INTEGRAL MODULAR FORMS AND SUMMATION FORMULAE

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Communicated by P. Hall

Received 30 May 1946

The purpose of this note is to point out that certain general summation formulae associated with integral modular forms follow immediately on combining results proved by Rankin, Wilton, and myself.

I have previously proved the following result*:

Lemma α. *If* (i) *there is a sequence of real numbers* $\{a_n\}$ *and a positive* β *such that*

$$\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is convergent for sufficiently large $R(s)$, *and can be continued analytically into the region* $R(s) \geqslant \frac{1}{2}\beta$,

(ii) *as* x *tends to infinity*

$$\sum_{1 \leqslant n \leqslant x} a_n = O(x^{\eta}),$$

where

$$\eta < \tfrac{1}{2}\beta,$$

(iii) $f(x)$ *is an integral, and both* $f(x)$ *and* $xf'(x)$ *belong to* $L^2(0, \infty)$, *then*

$$\sum_{n=1}^{\infty} a_n n^{\frac{1}{4}-\frac{1}{2}\beta} f(n) = \sum_{n=1}^{\infty} a_n n^{\frac{1}{4}-\frac{1}{2}\beta} g(n),$$

where

$$\int_0^x g(y)\,dy = \int_0^{\infty} f(y)\,\chi(xy)\frac{dy}{y}, \tag{1}$$

$$\chi(x) = \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-T}^{T} \frac{\psi(\frac{1}{2}\beta - it)}{\psi(\frac{1}{2}\beta + it)} \frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it}\,dt, \tag{2}$$

and $g(x)$ *is chosen so that it is the integral of its derivative.*

If $\tau(n)$ is Ramanujan's arithmetical function, defined by

$$x\left\{\prod_{n=1}^{\infty}(1-x^n)\right\}^{24} = \sum_{n=1}^{\infty} \tau(n)\,x^n \quad (|x|<1),$$

then Rankin† has proved that as x tends to infinity

$$\sum_{1 \leqslant n \leqslant x} \tau(n) = O(x^{59/10}). \tag{3}$$

Also, Wilton‡ has shown that

$$\phi(s) = \sum_{n=1}^{\infty} \tau(n)\,n^{-s} \quad \{R(s) > \tfrac{13}{2}\}$$

has an analytic continuation for all s which satisfies the functional equation

$$(2\pi)^{-s}\,\Gamma(s)\,\phi(s) = (2\pi)^{s-12}\,\Gamma(12-s)\,\phi(12-s). \tag{4}$$

* A. P. Guinand, *Quart J. Math.* 9 (1938), 53–67, Theorem 2 with $R_0(x) = 0$. $A(1-s)$ should be substituted for $A(s)$ in the formula (4·1).

† R. A. Rankin, *Proc. Cambridge Phil. Soc.* 36 (1940), 150–1.

‡ J. R. Wilton, *Proc. Cambridge Phil. Soc.* 25 (1928), 121–9.

https://doi.org/10.1017/S0305004100023276 Published online by Cambridge University Press

Hence the conditions of Lemma α are satisfied with $a_n = \tau(n), \beta = 12$, and (2) becomes

$$\chi(x) = \text{l.i.m.}_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{\Gamma(6+it)}{\Gamma(6-it)} \frac{(2\pi)^{-2it}x^{\frac{1}{2}-it}}{\frac{1}{2}-it}\,dt. \tag{5}$$

Now the Mellin transform of $J_{11}(x)$ is*

$$\frac{2^{s-1}\Gamma(\frac{11}{2}+\frac{1}{2}s)}{\Gamma(\frac{13}{2}-\frac{1}{2}s)}.$$

Hence
$$J_{11}(y) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{\Gamma(\frac{11}{2}+\frac{1}{2}s)}{\Gamma(\frac{13}{2}-\frac{1}{2}s)} 2^{s-1}y^{-s}\,ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma(6+it)}{\Gamma(6-it)}\left(\frac{y}{2}\right)^{-1-2it} dt.$$

Putting $y = 4\pi x^{\frac{1}{2}}$ and integrating with respect to x we find that (5) becomes

$$\chi(x) = 2\pi\int_0^x J_{11}(4\pi t^{\frac{1}{2}})\,dt.$$

For such $\chi(x)$ a simpler form of the transformation formula (1) holds. It is:

LEMMA β†. *If $f(x)$ is an integral, tends to zero at infinity, and $xf'(x)$ belongs to $L^2(0,\infty)$, then for $x>0$, $R(\nu)>-1$*

$$g(x) = 2\pi\int_0^{\to\infty} f(t)\,J_\nu(4\pi x^{\frac{1}{2}}t^{\frac{1}{2}})\,dt$$

converges, and
$$f(x) = 2\pi\int_0^{\to\infty} g(t)\,J_\nu(4\pi x^{\frac{1}{2}}t^{\frac{1}{2}})\,dt.$$

Further, $g(x)$ is also an integral, tends to zero at infinity, and $f(x)$, $g(x)$, and $xg'(x)$ all belong to $L^2(0,\infty)$.

Hence we have:

THEOREM 1. *If $f(x)$ is an integral, tends to zero at infinity, and $xf'(x)$ belongs to $L^2(0,\infty)$ then*

$$\sum_{n=1}^{\infty}\tau(n)\,n^{-11/2}f(n) = \sum_{n=1}^{\infty}\tau(n)\,n^{-11/2}g(n),$$

where
$$g(x) = 2\pi\int_0^{\to\infty} f(t)\,J_{11}(4\pi x^{\frac{1}{2}}t^{\frac{1}{2}})\,dt.$$

Now Lemma α can readily be extended to cover unsymmetrical summation formulae, Rankin‡ has proved an extension of (3) corresponding to more general integral modular forms, and Wilton's proof of (4) can be extended to cover the Dirichlet's series associated with these forms. Combining these results as in Theorem 1 we have:

* E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), 196.

† A. P. Guinand, *Annals of Math.* 42 (1941), 591–603, Lemma 4. The present lemma is proved in the same way. We use the notation

$$\int_a^{\to\infty} = \lim_{T\to\infty}\int_a^T.$$

‡ Loc. cit.

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THEOREM 2. *If $f(x)$ satisfies the conditions of Theorem* 1, *and $H(\tau)$ is an integral modular form of dimensions $-k$ $(k>0)$ and 'Stufe' N which vanishes at all the rational cusps of the fundamental region, and*

$$H(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau/N},$$

$$\tau^k H(\tau) = \sum_{n=1}^{\infty} b_n e^{-2\pi i n/N\tau}$$

where both series are absolutely convergent for $I(\tau)>0$, then

$$\sum_{n=1}^{\infty} a_n n^{\frac{1}{2}-\frac{1}{2}k} f(n) = (-i)^k \sum_{n=1}^{\infty} b_n n^{\frac{1}{2}-\frac{1}{2}k} g(n),$$

where

$$g(x) = \frac{2\pi}{N}\int_0^{\to\infty} f(t)\, J_{k-1}\left(\frac{4\pi}{N} x^{\frac{1}{2}} t^{\frac{1}{2}}\right) dt.$$

Further, it can be shown* that the functions

$$F(x) = x^{-\frac{1}{2}k-\frac{1}{4}} \sum_{1\leqslant n\leqslant x}{}' a_n,$$

$$G(x) = (-i)^k x^{-\frac{1}{2}k-\frac{1}{4}} \sum_{1\leqslant n\leqslant x}{}' b_n$$

belong to $L^2(0,\infty)$ and are a pair of transforms with respect to the kernel

$$\frac{2\pi}{N} J_{k+1}\left(\frac{4\pi}{N} x^{\frac{1}{2}}\right)$$

By using a general theorem due to Titchmarsh on the Hankel transforms of functions of $L^2(0,\infty)$ which are of bounded variation it follows that

$$F(x) = \frac{2\pi}{N}\int_0^{\to\infty} G(t)\, J_{k+1}\left(\frac{4\pi}{N} x^{\frac{1}{2}} t^{\frac{1}{2}}\right) dt,$$

and from this result we can deduce†:

THEOREM 3. *If $H(\tau)$ satisfies the conditions of Theorem* 2 *then*

$$\sum_{1\leqslant n\leqslant x}{}' a_n = (-i)^k \sum_{n=1}^{\infty} b_n \left(\frac{x}{n}\right)^{\frac{1}{2}k} J_k\left(\frac{4\pi}{N} n^{\frac{1}{2}} x^{\frac{1}{2}}\right) \tag{6}$$

The special case of Theorem 3 with $a_n = b_n = \tau(n)$ was proved by Hardy‡. Formulae obtainable by integrating (6) term by term were proved by Walfisz§.

* A. P. Guinand, *Quart. J. Math.* loc. cit. Theorem 1. The dashes indicate that the terms $n = x$ are to be halved if x is an integer.

† See E. C. Titchmarsh, loc. cit. pp. 265–7, where the method is applied to a similar problem.

‡ G. H. Hardy, *Proc. Cambridge Phil. Soc.* 34 (1938), 309–15.

§ A. Walfisz, *Math. Ann.* 108 (1933), 75–90.

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