalent to the Fokker–Planck equation; then to use the methods of Braginskii, so that a transport model can be developed that includes electron degeneracy, electron–electron collisions and a magnetic field. Such a regime is important in white dwarf stars and some extremely dense laboratory plasmas, such as might be established in inertial confinement and the radiative collapse of a Z-pinch.

Section 2 provides an overall view of transport theory, and shows how the transport coefficients can be calculated numerically from the first term in the distribution function expansion. The standard procedure for dealing with the $\ln\Lambda$ term in the degenerate region is analysed.

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Section 3 uses the quantum mechanical Boltzmann collision operator to derive zeroth- and first-order Braginskii-type operators. This is done by first creating a degenerate form of the Fokker–Planck operator.

2. Overview

2.1. The kinetic equation

The electron kinetic equation is given by

$$\frac{\partial f_e}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla} f_e-(\mathbf{a}+\mathbf{v}\times\boldsymbol{\omega}\cdot\boldsymbol{\nabla}_v f_e=C_{ee}+C_{ei}, \tag{2.1}$$

where $\mathbf{a}=e\mathbf{E}/m$ is the electron acceleration by an electric field $\mathbf{E}$, $\boldsymbol{\omega}$ is the electron cyclotron frequency in a magnetic field $\mathbf{B}$, $-e$ is the electron charge, and $C_{ee}+C_{ei}$ is the rate of change of F_e due to collisions.

The description of ions and electrons in terms of distribution functions is often more detailed than required, and it will normally be sufficient to describe a plasma in terms of macroscopic variables such as density, mean velocity, pressure, and so on. These may be obtained by taking velocity moments of the kinetic equation.

2.2. The Fokker–Planck equation

The collisional terms on the right-hand side of (2.1) are usually modelled by using the Fokker–Planck equation, which assumes that each particle interacts simultaneously with a large number of particles, and that the dominant effect is that of stochastic deflections resulting from distant collisions. Large deflections from close encounters are assumed to be relatively rare, and are ignored. It is assumed that there is a time Δt that is long enough compared with the collision time, but sufficiently small that the net mean square change in the velocity is small compared with the mean velocity. Defining $P(\mathbf{v};\Delta\mathbf{v})$ as the probability that v will change by an amount $\Delta\mathbf{v}$ in time Δt, the distribution function is given by

$$f(\mathbf{v},t)=\int d^3(\Delta\mathbf{v})\,f(\mathbf{v}-\Delta\mathbf{v},t-\Delta t)\,P(\mathbf{v}-\Delta\mathbf{v},\Delta\mathbf{v}). \tag{2.2}$$

The spatial dependence of f will be ignored, since we are deriving a collisional term that is essentially local, and implies that f does not vary over a Debye length, which is the limit of the long-range collision forces. Expanding in a Taylor approximation and using the definitions

$$\int d^3(\Delta\mathbf{v})\,P(\mathbf{v};\Delta\mathbf{v})=1, \tag{2.3}$$

$$\langle\Delta\mathbf{v}\rangle=\frac{1}{\Delta t}\int d^3(\Delta\mathbf{v})\,P(\mathbf{v};\Delta\mathbf{v})\,\boldsymbol{\Delta}\mathbf{v}, \tag{2.4}$$

$$\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle=\frac{1}{\Delta t}\int d^3(\Delta\mathbf{v})\,P(\mathbf{v};\Delta\mathbf{v})\,\Delta\mathbf{v}\,\Delta\mathbf{v}, \tag{2.5}$$

(2.2) becomes

$$\left(\frac{\partial f}{\partial t}\right)_c=-\frac{\partial}{\partial\mathbf{v}}(f\langle\Delta\mathbf{v}\rangle)+\frac{1}{2}\frac{\partial^2}{\partial\mathbf{v}\,\partial\mathbf{v}}:(f\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle), \tag{2.6}$$

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where higher terms in the expansion are neglected, as discussed by Robinson and Bernstein (1962). This assumption is valid so long as $\ln\Lambda$ is much larger than unity.

This is the Fokker–Planck collision operator, and is in two parts: the first representing friction and the second representing diffusion in velocity space.

2.3. Transport coefficients

The transport coefficients can be derived by linearization of the Fokker–Planck equation. It is assumed that the plasma is close to thermodynamic equilibrium. The distribution function can be expanded in Cartesian tensors, and truncated after two or three terms:

$$f(r,v,t) = f_0(\mathbf{r},\mathbf{v},t) + \mathbf{f}_1(\mathbf{r},v,t)\cdot\frac{\mathbf{v}}{v} + \mathbf{f}_2(\mathbf{r},v,t):\frac{\mathbf{vv}}{v^2}, \tag{2.7}$$

as shown by Johnston (1960). Substituting this into the Fokker–Planck equation (2.1). Shkarofsky *et al.* (1996) obtained, order by order,

$$\frac{\partial f_0}{\partial t} + \frac{v}{3}\boldsymbol{\nabla}\cdot\mathbf{f}_1 - \frac{1}{3v^2}\frac{\partial}{\partial v}v^2\mathbf{a}\cdot\mathbf{f}_1 - C_0 = 0, \tag{2.8}$$

$$\frac{\partial \mathbf{f}_1}{\partial t} + v\boldsymbol{\nabla}f_0 - \mathbf{a}\frac{\partial f_0}{\partial v} - \boldsymbol{\omega}\times\mathbf{f}_1 + \tfrac{2}{5}v\boldsymbol{\nabla}\cdot\mathbf{f}_2 - \frac{2}{5}\frac{1}{v^3}\frac{\partial}{\partial v}(v^3\mathbf{a}\cdot\mathbf{f}_2) - \mathbf{C}_1 = 0, \tag{2.9}$$

$$\left[\frac{\partial \mathbf{f}_2}{\partial t} + v(\boldsymbol{\nabla}\mathbf{f}_1 - \tfrac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{f}_1\,\mathbf{I}) - v\frac{\partial}{\partial v}\left(\frac{\mathbf{a}\mathbf{f}_1}{v} - \frac{\mathbf{a}\cdot\mathbf{f}_1}{3v}\mathbf{I}\right) - 2\boldsymbol{\omega}\times\mathbf{f}_2 - \mathbf{C}_2\right]_2 = 0, \tag{2.10}$$

where $\mathbf{I}$ is the unit tensor, and C_0, $\mathbf{C}_1$ and $\mathbf{C}_2$ are the collision terms. The bracket $[\mathbf{A}]_2$ acting on a tensor $\mathbf{A}$ denotes $\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. The collision term $\mathbf{C}_1$ is the sum of $\mathbf{C}_{ee1}$ and $\mathbf{C}_{ei1}$, the electron–electron and electron–ion collision terms respectively. In (2.10) terms in $\mathbf{f}_3$ are omitted.

The averages of a scalar $A(v)$, a vector $\mathbf{B}(v) = B(v)\,\mathbf{v}/v$ and a dyadic $\mathbf{CC}(v) = C^2(v)\,\mathbf{vv}/v^2$ are given by

$$\langle A\rangle = \frac{4\pi}{n}\int_0^\infty A f_0\, v^2\, dv, \tag{2.11}$$

$$\langle \mathbf{B}\rangle = \frac{4\pi}{3n}\int_0^\infty B\mathbf{f}_1\, v^2\, dv, \tag{2.12}$$

$$\langle \mathbf{CC}\rangle = \frac{4\pi}{3n}\int_0^\infty C^2 f_0\,\mathbf{I}v^2\, dv + \left(\frac{8\pi}{15n}\right)\int_0^\infty C^2\mathbf{f}_2\, v^2\, dv. \tag{2.13}$$

Using these definitions, it can be shown that the expressions for the current and heat flux are given as

$$\mathbf{j} = -\frac{4\pi e}{3}\int_0^\infty \mathbf{f}_1\, v^3\, dv, \tag{2.14}$$

$$\mathbf{q}_t = \frac{2\pi m}{3}\int_0^\infty \mathbf{f}_1\, v^5\, dv = \mathbf{q} - \frac{\frac{5}{4}mv^2}{e}\mathbf{j}, \tag{2.15}$$

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where $\mathbf{j}$ is the electric current density, $\mathbf{q}$ is the intrinsic heat flow and $\mathbf{q}_t$ is the total heat flow. Terms in Π, the stress tensor and $mn\langle v^2\rangle$ have been neglected.

Before these quantities are calculated, it is necessary to use (2.9) to calculate $\mathbf{f}_1$. It is assumed that the transport is linear, and that pressure anisotropies can be ignored for electrons. Thus we set $\mathbf{f}_2 = 0$. Temporal evolution will also be ignored. Using these approximations, (2.9) becomes (Epperlein and Haines 1986)

$$(\mathbf{G} + \boldsymbol{\omega}\times)\,\mathbf{f}_1 = \mathbf{g}, \tag{2.16}$$

where $\mathbf{g}$ is given by

$$\mathbf{g} = \left(v\boldsymbol{\nabla} - \mathbf{a}\frac{\partial}{\partial v}\right) f_0 \tag{2.17}$$

and $\mathbf{G}$ is defined by

$$\mathbf{G}\mathbf{f}_1 = \mathbf{C}_1. \tag{2.18}$$

Having solved (2.16) for $\mathbf{f}_1$, (2.14) and (2.15) can be used to calculate $\mathbf{j}$ and $\mathbf{q}$.

After some algebraic manipulation, the transport coefficients can be expressed in the form given by Braginskii:

$$en\mathbf{E} = -\boldsymbol{\nabla} p_e + \frac{\mathbf{j}\times\mathbf{B}}{c} + \frac{\boldsymbol{\alpha}\cdot\mathbf{j}}{ne} - n\boldsymbol{\beta}\cdot\boldsymbol{\nabla} T, \tag{2.19}$$

$$\mathbf{q} = -\boldsymbol{\kappa}\cdot\boldsymbol{\nabla} T + \boldsymbol{\beta}\cdot\mathbf{j}\frac{T}{e}. \tag{2.20}$$

The tensor quantities $\boldsymbol{\psi}$ (i.e. $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\kappa}$) are related to their driving forces $\mathbf{s}$ (i.e. $\mathbf{j}$ and $\mathbf{q}$) by

$$\boldsymbol{\psi}\cdot\mathbf{s} = \psi_\parallel \mathbf{b}(\mathbf{b}\cdot\mathbf{s}) + \psi_\perp \mathbf{b}\times(\mathbf{s}\times\mathbf{b}) \pm \boldsymbol{\psi}\times\mathbf{b}\times\mathbf{s}, \tag{2.21}$$

where the negative sign applies only to the case $\boldsymbol{\psi} = \boldsymbol{\alpha}$, and $\mathbf{b}$ is the unit magnetic field vector ($\mathbf{b} = \mathbf{B}/B$).

2.4. *Calculation of* $\ln\Lambda$

As in most transport theories, the calculation in this paper produces a logarithmic term, which is given by

$$\ln\Lambda = \tfrac{1}{2}\ln\left[1 + \left(\frac{b_{\max}}{b_{\min}}\right)^2\right], \tag{2.22}$$

where $b_{\max}$ and $b_{\min}$ are the maximum and minimum impact parameters of a collision. If $b_{\max}$ is allowed to remain finite, and $b_{\min}$ to tend to zero, $\ln\Lambda$ becomes infinite. Therefore the usual approach is to cut-off $b_{\min}$ at a non-zero value. As it will always be required that $b_{\max}$ be greater than $b_{\min}$, care must be taken in the choice of cut-offs. In this section we shall state the various forms that $b_{\max}$ and $b_{\min}$ can take, and in the next section examine the limiting values of $\ln\Lambda$ that the cut-offs produce.

2.4.1. Values of $b_{\max}$. Debye–Hückel (1923) theory states that the electron plasma screens out any effects beyond a certain characteristic length. It is

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therefore appropriate to take $b_{\max}$ as

$$b_{\max}^2 = \lambda_{De}^2 = \frac{4\pi kT}{ne^2}, \tag{2.23}$$

where λ_{De} is the Debye length due to electron screening. Spitzer (1954) generalized the calculation of the shielding length to arbitrary electron degeneracy (with the ions still non-degenerate). For weak screening, he derived a degeneracy correction, which involves multiplying (2.23) by $F'_{1/2}/F_{1/2}$, where $F_n(\alpha)$ is the standard Fermi integral $\int_0^\infty u^n[\exp(\alpha+u)+1]^{-1}\,du$, where $-\alpha kT$ is the chemical potential μ (see Clayton 1968), Brysk *et al.* (1974) showed that the approximation

$$\frac{1}{\lambda_{De}^2} = \frac{4\pi n_e e^2}{k(T_e^2+T_F^2)^{1/2}}, \tag{2.24}$$

where $T_F = \frac{1}{2}(3\pi^2 n)^{2/3}\hbar^2/m_e k$ is the Fermi temperature, matches Spitzer's expression to within 5%, thus avoiding the need to perform the computationally expensive Fermi integrals. In the case of an electron–ion plasma, (2.23) becomes

$$\frac{1}{\lambda_{DH}^2} = \frac{4\pi n_3 e^2}{k(T_e^2+T_F^2)^{1/2}} + \frac{4\pi n_i (Z^*)^2 e^2}{kT_i}, \tag{2.25}$$

where Z^* is the effective ionic charge and T_i the ion temperature, the ions being assumed non-degenerate.

The Debye model relies on the fact that there are many particles within the Debye sphere. However, at high densities the screening length can become less than the interparticle separation $R_0 = n^{-1/3}$. Therefore, at this point, $b_{\max}$ is given by

$$b_{\max} = R_0, \tag{2.26}$$

and the ions become strongly coupled.

For large magnetic fields such that the mean electron Larmor radius a_e is less than λ_{DH}, a_e should be employed for $b_{\max}$ (see Montgomery 1976).

2.4.2. Values of $b_{\min}$. In the classical regime, the minimum impact parameter is set by the distance for a 90° collision:

$$b_{\min} = b_0 = \frac{Ze^2}{mv^2}. \tag{2.27}$$

At low energies (2.27) can become greater than the interparticle separation. Under this condition, the concept of an impact parameter is not meaningful, since the distance to some particle must be less than the interparticle separation. If this happens then the plasma is in the regime of strong coupling, and the binary model used here is invalid. However, in order to produce some sort of model in this regime, we follow the approximation of Chittenden and Haines (1990), and demand that the minimum impact parameter be less than the interparticle separation:

$$b_{\min} = R_0. \tag{2.28}$$

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At high energies the classical impact parameter becomes less than the de Broglie wavelength, and at this point it is appropriate to replace the minimum impact parameter by half the de Broglie wavelength (see Spitzer 1962):

$$b_{\min} = \frac{\lambda}{2} = \frac{\hbar}{2mv}. \tag{2.29}$$

The Coulomb scattering cross-section for an unscreened charge is the same quantum mechanically as classically. However, for a screened charge the results differ for small-angle scattering – hence this correction.

In the degenerate limit the velocity must be calculated in terms of the Fermi energy E_F as $v = (2E_f/m)^{1/2}$. Brysk *et al.* (1974) relate the de Broglie wavelength and its Fermi counterpart in analogy with (2.24):

$$b_{\min}^2 = \frac{\hbar^2}{12 m_e k (T_e^2 + T_F^2)^{1/2}}. \tag{2.30}$$

2.5. *Regions of validity of* $\ln\Lambda$

The various regimes in which the different cut-offs are valid will now be examined. It will always be required that

$$\frac{b_{\max}}{b_{\min}} > 1. \tag{2.31}$$

Starting from the classical form of Λ (i.e. using (2.23) for $b_{\max}$ and (2.27) for $b_{\min}$) and with $3kT = mv^2$ gives

$$\frac{b_{\max}^2}{b_{\min}^2} = \frac{kT}{4\pi n e^2}\frac{9k^2T^2}{Z^2e^4}$$

$$= \frac{9}{4\pi}\frac{(kT)^3}{ne^6}. \tag{2.32}$$

Here it can be seen that on increasing the number density and decreasing the temperature, there will eventually be a point where the condition (2.31) will be violated.

If $b_{\max}$ in the above equation is replaced by the interparticle separation R_0 then Λ becomes

$$\frac{b_{\max}^2}{b_{\min}^2} = R_0\frac{mv^2}{Ze^2} = \frac{\text{kinetic energy}}{\text{potential energy}}. \tag{2.33}$$

This is a measure of the strong coupling. Therefore, as long as the plasma is weakly coupled and non-degenerate, (2.33) satisfies (2.31).

Before including the effects of degeneracy in $\ln\Lambda$, we shall first investigate some basic quantities in the absolute-zero limit. The Debye length at $T = 0$ becomes

$$\lambda_{DH}^2 = \frac{kT_F}{4\pi n e^2} \quad \Rightarrow \quad \lambda_{DH} \approx (R_0 a_0)^{1/2}. \tag{2.34}$$

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Therefore, in this limit, if the interparticle separation is greater than the Bohr radius a_0 then it is also greater than the Debye radius, and the Debye theory is no longer valid. In this case the Debye length must be replaced by the interparticle separation.

Now the strong-coupling parameter must be less than unity for the plasma to be weakly coupled:

$$\left.\begin{aligned} \frac{\text{potential energy}}{\text{kinetic energy}} &\ll 1, \\ \frac{e^2}{R_0}\frac{1}{3kT_F} &\ll 1, \\ \Rightarrow \quad R_0 &\ll a_0. \end{aligned}\right\} \tag{2.35}$$

This is quite a strange result, since (2.35) implies that increasing the density at $T = 0$ decreases the coupling. This occurs because increasing the density increases the Fermi energy more rapidly than the electrostatic energy. This is only true for the electrons on the Fermi surface: the electrons below the Fermi temperature will still remain strongly coupled. However, it is the electrons on the Fermi surface that will be responsible for the transport processes.

The effect of degeneracy will now be included in the calculation. The degenerate form of the Debye length (2.24) is used for $b_{\max}$, and the degenerate form of the de Broglie wavelength (2.30) for $b_{\min}$. As we are interested in the worst-case scenario, the $T = 0$ limit will be examined. At this point,

$$\begin{aligned} \frac{b_{\max}^2}{b_{\min}^2} &= \frac{kT_F}{4\pi ne^2}\frac{12m_e kT_F}{\hbar^2} \\ &= \frac{3^{7/3}\pi^{5/3}}{4}\frac{a_0}{R_0}, \end{aligned} \tag{2.36}$$

where a_0 is the Bohr radius. Equation (2.31) is satisfied provided that R_0 is less than the Bohr radius. As previously stated, if this criterion is breached then Debye theory is no longer valid, and the interparticle separation must be used as $b_{\max}$. Again using the degenerate form of the de Broglie wavelength (2.30) for $b_{\min}$ gives

$$\frac{b_{\max}^2}{b_{\min}^2} = R_0^2\frac{12m_e kT_F}{\hbar^2} = 6(3\pi^2)^{2/3}. \tag{2.37}$$

This ensures that (2.31) is always satisfied for high densities.

The regions where $b_{\max}$ and $b_{\min}$ are used are given in Figs 1 and 2 respectively.

When the particles are separated by the Bohr orbit, the particle number density is 1.6×10^{30} m^{-3}. It can be seen from Fig. 3 that below this number density, $\ln\Lambda$ is constant, and above this density, $\ln\Lambda$ increases, which is in agreement with the theory given in this section.

Graphs of $\ln\Lambda_{ee}$ and $\ln\Lambda_{ei}$ as a function of number density and temperature are given in Figs 4 and 5.

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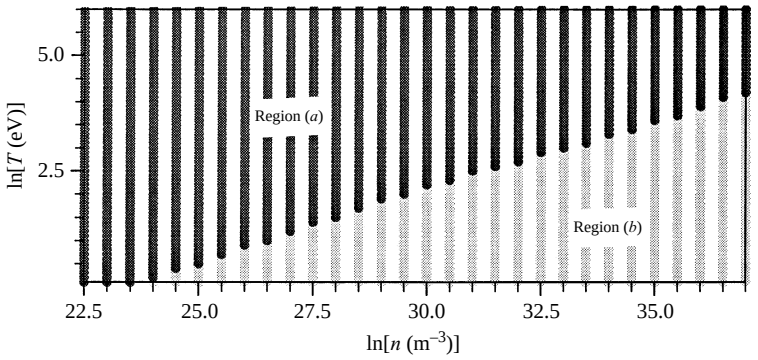


Figure 1. Temperature–density regions showing where the different values of $b_{\max}$ are used: in region (a) the Debye length given by (2.24) is used; in region (b) the interparticle separation is used.

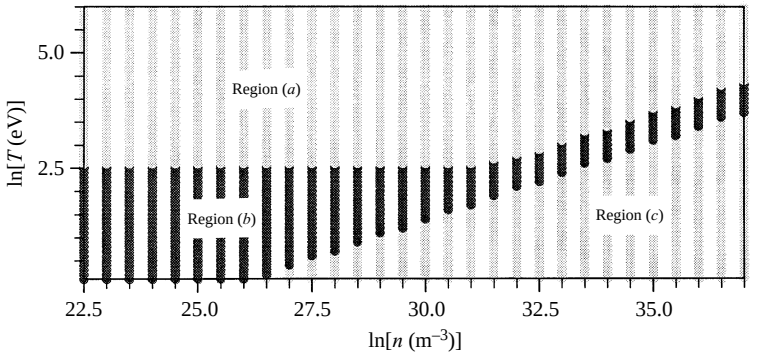


Figure 2. Temperature–density regions showing where the different values of $b_{\min}$ are used: in region (a) the de Broglie wavelength given by (2.47) is used; in region (b) the classical impact parameter (2.45) is used; in region (c) the interparticle separation is used.

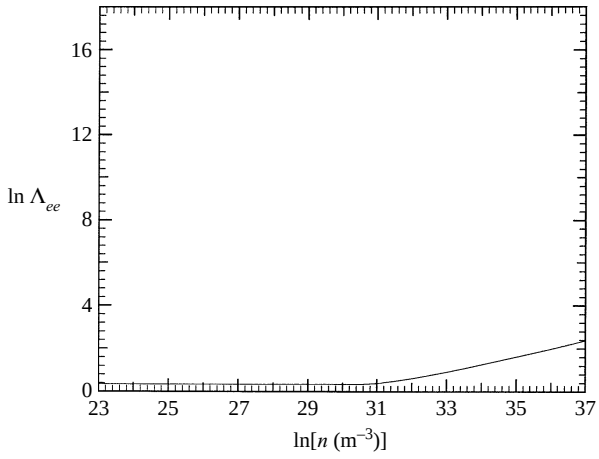


Figure 3. $\ln \Lambda_{ee}$ as a function of number density at $T = 0$.

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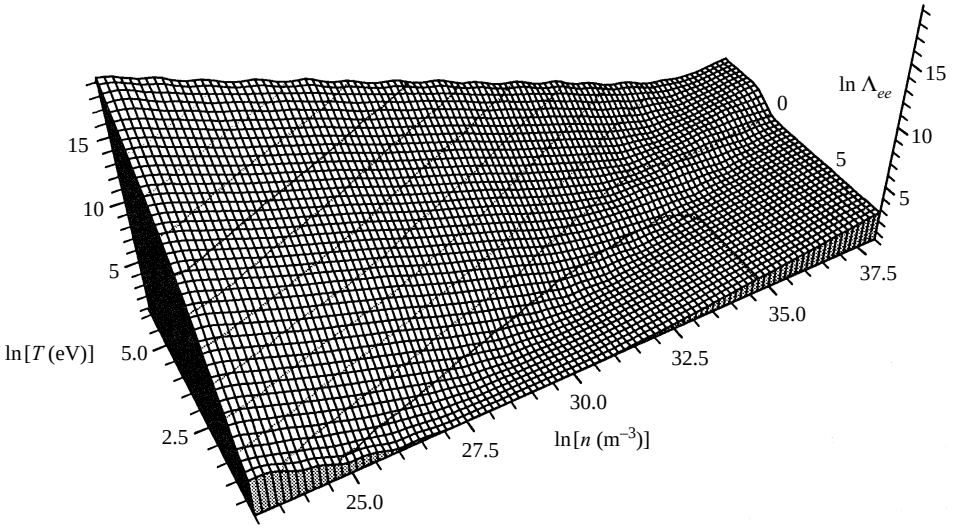


Figure 4. $\ln \Lambda_{ee}$ as a function of number density and temperature.

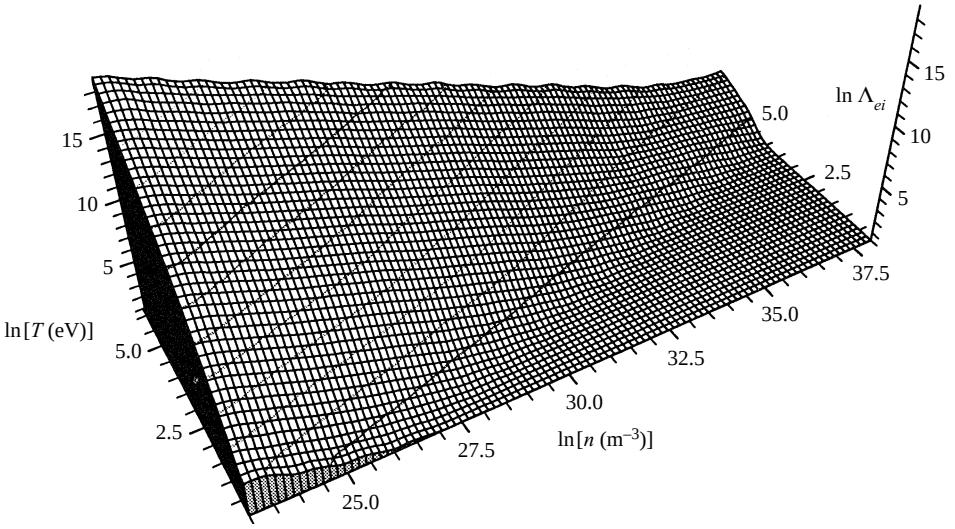


Figure 5. $\ln \Lambda_{ei}$ as a function of number density and temperature.

3. Derivation of a modified Fokker–Planck collision operator for degenerate electrons

3.1. The relaxation operator

The most basic approach to solving the kinetic equation uses the Krook or relaxation model, which assumes that the collision operator in the Fokker–Planck equation is given by

$$\frac{\partial f}{\partial t} = \frac{f_0 - f_1}{\tau_c}, \tag{3.1}$$

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where τ_c is the electron relaxation time. This is the model that is used in most solid-state investigations (e.g. Ziman 1960), where electron–electron collisions can be neglected, and it has the advantage that it yields simple analytic transport coefficients in the $T = 0$ limit. Lee and More (1984) extended the relaxation method in a numerical code to calculate the transport coefficients for non-zero temperature. As will be shown later, (3.1) is suitable for collisions involving only electron–ion collisions, where it can be assumed that the ion has infinite mass and negligible velocity. However, (3.1) is no longer valid in low-Z materials. In order to derive a collision operator that includes degeneracy and electron–electron collisions, we shall first examine the derivation of the classical collision operator given by Shkarofsky (1963).

Shkarofsky's formulation starts from the definition that if the electron distribution at time t is $f(\mathbf{v}-\Delta\mathbf{v}, t)$, and the probability of changing its velocity by $\Delta\mathbf{v}$ is $p(\mathbf{v}-\Delta\mathbf{v}; \Delta\mathbf{v})$, then the distribution function at time $t+\Delta t$ is given by

$$f(\mathbf{v}, t+\Delta t) = \int f(\mathbf{v}-\Delta\mathbf{v}, t)\, p(\mathbf{v}-\Delta\mathbf{v}, t)\, d^3v. \tag{2.2′}$$

As has already been shown, this can be expanded in a Taylor series, and leads to the Fokker–Planck collision operator. It should be noted that the expansion is only valid if it is assumed that $\Delta\mathbf{v}$ is small.

Equation (3.1) seems to be a rather *ad hoc* definition, and it is unclear how to develop it for the degenerate case. We shall therefore proceed from the Boltzmann collision operator.

3.2. Boltzmann collision operator

The Boltzmann collision operator in a non-degenerate plasma is given by

$$\frac{\delta_e f_1}{\delta t} d^3v_1 = \left[\int\int f_1(\mathbf{v}_1') f_2(\mathbf{v}_2')\, \rho_{12}(|\mathbf{v}_1'-\mathbf{v}_2'|, \chi, \psi)\, d^2\Omega\, d^3v_2'\right] d^3v_1' - \left[\int\int f_1(\mathbf{v}_1) f_2(\mathbf{v}_2)\, \rho_{12}(|\mathbf{v}_1-\mathbf{v}_2|, \chi, \psi)\, d^2\Omega\, d^3v_2\right] d^3v_1, \tag{3.2}$$

which represents the number of particles 1 per unit time entering d^3v_1 at $\mathbf{v}_1$ from collisions of particles 1 in d^3v_1' with an initial velocity $\mathbf{v}_1'$, minus the number leaving through collisions in d^3v_1. This entire quantity is then integrated over all scattering angles and all scatterers, including other particles of type 1, that is, over $d^2\Omega = \sin\chi\, d\chi\, d\psi$, d^3v_2' and d^3v_2 respectively. Here χ and ψ are the polar and azimuthal angles in a spherical coordinate system with axis along the initial velocity measured in a rest frame. Defining the initial velocity from $\mathbf{v}_1, \chi, \psi$ and the conservation of momentum and kinetic energy allows us to eliminate $\mathbf{v}_2$. The quantity ρ_{12} gives the rate of flow per unit solid angle per number flow rate of incoming particles of type 1 scattered by angles χ and ψ. If the right-hand side of (3.2) is zero (i.e. the plasma is in equilibrium) it can be shown (Chapman and Cowling 1961) that it leads to a Maxwellian distribution function. This is because the electrons are permitted to scatter into any state, even if that state is already occupied, in contradiction to the Pauli exclusion principle. It should be noted that (2.2), the fundamental equation from which Shkarofsky calculates his transport coefficients, does not include the Pauli principle, and it is for this reason that Macdonald *et al.* (1957) find that the Shkarofsky transport

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coefficients force a plasma with an initial Gaussian distribution function to relax to a Maxwellian distribution function.

To correct the transport coefficients for quantum statistics it is necessary to include the Pauli exclusion principle. Consider, for example, a gas consisting of fermions (proteons or electrons). According to quantum theory, no more than $\beta d|\mathbf{c}|\,(mL/h)^3$ such particles in a volume L^3 can possess velocities in the range $d\mathbf{c}$, where m and β are the mass and the statistical weight of the particle (Chapman and Cowling 1961), and h is Planck's constant. Hence, if in the velocity range there is already this number of particles, and the result of a collision would be that a further particle would enter the range, the collision cannot happen. More generally, if $L^3 f(\mathbf{c})\,d\mathbf{c}$ have velocities in $d\mathbf{c}$, the result of a collision that would result in a particle entering the velocity range is reduced in the ratio $1-f(\mathbf{c})\,h^3/m^3\beta$. Accordingly, (3.2) must be replaced by†

$$\begin{aligned}\frac{\delta_e f_1}{\delta t}\,d^3v_1 = &\left\{\iint f_1(\mathbf{v}_1')f_2(\mathbf{v}_2')\left[1-\frac{f_1(\mathbf{v}_1)\,h^3}{m_1^3\beta_1}\right]\right.\\ &\left.\times\left[1-\frac{f_2(\mathbf{v}_2)\,h^3}{m_2^3\beta_2}\right]\rho_{12}(|\mathbf{v}_1'-\mathbf{v}_2'|,\chi,\psi)\,d^2\Omega\,d^3v_2'\right\}d^3v_1'\\ &-\left\{\iint f_1(\mathbf{v}_1)f_2(\mathbf{v}_2)\left[1-\frac{f_1(\mathbf{v}_1')\,h^3}{m_1^3\beta_1}\right]\right.\\ &\left.\times\left[1-\frac{f_2(\mathbf{v}_2')\,h^3}{m_2^3\beta_2}\right]\rho_{12}(|\mathbf{v}_1-\mathbf{v}_2|,\chi,\psi)\,d^2\Omega\,d^3v_2\right\}d^3v_1. \qquad (3.3)\end{aligned}$$

It can be shown that in an equilibrium situation (i.e. when the collision operator is equal to zero) the resultant distribution function is a Fermi–Dirac (Chapman and Cowling 1961). Therefore it is expected that any transport coefficients derived from this collision operator will force a plasma to relax to a Fermi–Dirac distribution function, in contrast to the Shkarofsky coefficients. To simplify the above equation, we normalize as follows:

$$\begin{aligned}\frac{\delta_e f_1^n}{\delta t}\,d^3v_1 = &\left\{\iint f_1^n(\mathbf{v}_1')f_2^n(\mathbf{v}_2')\,[1-f_1^n(\mathbf{v}_1)]\,[1-f_2^n(\mathbf{v}_2)]\right.\\ &\left.\times\rho_{12}^n(|\mathbf{v}_1'-\mathbf{v}_2'|,\chi,\psi)\,d^2\Omega\,d^3v_2'\right\}d^3v_1'\\ &-\left\{\iint f_1^n(\mathbf{v}_1)f_2^n(\mathbf{v}_2)\,[1-f_1^n(\mathbf{v}_1')]\,[1-f_2^n(\mathbf{v}_2')]\right.\\ &\left.\times\rho_{12}^n(|\mathbf{v}_1-\mathbf{v}_2|,\chi,\psi)\,d^2\Omega\,d^3v_2\right\}d^3v_1, \qquad (3.4)\end{aligned}$$

where

$$f_1^n = f_1\frac{h^3}{m_1^3\beta_1}, \qquad f_2^n = f_2\frac{h^3}{m_2^3\beta_2}, \qquad (3.5)$$

$$\rho_{12}^n(g,\chi) = \rho_{12}(g,\chi)\frac{m_2^3\beta_2}{h^3}. \qquad (3.6)$$

(The renormalization means that in the completely degenerate limit $f\to 1$, and

† The statistical weight is the number of independent quantum states in which the particle can possess the same internal energy: for an electron it is 2, corresponding to the two possible values of the spin.

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in the completely non-degenerate limit $f \to 0$.) From now on it should be remembered that the equation has been normalized, and the n superscript should be dropped.

We shall now proceed by multiplying (3.4) by $X(v)$, where X can be *any* function of v, and integrating over d^3v. We also introduce the notation $F(\mathbf{V})$ for the distribution function of the scattering particles of mass M, and $f(\mathbf{v})$ for the distribution function of the scattering particles of mass m:

$$\int \frac{\delta f}{\delta t} X\, d^3\mathbf{v} = \iiint XF(\mathbf{V}')f(\mathbf{v}')\,[1-f(\mathbf{v})]\,[1-F(\mathbf{V})]\,\rho\, d^2\Omega\, d^3\mathbf{V}'\, d^3\mathbf{v}' - \iiint XF(\mathbf{V})f(\mathbf{v})\,[1-f(\mathbf{v}')]\,[1-F(\mathbf{V}')]\,\rho\, d^2\Omega\, d^3\mathbf{V}\, d^3\mathbf{v}. \quad (3.7)$$

As the scattering is assumed to be elastic (and therefore reversible), the primed and unprimed vectors in the first integral on the right-hand side can be interchanged without affecting the value of the integral. This leads to

$$\int \frac{\delta f}{\delta t} X\, d^3v = \iiint (X'-X)\,F(\mathbf{V})f(\mathbf{v})\,[1-F(\mathbf{V}')]\,[1-f(\mathbf{v}')]\,\rho\, d^2\Omega\, d^3V\, d^3v \quad (3.8)$$

It is now assumed that both the scattered and scattering distribution functions are slightly perturbed from their equilibrium values. Then writing

$$\mathbf{v}' = \mathbf{v} - \Delta\mathbf{v}, \qquad \mathbf{V}' = \mathbf{V} - \Delta\mathbf{V}, \quad (3.9)$$

$F(\mathbf{V}')$, $f(\mathbf{v}')$ and X can be expanded in Taylor series. For example

$$X(\mathbf{v}-\Delta\mathbf{v}) = X(\mathbf{v}) - \sum_{\mu} \frac{\partial(X)}{\partial v_\mu}\Delta v_\mu + \frac{1}{2}\sum_{\mu,\nu} \frac{\partial^2(X)}{\partial v_\mu \partial v_\nu}\Delta v_\mu \Delta v_\nu + \ldots, \quad (3.10)$$

Using conservation of momentum, we also have

$$\Delta\mathbf{V}_\alpha = -\frac{m}{M}\Delta\mathbf{v}_\alpha. \quad (3.11)$$

Expanding (3.8) to second order and integrating by parts produces

$$\begin{aligned}\int \frac{\delta f}{\delta t} X\, d^3v = &-\iiint X \frac{\partial}{\partial v_\mu}[fF(1-f)\,(1-F)\,\rho\Delta v_\mu]\, d^3v\, d^2\Omega\, d^3V \\ &+\frac{1}{2}\iiint X \frac{\partial^2}{\partial v_\mu \partial v_\nu}[fF(1-f)\,(1-F)\,\rho\,\Delta v_\mu \Delta v_\nu]\, d^3v\, d^2\Omega\, d^3V \\ &+\iiint X \frac{\partial}{\partial v_\mu}\left[F(1-F)\frac{\partial f}{\partial v_\nu} f\,\Delta v_\mu \Delta v_\nu \rho\right] d^2\Omega\, d^3V\, d^3v \\ &-\iiint X \frac{\partial}{\partial v_\mu}\left[F\frac{\partial F}{\partial V_\nu}\frac{m}{M} f(1-f)\,\rho\Delta v_\mu \Delta v_\nu\right] d^2\Omega\, d^3V\, d^3v. \end{aligned} \quad (3.12)$$

To the best of our knowledge, this is a new result. This is of the form

$$\int X f_1\, d^3v = \int X f_2\, d^3v.$$

It must be noted that X can be any function of v, and the above identity must

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be true for all $X(v)$. The only way that this is possible is if $f_1 = f_2$. Therefore the integration over $X(v)\,d^3v$ on both sides of (3.12) can be dropped.

Equation (3.12) should be compared with (2.6). It can be seen that as $f \to 0$ and $F \to 0$ (the non-degenerate limit), (3.12) tends to (2.6). However, as $f \to 1$ and $F \to 1$, the first two terms in (3.12), i.e. the degenerate analogues of the frictional and diffusional terms, tend to zero, and the non-Fokker–Planck terms in (3.12) dominate. An interesting feature of (3.12) is that if scattering is performed by identical particles, and $f = F$ and $m = M$, then the last two terms in the equation are equal in magnitude and cancel identically. This only leaves the first two terms, which are the degenerate forms of (2.6).

The rate at which electrons are scattered is related to the cross-section α by the relation

$$\rho = \mathscr{V}\alpha(\mathscr{V}), \qquad \mathscr{V} \equiv |\mathbf{V}-\mathbf{v}|. \tag{3.13}$$

It is assumed that the cross-section for scattering between charged particles is given by the (normalized) Rutherford formula

$$\alpha(\mathscr{V}) = \left(\frac{ZZ'e^2}{8\pi\epsilon_0 m'\mathscr{V}^2 \sin^2 \tfrac{1}{2}\chi_{cm}}\right)^2 \frac{m_2^3\beta_2}{h^3}, \tag{3.14}$$

where $m' = mM/(m+m)$ is the reduced mass, and χ_{cm} is the scattering angle in the centre-of-mass system.

Defining the integrations over all solid angles as

$$[\Delta\mathscr{V}_i]_\Omega = \int_0^\pi \int_0^{2\pi} \Delta\mathscr{V}_i \mathscr{V}\alpha(\mathscr{V}, \chi_{cm}) \sin\chi_{cm}\, d\chi_{cm}\, d\phi_{cm}, \tag{3.15}$$

$$[\delta\mathscr{V}_{ij}]_\Omega = \int_0^\pi \int_0^{2\pi} \Delta\mathscr{V}_i \Delta\mathscr{V}_j \mathscr{V}\alpha(\mathscr{V}, \chi_{cm}) \sin\chi_{cm}\, d\chi_{cm}\, d\phi_{cm}, \tag{3.16}$$

it can be shown (Shkarofsky 1963) that

$$[\Delta v_\mu]_\Omega = Y\frac{M+m}{M}\frac{\partial}{\partial v_\mu}\frac{1}{\mathscr{V}}, \tag{3.17}$$

$$[\Delta v_\mu \Delta v_\nu]_\Omega = Y\frac{\partial^2}{\partial v_\mu \partial v_\nu}\mathscr{V}, \tag{3.18}$$

where Y is given by

$$Y = 4\pi\frac{ZZ'e^2}{4\pi\epsilon_0 m}(\ln\Lambda)\frac{M^3\beta_2}{h^3} \tag{3.19}$$

and Λ is given by

$$\ln\Lambda = \frac{1}{2}\ln\left(1+\frac{b_{\max}^2}{b_{\min}^2}\right). \tag{3.20}$$

Inserting this into (3.13)–(3.20), two modified Rosenbluth potentials can be defined. First,

$$\mathscr{H}(v) = \frac{M+m}{M}\int\frac{F(1-F)}{\mathscr{V}}d^3V \tag{3.21}$$

$$\equiv {}_1\mathscr{H} - {}_2\mathscr{H}, \tag{3.22}$$

where

$${}_1\mathscr{H}(v) = \frac{M+m}{M}\int\frac{F}{\mathscr{V}}d^3V, \qquad {}_2\mathscr{H} = \frac{M+m}{M}\int\frac{F^2}{\mathscr{V}}d^3V, \tag{3.23}$$

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and secondly,

$$\mathscr{G}(v) = \int F(1-F)\,\mathscr{V}\,d^3V, \tag{3.24}$$

$$\equiv {}_1\mathscr{G} - {}_2\mathscr{G}, \tag{3.25}$$

where

$${}_1\mathscr{G}(v) = \int F\mathscr{V}\,d^3V, \qquad {}_2\mathscr{G}(v) = \int FF\mathscr{V}\,d^3V. \tag{3.26}$$

Using (3.22) and (3.25) in (3.12) enables the collision term to be written as

$$\frac{1}{Y}\frac{\delta f}{\delta t} = \frac{1}{Y}\left(\frac{\delta f}{\delta t}\right)_1 + \frac{1}{Y}\left(\frac{\delta f}{\delta t}\right)_2, \tag{3.27}$$

where

$$\frac{1}{Y}\left(\frac{\delta f}{\delta t}\right)_1 = -\frac{\partial}{\partial v_\mu}\left[f(1-f)\frac{\partial}{\partial v_\mu}\mathscr{H}(v)\right] + \frac{1}{2}\frac{\partial^2}{\partial v_\mu \partial v_\nu}\left[f(1-f)\frac{\partial^2}{\partial v_\mu \partial v_\nu}\mathscr{G}(v)\right], \tag{3.28}$$

$$\frac{1}{Y}\left(\frac{\delta f}{\delta t}\right)_2 = \frac{\partial}{\partial v_\mu}\left[f\frac{\partial f}{\partial v_\nu}\frac{\partial^2 \mathscr{G}(v)}{\partial v_\mu \partial v_\nu}\right] - \frac{1}{2}\frac{m}{M}\frac{\partial}{\partial v_\mu}\left[f(1-f)\frac{\partial^2}{\partial v_\mu \partial v_\nu}\frac{\partial}{\partial v_\nu}{}_2\mathscr{G}(v)\right], \tag{3.29}$$

It is interesting to compare this new operator with the Fokker–Planck equation given by Shkarofsky *et al.* (1966):

$$\frac{1}{Y}\left(\frac{\delta f}{\delta t}\right)_{Shk} = \frac{\partial}{\partial v_\mu} f\frac{\partial_1 \mathscr{H}}{\partial v_\mu} + \frac{1}{2}\frac{\partial^2}{\partial v_\nu \partial v_\mu} f\frac{\partial^2 {}_1\mathscr{G}}{\partial v_\nu \partial v_\mu}. \tag{3.30}$$

Here it can seen that in the non-degenerate limit ($f, F \to 0$), the second term in (3.27) tends to zero, and the first term reverts to Shkarofsky's operator (3.30). Also in the degenerate limit ($f, F \to 1$), (3.28) tends to zero, and then (3.29) becomes the dominant term in the collision operator. It is therefore to be expected that the new transport coefficients will differ from the Braginskii transport coefficients at high degeneracies.

3.3. Modified Rosenbluth potentials

Although the above potentials are very similar to those given by Rosenbluth, they differ in that they contain a factor FF, whereas the ordinary Rosenbluth potentials only contain a factor F. As will be seen later, this means that the spherical-harmonic expansion, which is used to solve $\mathscr{G}$ and $\mathscr{H}$ in potential theory, needs careful consideration. However, the relationships between the $\mathscr{G}$ and $\mathscr{H}$ functions can be written as follows.†

From (3.24) for $\mathscr{G}$ and (3.13) for $\mathscr{V}$, we have

$$\nabla\mathscr{G} = \int d^3V F(1-F)\frac{\mathbf{v}-\mathbf{V}}{\mathscr{V}}. \tag{3.31}$$

† In this section, unless specified to the contrary, ∇ is an operator on $\mathbf{v}$ only, i.e. $\nabla_{\mathbf{v}}$.

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Using (3.31) and (3.21) for $\mathscr{H}$ gives

$$\nabla^2\mathscr{G} = 2\int\frac{d^3V\,F(1-F)}{\mathscr{V}} \tag{3.32}$$

$$= \frac{2M\mathscr{H}}{m+M}. \tag{3.33}$$

Now,

$$\nabla^2\mathscr{H} = -4\pi\frac{M+m}{M}F(1-F). \tag{3.34}$$

Substituting this into (3.33) produces

$$\nabla^4\mathscr{G} = -8\pi F(1-F). \tag{3.35}$$

Similarly, it can be shown that

$$\nabla^4{}_2\mathscr{G} = 8\pi FF. \tag{3.36}$$

Transforming (3.28) and (3.29) into vector notation, and using (3.31)–(3.36) gives

$$\frac{1}{Y}\left(\frac{\delta f}{\delta t}\right)_1 = 4\pi fF(1-f)\,(1-F) + (1-2f)\frac{M-m}{M+m}\nabla f\cdot\nabla\mathscr{H}$$

$$+\tfrac{1}{2}(1-2f)\,\nabla\nabla f\colon\nabla\nabla\mathscr{G} - \nabla f\nabla f\colon\nabla\nabla\mathscr{G}, \tag{3.37}$$

$$\frac{1}{Y}\left(\frac{\delta f}{\delta t}\right)_2 = \nabla f\nabla f + f\nabla\nabla f\colon\nabla\nabla\mathscr{G} + 2\frac{M}{M+m}\nabla f\cdot\nabla\mathscr{H}$$

$$-\frac{m}{M+m}(1-2f)\,\nabla f\cdot\nabla\mathscr{H}_2 + 4\pi\frac{m}{M}f(1-f)\,F^2. \tag{3.38}$$

3.4. Calculation of potentials

To calculate the modified Rosenbluth potentials, the distribution function is expanded in spherical harmonics $Y_{lms} = P^l_m(\cos\theta')\,(\delta_{0s}\cos m\theta' + \delta_{1s}\sin\theta')$, where P^L_m is an associated Legendre polynomial and δ is a Kronecker delta function:

$$F(\mathbf{V}) = \sum F_{lms}(V)\,Y_{lms}(\theta',\phi'), \tag{3.39}$$

$$= F_0 + \mathbf{F}_1\cdot\frac{\mathbf{V}}{V} + \frac{F_2\colon\mathbf{VV}}{V^2} + \ldots, \tag{3.40}$$

the equivalence of the tensor expansion to the spherical–harmonic expansion having been shown by Johnston (1960).

The expressions $1/V$ in $\mathscr{H}$, and V in $\mathscr{G}$, can be expanded in Legendre polynomials in terms of γ, which is equal to the cosine of the angle between the velocities $\mathbf{v}$ and $\mathbf{V}$:

$$\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'); \tag{3.41}$$

θ and θ' are the polar angles in a spherical coordinate system, and ϕ and ϕ' are

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the azimuthal angles of $\mathbf{v}$ and $\mathbf{V}$. It can be shown that

$$\frac{1}{\mathscr{V}} = \sum_{n=0}^{\infty} \frac{v_<^n}{v_>^{n+1}} P_n(\gamma), \tag{3.42}$$

$$\mathscr{V} = \sum_{n=0}^{\infty} \left[\frac{(v_</v_>)^2}{2n+3} - \frac{1}{2n-1} \right] \frac{v_<^n}{v_>^{n-1}} P_n(\gamma), \tag{3.43}$$

where $v_>$ is the greater and $v_<$ the lesser of v and V, and P_n is a Legendre polynomial.

The tensor expansions of ${}_1\mathscr{H}$ and ${}_1\mathscr{G}$ were given by Shkarofsky (1963), and so we concentrate on the derivation of ${}_2\mathscr{H}$ and ${}_2\mathscr{G}$.

Inserting (3.42) into (3.23), it follows that ${}_2\mathscr{H}$ is given by

$${}_2\mathscr{H} = \frac{M+m}{M} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \left[\int_0^v V^2\, dV \sin\theta'\, d\theta'\, d\phi'\, F(\mathbf{V}) F(\mathbf{V}) \sum_{n=0}^{\infty} \frac{V^n}{v^{n+1}} P_n(z) \right.$$
$$\left. + \int_v^{\infty} V^2\, dV \sin\theta'\, d\theta'\, d\phi'\, F(\mathbf{V}) f(\mathbf{V}) \sum_{n=0}^{\infty} \frac{v^n}{V^{n+1}} P_n(z) \right]. \tag{3.44}$$

We insert the spherical-harmonic expression for F given in (3.39), and use the orthogonality theorem for spherical harmonics:

$$\int_0^{2\pi} d\phi' \int_0^{2\pi} \sin\theta'\, d\theta'\, Y_{lms}(\theta', \phi')\, P_n(\gamma) = \frac{4\pi}{2n+1} Y_{nms}(\theta, \phi)\, \delta_{ln}. \tag{3.45}$$

This orthogonality condition cannot be directly used to solve ${}_2\mathscr{H}$, since the integrals have the form

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta'\, d\theta'\, Y_{l'm's'}(\theta', \phi')\, Y_{lms}(\theta', \phi')\, P_n(\gamma). \tag{3.46}$$

In general, the $Y_{l'm's'} Y_{lms}$ must be expressed as a single higher-order spherical harmonic, a process that leads to great complexity. However, if it is remembered that $Y_{000} = 1$ (i.e. it has no angular dependence), the terms involving $F_0 = F_{000} Y_{000}$ revert to the integral of the form (3.45) multiplied by F_0. Using this fact, ${}_2\mathscr{H}$ can be expanded up to first order as

$${}_2\mathscr{H} = 4\pi \frac{M+m}{M} \left(\frac{1}{v} \int_0^v dV\, F_0^2\, V^2 + \frac{2}{3v^2} \int_0^v dV\, F_0\, \mathbf{F}_1 \cdot \frac{\mathbf{v}}{v}\, V^3 + \ldots \right.$$
$$\left. + \int_v^{\infty} F_0^2\, V\, dV + \frac{2v}{3} \int_v^{\infty} F_0\, \mathbf{F}_1 \cdot \frac{\mathbf{v}}{v}\, dV + \ldots \right). \tag{3.47}$$

The following integrals are now introduced:

$$\mathbf{I}_p^l \equiv \frac{4\pi}{v^p} \int_0^v \mathbf{F}_l\, V^{p+2}\, dV, \qquad \mathbf{J}_p^l \equiv \frac{4\pi}{v^p} \int_v^{\infty} \mathbf{F}_l\, V^{p+2}\, dV, \tag{3.48}$$

i.e. the usual Allis integrals, and also

$$\mathbf{I}_p^{lm} \equiv \frac{4\pi}{v^p} \int_0^v \mathbf{F}_l \mathbf{F}_m\, V^{p+2}\, dV, \qquad \mathbf{J}_p^{lm} \equiv \frac{4\pi}{v^p} \int_v^{\infty} \mathbf{F}_l \mathbf{F}_m\, V^{p+2}\, dV. \tag{3.49}$$

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These obey the relations

$$\left.\begin{aligned}\frac{\partial}{\partial v}\mathbf{I}_p^l &= 4\pi v^2\mathbf{F}_l - \frac{p\mathbf{I}_p^l}{v},\\ \frac{\partial}{\partial v}\mathbf{J}_p^l &= -4\pi v^2\mathbf{F}_l - \frac{p\mathbf{J}_p^l}{v},\end{aligned}\right\} \tag{3.50}$$

and similarly, for $\mathbf{I}_p^{lm}$ and $\mathbf{J}_p^{lm}$,

$$\left.\begin{aligned}\frac{\partial}{\partial v}\mathbf{I}_p^{lm} &= 4\pi v^2\mathbf{F}_l\mathbf{F}_m - \frac{p\mathbf{I}_p^{lm}}{v},\\ \frac{\partial}{\partial v}\mathbf{J}_p^{lm} &= -4\pi v^2\mathbf{F}_l\mathbf{F}_m - \frac{p\mathbf{J}_p^{lm}}{v}.\end{aligned}\right\} \tag{3.51}$$

Using Shkarofsky's expressions for ${}_1\mathscr{H}$, and the above expressions for ${}_2\mathscr{H}$, it is found from (3.21) that

$$\mathscr{H}(v) = \frac{M+m}{Mv}\sum\left[I_0^0 + J_1^0 - I_0^{00} - J_{-1}^{00} + (\mathbf{I}_1^1 + \mathbf{J}_{-2}^1 - 2\mathbf{I}_1^{01} - 2\mathbf{J}_{-2}^{01})\cdot\frac{\mathbf{v}}{3v}\right] \tag{3.52}$$

$$\equiv \mathscr{H}_0 + \boldsymbol{\mathscr{H}}_1\cdot\frac{\mathbf{v}}{v} + \boldsymbol{\mathscr{H}}_2\cdot\frac{\mathbf{vv}}{v^2} + \ldots, \tag{3.53}$$

It can also be shown from (3.43) for $\mathscr{V}$ and (3.24) for $\mathscr{G}$ that

$$\begin{aligned}\mathscr{G}(v) = v\Bigg[&\frac{I_2^0}{3} + I_0^0 + \frac{J_{-1}^0}{3} + J_1^0 - \frac{I_2^{00}}{3} - I_0^{00} - \frac{J_{-1}^{00}}{3} - J_1^{00}\\ &+\left(\frac{\mathbf{I}_3^1}{15} - \frac{\mathbf{I}_1^1}{3} + \frac{\mathbf{J}_{-2}^1}{15} - \frac{\mathbf{J}_0^1}{3} - \frac{2\mathbf{I}_3^{01}}{15} + \frac{2\mathbf{I}_1^{01}}{3} - \frac{2\mathbf{J}_{-2}^{01}}{15} + \frac{2\mathbf{J}_0^{01}}{3}\right)\cdot\frac{\mathbf{v}}{v}\Bigg]\end{aligned} \tag{3.54}$$

$$\equiv \mathscr{G}_0 + \boldsymbol{\mathscr{G}}_1\cdot\frac{\mathbf{v}}{v} + \boldsymbol{\mathscr{G}}_2\cdot\frac{\mathbf{vv}}{v^2} + \ldots, \tag{3.55}$$

3.5. Linearization

In analogy with (3.40), f is expanded as

$$f = f_0 + \mathbf{f}_1\cdot\frac{\mathbf{v}}{v} + \frac{\mathbf{f}_2:\mathbf{vv}}{v^2}. \tag{3.56}$$

Using this and $\mathscr{G}$ and $\mathscr{H}$ from (3.52) and (3.54), the collision operator (3.27) is linearized to obtain expressions for $\delta f_0/\delta t$, and $\delta\mathbf{f}_1/\delta t$.

3.5.1. The $\delta f_0/\delta t$ relation. Equations (3.27), (3.37) and (3.38) yield

$$\frac{1}{Y}\frac{\delta f_0}{\delta t} = \frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right)_1 + \frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right)_2, \tag{3.57}$$

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where

$$\frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right)_1 = -(1-2f_0)\,\boldsymbol{\nabla}\mathscr{H}_0\cdot\boldsymbol{\nabla}\mathscr{H}_0 - f_0(1-f_0) - \boldsymbol{\nabla}f_0\,\boldsymbol{\nabla}f_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0$$
$$+\tfrac{1}{2}(1-2f_0)\,\boldsymbol{\nabla}\boldsymbol{\nabla}f_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0$$
$$+(1-2f_0)\,\boldsymbol{\nabla}f_0\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0)$$
$$+\tfrac{1}{2}f_0(1-f_0)\boldsymbol{\nabla}\boldsymbol{\nabla} : \boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0, \tag{3.58}$$

$$\frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right)_2 = \boldsymbol{\nabla}f_0\,\boldsymbol{\nabla}f_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0 + f_0\,\boldsymbol{\nabla}\boldsymbol{\nabla} : \boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0 + f_0\,\boldsymbol{\nabla}f_0\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0)$$
$$-\frac{1}{2}\frac{m}{M}[(1-2f_0)\,\boldsymbol{\nabla}f_0\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}_2\mathscr{G}_0) + f_0(1-f_0)\,\nabla^4{}_2\mathscr{G}_0]. \tag{3.59}$$

Now using the expressions for $\mathscr{H}_0$ from (3.52) and $\mathscr{G}_0$ from (3.54) gives†

$$\frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right)_1 = \frac{1}{3v^2}\frac{\partial}{\partial v}\left\{\frac{3m}{M}f_0(1-f_0)\,(I_0^0 - I_0^{00}) + v(I_2^0 - I_2^{00} + J_{-1}^0 - J_{-1}^{00})\frac{\partial}{\partial v}[f_0(1-f_0)]\right\} \tag{3.60}$$

$$\frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right)_2 = \frac{1}{3v^2}\frac{\partial}{\partial v}\left[v(I_2^0 - I_2^{00} + J_{-1}^0 - J_{-1}^{00})\frac{\partial f_0^2}{\partial v} + \frac{3m}{M}I_0^{00}f_0(1-f_0)\right]. \tag{3.61}$$

Therefore

$$\frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right) = \frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{3m}{M}f_0(1-f_0)\,I_0^0 + v(I_2^0 - I_2^{00} + J_{-1}^0)\frac{\partial f_0}{\partial v}\right]. \tag{3.62}$$

In equilibrium, the left-hand side of this equation is equal to zero. On the right-hand side, the i^{00} and J^{00} terms cannot be easily evaluated by analytical means. For the present we shall assume that the scattering is from non-degenerate ions, and that the I^{00} and J^{00} terms (which represent degeneracy) can be disregarded. It will also be assumed that the Lorentz approximation (that is, the ratio of the ion mass to the electron mass is very large, and the ion has negligible velocity) is valid. With these assumptions, (3.62) becomes

$$\frac{3m}{M}f_0(1-f_0) + v(I_2^0 + J_{-1}^0)\frac{\partial f_0}{\partial v} = K, \tag{3.63}$$

where K is a constant. Now assuming that the ions have a Maxwellian distribution function, and that the electron velocity is large, the I and J integrals can be evaluated using error functions (see e.g. Boas 1983). In this case

$$I_0^0 \to n_+,$$
$$I_2^0 \to \frac{3n_+ kT}{2mv^2},$$
$$J_{-1}^0 \to 0,$$

† The tensor products for the F_0 and F_1 equations are given in the appendix.

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where n_+ is the ion number density. Using these relations, (3.63) becomes

$$\frac{m}{kT} f_0(1-f_0) + \frac{1}{v}\frac{\partial f_0}{\partial v} = K. \tag{3.64}$$

This is satisfied provided that f_0 is a Fermi–Dirac distribution function. Therefore, in thermodynamic equilibrium, the collision operator produces a Fermi–Dirac distribution function as required.

3.5.2. The $\delta\mathbf{f}_1/\delta t$ relation. From (3.27) (3.37) and (3.38), the *total* collision operator is

$$\begin{aligned}\frac{1}{Y}\left(\frac{\delta \mathbf{f}_1}{\delta t}\right) &= 4\pi\frac{m}{M}[\mathbf{F}_1 f_0(1-f_0) + \mathbf{f}_1 F_0(1-2f_0)] \\ &+ \frac{m}{m+M}(\boldsymbol{\nabla}\mathbf{f}_1\cdot\boldsymbol{\nabla}\mathscr{H}_0) + \frac{M}{m+M}(\boldsymbol{\nabla} f_0\cdot\boldsymbol{\nabla}\mathscr{G}_1) \\ &- \frac{m}{m+M}[\boldsymbol{\nabla}\mathbf{f}_1\cdot\boldsymbol{\nabla}_1\mathscr{H}_0(1-2f) + \boldsymbol{\nabla} f_0\cdot\boldsymbol{\nabla}_1\mathscr{G}_1(1-sf_0) - 2\mathbf{f}_1\boldsymbol{\nabla} f_0\cdot\boldsymbol{\nabla}_1\mathscr{G}_1] \\ &+ \tfrac{1}{2}\boldsymbol{\nabla}\boldsymbol{\nabla}:\boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0 + \tfrac{1}{2}\boldsymbol{\nabla}\boldsymbol{\nabla} f_0:\boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{H}_1.\end{aligned} \tag{3.65}$$

Using the definitions of $\mathscr{H}_1$ and $\mathscr{G}_1$ from (3.52) and (3.54) gives

$$\begin{aligned}\frac{1}{Y}\frac{\delta \mathbf{f}_1}{\delta t} &= \frac{1}{3v}\frac{\partial^2 \mathbf{f}_1}{\partial v^2}\{I_2^0 - I_2^{00} + J_{-1}^0 - J_{-1}^{00}\} \\ &+ \frac{1}{3v^2}\frac{\partial \mathbf{f}_1}{\partial v}\left[\frac{3m}{M}(1-2f_0)I_0^0 - I_2^0 + I_2^{00} + 2J_{-1}^0 - 2J_{-1}^{00}\right] \\ &+ \frac{1}{3v^3}\mathbf{f}_1(-3I_0^0 + 3I_0^{00} + I_2^0 - I_2^{00} + 2J_{-1}^0 - 2J_{-1}^{00}) \\ &+ 4\pi\frac{m}{M}[\mathbf{F}_1 f_0(1-f_0) + \mathbf{f}_1 F_0(1-2f_0)] \\ &+ \frac{1}{5v}\frac{\partial^2 f_0}{\partial v^2}(\mathbf{I}_3^1 - 2\mathbf{I}_3^{01} + \mathbf{J}_{-2}^1 - 2\mathbf{J}_{-2}^{01}) \\ &+ \frac{1}{15v}\frac{\partial f_0}{\partial v}\left\{-3\mathbf{I}_3^1 + 6\mathbf{I}_3^{01} + \left[7 - 5(1-2f_0)\frac{m}{M}\right]\mathbf{J}_{-2}^1\right. \\ &+ \left.\left[-5 + 10\frac{m}{M}(1-2f_0)\right]\mathbf{I}_1^1 + 10\mathbf{I}^{01} - 14\mathbf{J}_{-2}^{01}\right\} \\ &- 2\frac{m}{Mv^2}\mathbf{f}_1 I_0^0\frac{\partial f_0}{\partial v}.\end{aligned} \tag{3.66}$$

If the above equation is applied to electron–ion collisions then it can be assumed first that the Lorentz approximation is valid, secondly that F_0 is Maxwellian,

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and thirdly that F is isotropic (i.e. that $\mathbf{F}_1 = 0$). Then (3.66) becomes

$$\frac{1}{Y}\frac{\partial \mathbf{f}_1}{\partial t} = \frac{\mathbf{f}_1 n_+}{3v^3}. \tag{3.67}$$

This is the same equation as that obtained from (3.1) if τ is calculated using the Rutherford scattering formula (see Lee and More 1984). Therefore (3.1) is suitable for electron–ion scattering, but unsuitable for electron–electron scattering.

To calculate the electron–electron collision operator in the above equation, the following substitutions have to be made: $m = M$, $f_0 = F_0$ and $Z = Z' = 1$. The **I** and **J** integrals also need to be redefined as follows:

$$\mathbf{I}_p^l \equiv \frac{4\pi}{v^p}\int_0^v \mathbf{f}_l\, v^{p+2}\, dv, \qquad \mathbf{J}_p^l \equiv \frac{4\pi}{v^p}\int_v^\infty \mathbf{f}_l\, v^{p+2}\, dv, \tag{3.68}$$

$$\mathbf{I}_p^{lm} \equiv \frac{4\pi}{v^p}\int_0^v \mathbf{f}_l \mathbf{f}_m\, v^{p+2}\, dv, \qquad \mathbf{j}_p^{lm} \equiv \frac{4\pi}{v^p}\int_v^\infty \mathbf{f}_l \mathbf{f}_m\, v^{p+2}\, dv. \tag{3.69}$$

3.6. Conservation equations

For (3.57) and (3.66) to be valid, it is necessary to show that they satisfy the conservation laws of number density, momentum and energy.

If we expand $f = f_0 + \mathbf{v}\cdot\mathbf{f}_1/v + \ldots$ then the average of a scalar $\phi(v)$ is

$$\bar{\phi} = \frac{4\pi\gamma}{3n}\int \phi f_0\, v^2\, dv \tag{3.70}$$

where $\gamma = 2m^3/h^3$ is the normalization due to (3.5).

The average of a vector $\mathbf{A}(v) \equiv A\mathbf{v}/v$ is

$$\bar{\mathbf{A}} = \frac{4\pi\gamma}{3n}\int \mathbf{f}_1 A v^2 = dv. \tag{3.71}$$

Setting $\phi = 1$ in (3.70), using $\delta f_0/\delta t$ from (3.62), and integrating over all v gives

$$\int \frac{\delta f_0}{\delta t} d^3v = 0 \tag{3.72}$$

i.e. the particle number density is conserved. Now setting $\phi = \frac{1}{2}mv^2$ in (3.70) and using (3.62) results in:

$$\frac{1}{Y}\int \frac{1}{2}mv^2 \frac{\partial f}{\partial t} d^3v = (4\pi)^2 m \int_0^\infty F_0(1-F_0)\, V\, dV \int_0^V f_0\, v^2\, dv$$
$$-(4\pi)^2 \frac{m^2}{M}\int_0^\infty f_0(1-f_0)\, v\, dv \int_0^v F_0\, V^2\, dV. \tag{3.73}$$

It can be seen from this, first that energy is conserved in collisions between like particles, and, secondly, that collisions between particles a and b give rise to the opposite effect as those between b and a, since $M^2 Y_{ab} = m^2 Y_{ba}$. Therefore energy is correctly conserved.

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Conservation of momentum can be demonstrated by setting $\phi = mv$ in (3.71) and using (3.66):

$$\frac{1}{Y}\int m\mathbf{v}\frac{\delta f}{\delta t}d^3v = \frac{(4\pi)^2}{3}m\left(1+\frac{m}{M}\right)\int\left(\mathbf{F}_1\,dV\int_0^V f_0\,v^2\,dv - \mathbf{f}_1\,dv\int_0^v F_0\,V^2\,dV\right)$$
$$+\frac{(4\pi)^2\,m}{3}\left(\int_0^\infty F_0 F_0\,V^2\,dV\int_0^V \mathbf{f}_1\,dv\right.$$
$$\left.-\frac{m}{M}\int_0^\infty f_0 f_0\,v^2\,dv\int_0^v \mathbf{F}_1\,dV\right)$$
$$-2\frac{(4\pi)^2\,m}{3}\left(\int_0^\infty F_0\,\mathbf{F}_1\,dV\int_0^V f_0\,v^2\,dv\right.$$
$$\left.-\frac{m}{M}\int_0^\infty f_0\,\mathbf{f}_1\,dv\int_0^v F_0\,V^2\,dV\right). \tag{3.74}$$

Here it can be seen that there is no net transfer of momentum between like particles, and collisions between particles a and b produce the opposite effect as those between b and a. Therefore we see that momentum is also conserved.

Equation (3.62) is in a similar form to the Braginskii collision operator, which is already used in Fokker–Planck numerical codes (see e.g. Epperlein *et al.* 1988). Therefore it should be relatively simple to adapt such codes so that they correctly model degeneracy using the above operator.

Equation (3.66) is a second-order integro-differential operator acting upon $\mathbf{f}_1$. Using the numerical methods developed by Epperlein and Haines (1986), this equation can be used to develop a full set of fluid transport coefficients. This will be the subject of a future paper.

Appendix

A. 1. Zeroth-order terms

To calculate the f_0 equation, the following identities are needed:

$$\boldsymbol{\nabla}\phi(v) = \frac{1}{v}\frac{\partial\phi}{\partial v}\mathbf{v}, \qquad \boldsymbol{\nabla}[\phi(v)\,\mathbf{v}] = \frac{1}{v} = \mathbf{v}\mathbf{v} + \mathbf{I}_2\,\phi, \tag{A 1}$$

where ϕ is a scalar function of magnitude v, and $\mathbf{I}_2$ is the unit tensor. We then use (3.52) and (3.54) to obtain

$$\boldsymbol{\nabla} f_0 = \frac{1}{v}\frac{\partial f_0}{\partial v}\mathbf{v}, \tag{A 2}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla} f_0 = \left(\frac{1}{v^2}\frac{\partial^2 f_0}{\partial v^2} - \frac{1}{v^3}\frac{\partial f_0}{\partial v}\right) + \frac{1}{v}\frac{\partial f_0}{\partial v}\mathbf{I}_2, \tag{A 3}$$

$$\mathscr{H}_0 = \frac{(M+m)\,(I_0^0 + J_{-1}^0 - I_0^{00} - J_{-1}^{00})}{Mv}, \tag{A 4}$$

$$\boldsymbol{\nabla}\mathscr{H}_0 = -\frac{M+m}{Mv^3}(I_0^0 - I_0^{00})\,\mathbf{v}, \tag{A 5}$$

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$$\boldsymbol{\nabla}_2 \mathscr{H}_0 = -\frac{M+m}{Mv^3} I_0^{00}\,\mathbf{v}, \tag{A 6}$$

$$\mathscr{G}_0 = v[\tfrac{1}{3}(I_2^0 - I_2^{00} + J_{-1}^0 - J_{-1}^{00}) + I_0^0 - I_0^{00} + J_1^0 - J_1^{00}], \tag{A 7}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0 = \frac{1}{v^3}(I_2^0 - I_2^{00} - I_0^0 + I_0^{00})\,\mathbf{vv} + \frac{1}{v}(-I_2^0 + I_2^{00} + 2J_{-1}^0 - 2J_{-1}^{00} + 3I_0^0 - 3I_0^{00})\,\mathbf{I}_2. \tag{A 8}$$

The products required in the f_0 equation are

$$\mathbf{Av}:\mathbf{Bv} = (\mathbf{A}\cdot\mathbf{v})\,(\mathbf{B}\cdot\mathbf{v}), \qquad \mathbf{Av}:\mathbf{I}_2 = \mathbf{A}\cdot\mathbf{v}, \qquad \mathbf{I}_2:\mathbf{I}_2 = 3. \tag{A 9}$$

The products are as follows:

$$\boldsymbol{\nabla} f_0 \cdot \mathscr{H}_0 = -\frac{M+m}{Mv^2}(I_0^0 - I_0^{00})\frac{\partial f_0}{\partial v}, \tag{A 10}$$

$$\boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla}_2 \mathscr{H}_0 = -\frac{1}{v^2}\frac{M+m}{M}\frac{\partial f_0}{\partial v} I_0^{00}, \tag{A 11}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla} f_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0 = \frac{2}{3}\frac{1}{v}\frac{\partial^2 f}{\partial v^2}(I_2^0 - I_2^{00} + J_{-1}^0 + J_{-1}^0 - J_{-1}^{00}) + \frac{1}{v^2}\frac{\partial f_0}{\partial v}(-\tfrac{2}{3}I_2^0 + \tfrac{2}{3}I_2^{00} + \tfrac{4}{3}J_{-1}^0 - 4J_{-1}^{00} + 2I_0^0 - 2I_0^{00}), \tag{A 12}$$

$$\boldsymbol{\nabla} f_0\,\boldsymbol{\nabla} f_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}_0 = \frac{2}{3v}(I_2^0 - I_2^{00} + J_{-1}^0 - J_{-1}^{00})\frac{\partial f_0}{\partial v}\frac{\partial f}{\partial v}, \tag{A 13}$$

$$f_0\,\boldsymbol{\nabla} f_0 \cdot (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\mathscr{G}) = -\frac{2f_0}{v_2}(I_0^0 - I_0^{00})\frac{\partial f_0}{\partial v}. \tag{A 14}$$

A. 2. First-order terms

To calculate the $\mathbf{f}_1$ equation, the following identities are employed:

$$\left.\begin{aligned}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{v}) &= \frac{1}{v}\frac{\partial \mathbf{A}}{\partial v}\cdot\mathbf{vv} + \mathbf{A},\\ \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{vv}) &= \frac{1}{v}\frac{\partial \mathbf{A}}{\partial v}\cdot\mathbf{vvv} + \mathbf{Av} + \mathbf{I}_2\,\mathbf{A}\cdot\mathbf{v},\end{aligned}\right\} \tag{A 15}$$

$$\boldsymbol{\nabla}\mathbf{A} = \frac{1}{v}\mathbf{v}\frac{\partial \mathbf{A}}{\partial v}, \tag{A 16}$$

and using (3.52) and (3.54) gives

$$\boldsymbol{\nabla}\left(\frac{\mathbf{f}_1\cdot\mathbf{v}}{v}\right) = \left(\frac{\partial \mathbf{f}_1}{\partial v} - \frac{\mathbf{f}_1}{v}\right)\cdot\frac{\mathbf{vv}}{v^2} + \frac{\mathbf{f}_1}{v}, \tag{A 17}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\left(\frac{\mathbf{f}_1\cdot v}{v}\right) = \left(\frac{\partial^2 \mathbf{f}_1}{\partial v^2} - \frac{3}{v}\frac{\partial \mathbf{f}_1}{\partial v} + \frac{3}{v^2}\mathbf{f}_1\right)\cdot\frac{\mathbf{vvv}}{v^3} + \left(\frac{\partial \mathbf{f}_1}{\partial v} - \frac{\mathbf{f}_1}{v}\right)\frac{\mathbf{v}}{v^2} + \frac{\mathbf{v}}{v^2}\left(\frac{\partial \mathbf{f}_1}{\partial v} - \frac{\mathbf{f}_1}{v}\right) + \mathbf{I}_2\left(\frac{\partial \mathbf{f}_1}{\partial v} - \frac{\mathbf{f}_1}{v}\right)\cdot\frac{\mathbf{v}}{v^2}, \tag{A 18}$$

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$$\boldsymbol{\nabla}\left(\frac{{}_1\mathscr{H}_1\cdot\mathbf{v}}{v}\right)=\frac{M+m}{Mv^2}\left(-\frac{\mathbf{I}^1_1\cdot\mathbf{vv}}{v^2}+\frac{\mathbf{I}^1_1+\mathbf{J}^1_{-2}}{3}\right), \tag{A 19}$$

$$\boldsymbol{\nabla}_1\mathscr{H}_0=-\frac{(M+m)\,I^0_0\,\mathbf{v}}{Mv^3}, \tag{A 20}$$

$$\begin{aligned}\boldsymbol{\nabla\nabla}\left(\frac{\mathscr{G}_1\cdot\mathbf{v}}{v}\right)=&\frac{(\mathbf{I}^1_3-\mathbf{I}^{01}_3-\mathbf{I}^1_1+\mathbf{I}^{01}_1)\cdot\mathbf{vvv}}{v^4}\\&+\frac{(-3\mathbf{I}^1_3+3\mathbf{I}^{01}_3+2\mathbf{J}^1_{-2}-2\mathbf{J}^{01}_{-2}+5\mathbf{I}^1_1-5\mathbf{I}^{01}_1)\,\mathbf{v}}{15v^2}\\&+\frac{\mathbf{v}(-3\mathbf{I}^1_3+3\mathbf{I}^{01}_3+2\mathbf{J}^1_{-2}-2\mathbf{J}^{01}_{-2}+5\mathbf{I}^1_1-5\mathbf{I}^{01}_1)}{15v^2}\\&+\frac{\mathbf{I}_2(-3\mathbf{I}^1_3+3\mathbf{I}^{01}_3+2\mathbf{J}^1_{-2}-2\mathbf{J}^{01}_{-2}+5\mathbf{I}^1_1-5\mathbf{I}^{01}_1)\cdot\mathbf{v}}{15v^2}.\end{aligned} \tag{A 21}$$

Using these relationships as well as those given in the previous section, we obtain

$$\boldsymbol{\nabla}_1\mathscr{H}_0\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{f}_1\cdot\mathbf{v}}{v}\right)=-\frac{M+m}{mv^2}I^0_0\frac{\partial\mathbf{f}_1}{\partial v}\cdot\frac{\mathbf{v}}{v}, \tag{A 22}$$

$$\boldsymbol{\nabla}\left(\frac{{}_1\mathscr{H}_1\cdot\mathbf{v}}{v}\right)f_0=\frac{M+m}{Mv^2}\frac{\partial f_0}{\partial v}(-\tfrac{2}{3}\mathbf{I}^1_1+\tfrac{1}{3}\mathbf{J}^1_{-2})\cdot\frac{\mathbf{v}}{v} \tag{A 23}$$

$$\begin{aligned}\tfrac{1}{2}\boldsymbol{\nabla\nabla}\left(\frac{\mathbf{f}_1\cdot\mathbf{v}}{v}\right):\boldsymbol{\nabla\nabla}\mathscr{G}_0=&\frac{1}{3v}\frac{\partial^2\mathbf{f}_1}{\partial v^2}(I^0_2-I^{00}_2+J^0_{-1}-J^{00}_{-1})\cdot\frac{\mathbf{v}}{v}\\&+\frac{1}{3v^2}(3I^0_0-3I^{00}_0-I^0_2+I^{00}_2+2J^0_{-1}-2J^{00}_{-1})\times\left(\frac{\partial\mathbf{f}_1}{\partial v}-\frac{\mathbf{f}_1}{v}\right)\cdot\frac{\mathbf{v}}{v},\end{aligned} \tag{A 24}$$

$$\begin{aligned}\tfrac{1}{2}\boldsymbol{\nabla\nabla}\left(\frac{\mathscr{G}_1\cdot\mathbf{v}}{v}\right)=&\frac{1}{5v}\frac{\partial^2 f_0}{\partial v^2}(\mathbf{I}^1_3-\mathbf{I}^{01}_3+\mathbf{J}^1_{-2}-\mathbf{J}^{01}_{-2})\cdot\frac{\mathbf{v}}{v}\\&\frac{1}{15v^2}\frac{\partial f_0}{\partial v}(-3\mathbf{I}^1_3+3\mathbf{I}^{01}_3+2\mathbf{J}^1_{-2}-2\mathbf{J}^{01}_{-2}+5\mathbf{I}^1_1-5\mathbf{I}^{01}_1)\cdot\frac{\mathbf{v}}{v}.\end{aligned} \tag{A 25}$$

References

Allis, W. P. 1956 *Handbuch der Physik*, Vol. 4, Part 21, p. 383. Springer-Verlag, Berlin.

Boas, M. L. 1983 *Mathematical Methods in the Physical Sciences*, 2nd edn. Wiley, New York.

Braginskii, S. I. 1958 *Soviet Phys. JETP* **6**, 358.

Brysk, H., Campell, P. M. and Hammerling, P. 1975 *Plasma Phys.* **17**, 473.

Chapman, S. and Cowling, T. G. 1939 *Mathematical Theory of Non-Uniform Gases*. Cambridge University Press.

Chittenden, J. P. and Haines, M. G. 1990 *Phys. Fluids* **B2**, 1889.

Clayton, D. D. 1968 *Principles of Stellar Evolution and Nucleosynthesis*. University of Chicago Press.

Debye, P. and Hückel, E. 1923 *Phys. Z.* **24**, 190.

Epperlin, E. M. and Haines, M. G. 1986 *Phys. Fluids* **29**, 1029.

https://doi.org/10.1017/S0022377897006041 Published online by Cambridge University Press

Epperlein, E. M., Rickard, G. J. and Bell, A. R. 1988 *Comput. Phys. Commun.* **52**, 7.

Haines, M. G. 1989 *Plasma Phys. Contr. Fusion* **31**, 759.

Hubbard, W. B. and Lampe, M. 1969 *Astrophys. J. Suppl.* **163**, 297.

Hugill, J. 1981 *Plasma Physics and Nuclear Fusion Research*. Academic Press, New York.

Johnston, T. W. 1960 *Phys. Rev.***120**, 1103.

Lampe, M. 1967 *Phys. Rev.* **170**, 306.

Lee, Y. T. and More, R. M. 1984 *Phys. Fluids* **27**, 1273.

Liboff, R. L. 1990 *Kinetic Theory*. Prentice-Hall, Englewood Cliffs, NJ.

Macdonald, W. M., Rosenbluth, M. N. and Chuck, W. 1957 *Phys. Rev.* **107**, 350.

Mestel, L. 1950 *Proc. Camb. Phil. Soc.* **46**, 331.

Montgomery, D. 1976 *Physica* **82C**, 111.

Robinson, B. B. and Bernstein, I. B. 1962 *Ann. Phys. (NY)* **18**, 110.

Shkarovsky, I. P. 1963 *Can. J. Phys.* **41**, 1743.

Shkarofsky, I. P., Johnston, T. W. and Bachinskii, M. P. 1966 *The Particle Kinetics of Plasmas*. Addison-Wesley, Reading, MA.

Sivukhin, D. V. 1966 *Reviews of Plasma Physics*, vol. 4 (ed. M. A. Leontovich), p. 109. Consultants Bureau, New York.

Spitzer, L. 1962 *Physics of Fully Ionised Gases*, 2nd edn. Interscience, New York.

Ziman, J. M. 1961 *Phil. Mag.* **6**, 1013.

https://doi.org/10.1017/S0022377897006041 Published online by Cambridge University Press