A theorem on transitive groups. By GARRETT BIRKHOFF*, Trinity College. (Communicated by Mr F. P. WHITE.)

[*Received* 13 February, *read* 6 March 1933.]

Let $\mathfrak{G}$ be any transitive permutation group on the n symbols $1, \ldots, n$. Let $\mathfrak{H}_i$ be the subgroup of $\mathfrak{G}$ whose elements leave i fixed. Let $\mathfrak{G}'$ be the *normalizer* of $\mathfrak{G}$, i.e., the subgroup of the symmetric group $\mathfrak{S}_n$ on $1, \ldots, n$ transforming $\mathfrak{G}$ into itself. Let G', G_1', G_2', etc., denote elements of $\mathfrak{G}'$. Finally, let $\mathfrak{G}''$ be the *centralizer* of $\mathfrak{G}$, i.e., the subgroup in $\mathfrak{S}_n$ transforming every element of $\mathfrak{G}$ into itself.

$(G')^{-1}\mathfrak{G}G'$ is an automorphism of $\mathfrak{G}$, and $(G_1'G_2')^{-1}\mathfrak{G}(G_1'G_2')$ is identical with $G_2'^{-1}(G_1'^{-1}\mathfrak{G}G_1')G_2'$. Hence the elements of $\mathfrak{G}'$ are isomorphic† with a subgroup of the automorphisms of $\mathfrak{G}$. But $\mathfrak{G}''$ is by definition the set of elements of $\mathfrak{G}'$ isomorphic with the automorphism identity; hence $\mathfrak{G}'/\mathfrak{G}''$ is (1, 1) *isomorphic* with a subgroup of the automorphisms of $\mathfrak{G}$.

Suppose that H^i is an element of $\mathfrak{H}_i$ as defined above, and let $t(i)$ be the transform of i under G'. Then $G'^{-1}H^iG'$ permutes

$$t(i) \to i \to i \to t(i);$$

hence $G'^{-1}\mathfrak{H}_iG' = \mathfrak{H}_{t(i)}$. We have therefore

THEOREM 1: *$\mathfrak{G}'/\mathfrak{G}''$ is* (1, 1) *isomorphic with the subgroup of those automorphisms of $\mathfrak{G}$ which* (1) *permute the subgroups $\mathfrak{H}_i$ among themselves, and* (2) *can be obtained through transformations of the form $G'^{-1}\mathfrak{G}G'$.*

Since $\mathfrak{G}$ is transitive, we can assign to each symbol i a T_i of $\mathfrak{G}$ permuting $1 \to i$. Evidently $T_i\mathfrak{H}_iT_i^{-1} = \mathfrak{H}_1$. Moreover $\mathfrak{H}_1T_i$ is the subset permuting $1 \to i$; consequently (*a*) G takes $i \to j$ if, and only if, $\mathfrak{H}_1T_iG = \mathfrak{H}_1T_j$, and (*b*) if G permutes $1 \to i$, G has a *single* representation HT_k ($H \subset \mathfrak{H}_1$), namely, $k = i$ and $H = GT_i^{-1}$.

Let now $\alpha : G \to \alpha(G)$ be any automorphism of $\mathfrak{G}$ such that $\alpha(\mathfrak{H}_1) = \mathfrak{H}_\tau$. If $H \subset \mathfrak{H}_1$, $H' \equiv T_\tau\alpha(H)T_\tau^{-1}$ takes $1 \to \tau \to \tau \to 1$. Therefore $H' \subset \mathfrak{H}_1$, and, representing an arbitrary $G \subset \mathfrak{G}$ by HT_i as in (*b*), we have

$$T_\tau\alpha(HT_i)T_\tau^{-1} = T_\tau\alpha(H)T_\tau^{-1}T_\tau\alpha(T_i)T_\tau^{-1} = H'H_iT_{\beta(i)} \subset \mathfrak{H}T_{\beta(i)},$$

where $\beta(i)$ depends only on i and α.

That is, the automorphism $\beta : G \to \beta(G) = T_\tau\alpha(G)T_\tau^{-1}$ transforms $\mathfrak{H}_1T_i \to \mathfrak{H}_1T_{\beta(i)}$. But since β is a permutation of the

* Henry Fellow. The theorem was kindly suggested to me by Mr P. Hall of King's College.

† Isomorphic means *isomorph* in the sense of Speiser.

https://doi.org/10.1017/S0305004100011051 Published online by Cambridge University Press

elements of $\mathfrak{G}$, it must have an inverse; therefore $i \to \beta(i)$ must also have an inverse, and hence must be a permutation P on the n symbols.

Let us compare $P^{-1}GP$ with $T_\tau \alpha(G)\, T_\tau^{-1}$. $P^{-1}GP$ takes an arbitrary symbol $\beta(k) \to k \to G(k) \to \beta\{G(K)\}$. But, by (a),

$$\mathfrak{H}_1 T_k G = \mathfrak{H}_1 T_{G(k)};$$

hence $\mathfrak{H}_1 T_{\beta(k)} \beta(G) = \beta(\mathfrak{H}_1 T_k)\, \beta(G) = \beta(\mathfrak{H}_1 T_k G)$

$$= \beta(\mathfrak{H}_1 T_{G(k)}) = \mathfrak{H}_1 T_{\beta\{G(k)\}}.$$

Or, again by (a), $\beta(G)$ also takes $\beta(k) \to \beta\{G(k)\}$. Therefore $P^{-1}GP$ is identical with

$$\beta(G) = T_\tau \alpha(G)\, T_\tau^{-1}, \text{ and } (PT_\tau)^{-1}\mathfrak{G}(PT_\tau) = \alpha(\mathfrak{G}).$$

Thus we have

THEOREM 2: *Any automorphism $\alpha(\mathfrak{G})$ which permutes $\mathfrak{H}_1 \to \mathfrak{H}_\tau$ can be obtained by an automorphism of the form $G'^{-1}\mathfrak{G}G'$.*

Let G'' be any element of $\mathfrak{G}''$; suppose that G'' permutes $1 \to \alpha$. Since G'' commutes with each T_k, $G'' = T_k^{-1} G'' T_k$ must permute each $k \to 1 \to \alpha \to k(\alpha)$, where, by (a), $\mathfrak{H}_1 T_{k(\alpha)} = \mathfrak{H}_1 T_\alpha T_k$. Therefore α completely defines G''. But since G'' commutes with every HT_k $(H \subset \mathfrak{H}_1)$, and G'' permutes k in just one way, every HT_k [fixed k] must permute α in the same way; conversely, if every HT_k permutes α in the same way, we can define $G_\alpha'' \subset \mathfrak{S}_n$ permuting $k \to k(\alpha)$.

But, by (a), HT_k permutes α into the symbol $\alpha(k)$ defined by that coset $\mathfrak{H}_1 T_{\alpha(k)} = \mathfrak{H}_1 T_\alpha H T_k = \mathfrak{H}_1 (T_\alpha H T_\alpha^{-1})\, T_\alpha T_k$ of $\mathfrak{H}_1$ whose elements permute $1 \to \alpha(k)$. This is the same for all H if, and only if, $(T_\alpha H T_\alpha^{-1})$ is in the same coset of $\mathfrak{H}$ for all H. But the permutation identity is in $\mathfrak{H}_1$, and it is transformed into itself under $T_\alpha \mathfrak{G} T_\alpha^{-1}$; hence this coset must be $\mathfrak{H}_1$ itself. Thus a necessary and sufficient condition that α should define a permutation is that $T_\alpha^{-1} \mathfrak{H}_1 T_\alpha = \mathfrak{H}_1$; i.e., that $\mathfrak{H}_1 T_\alpha$ should belong to $\mathfrak{N}_1$, the normalizer of $\mathfrak{H}_1$ in $\mathfrak{G}$.

Moreover under this restriction on α, the permutation $k \to k(\alpha)$ is in $\mathfrak{G}''$. For $G_\alpha''^{-1} G G_\alpha''$ takes $k(\alpha) \to k \to G(k) \to \phi$ according as

$$\mathfrak{H}_1 T_\phi = \mathfrak{H}_1 T_\alpha T_{G(k)} \subset \mathfrak{H}_1 T_\alpha \mathfrak{H}_1 T_k G = \mathfrak{H}_1 T_\alpha T_k G = \mathfrak{H}_1 T_{k(\alpha)} G.$$

Hence, by (a), $G_\alpha''^{-1} G G_\alpha''$ is identical with G.

We can therefore establish a (1, 1) correspondence between the elements G'' of $\mathfrak{G}''$ and the elements $\mathfrak{H}_1 T_\alpha$ of $\mathfrak{N}_1/\mathfrak{H}_1$. But G_β'' commutes with T_α; hence $G_\beta'' = T_\alpha^{-1} G_\beta'' T_\alpha$ permutes

$$\alpha \to 1 \to \beta \to \beta(\alpha).$$

https://doi.org/10.1017/S0305004100011051 Published online by Cambridge University Press

Thus $G_\alpha'' G_\beta''$ permutes $1 \to \alpha \to \beta(\alpha)$, and $G_\alpha'' G_\beta'' = G_{\alpha(\beta)}''$. Similarly $\mathfrak{H}_1 T_\beta \mathfrak{H}_1 T_\alpha = \mathfrak{H}_1 \mathfrak{H}_1 T_\beta T_\alpha = \mathfrak{H}_1 T_{\alpha(\beta)}$, by definition. Thus right-handed combination of the elements of $\mathfrak{G}''$ is equivalent to left-handed combination of the elements of $\mathfrak{A}_1/\mathfrak{H}_1$.

It follows that the correspondence $(G_\alpha'')^{-1} \rightleftharpoons \mathfrak{H}_1 T_\alpha$ defines a (1, 1) isomorphism between $\mathfrak{G}''$ and $\mathfrak{A}_1/\mathfrak{H}_1$. And we need merely to remark that $\mathfrak{G}''$ is abstractly invariant under permutations of the symbols, to extend this (1, 1) isomorphism to the general case of $\mathfrak{A}_i/\mathfrak{H}_i$.

We have now essentially proved

THEOREM 3: *Under the definitions laid down,*

(1) $\mathfrak{G}'/\mathfrak{G}''$ *is* (1, 1) *isomorphic with the subgroup of the automorphisms of* $\mathfrak{G}$ *which permute the* $\mathfrak{H}_i$;

(2) $\mathfrak{G}''$ *is* (1, 1) *isomorphic with* $\mathfrak{A}_i/\mathfrak{H}_i$.

For conclusion (1) is merely the obvious combination of Theorems 1 and 2, while conclusion (2) has just been established.

Let us now remark that if $\mathfrak{G}$ is *primitive*, $\mathfrak{H}_1$ is a greatest subgroup of $\mathfrak{G}$. Hence $\mathfrak{A}_1$ must be either $\mathfrak{H}_i$ or $\mathfrak{G}$. But it cannot be $\mathfrak{G}$, as if it were, $\mathfrak{H}_1$ would be a normal subgroup of $\mathfrak{G}$, which is impossible for a transitive group. Hence it must be $\mathfrak{H}_i$. We hence obtain immediately the following

Corollary: If in the hypotheses of Theorem 3 we substitute *primitivity* for *transitivity*, we obtain the conclusion:

(1) $\mathfrak{G}'$ *is* (1, 1) *isomorphic with the subgroup of the automorphisms of* $\mathfrak{G}$ *which permute the conjugates of* $\mathfrak{H}_1$.

This result can be obtained independently.

https://doi.org/10.1017/S0305004100011051 Published online by Cambridge University Press