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The Maximal Solution and Comparison Theorems for the Periodic Discrete-Time Riccati Equation

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Abstract. Properties and comparison theorems for the maximal solution of the periodic discrete-time Riccati equation are supplemented by an extension of some earlier results and analysis, for the discrete-time Riccati equation to the periodic case.

AMS subject classifications: 65F15, 65F99

Key words: The maximal solution, pd-stabilisable, p-observable, comparison theorems.

1. Introduction

The following periodic discrete-time Riccati equation (PDRE) is considered:

$$X_{j-1} = A_j^* X_j A_j - (B_j^* X_j A_j + C_j)^* (R_j + B_j^* X_j B_j)^{-1} (B_j^* X_j A_j + C_j) + Q_j\,, \quad j = 1, \cdots, p, \tag{1.1}$$

where $X_0 = X_p$ and the superscript $*$ denotes the matrix conjugate transpose, and the matrices $A_j \in \mathbb{C}^{n_{j+1} \times n_j}$, $B_j \in \mathbb{C}^{n_{j+1} \times m_j}$, $C_j \in \mathbb{C}^{m_j \times n_j}$, $R_j = R_j^* \in \mathbb{C}^{m_j \times m_j}$, $Q_j = Q_j^* \in \mathbb{C}^{n_j \times n_j}$ have period $p \geq 1$ — i.e. $A_{j+p} = A_j$, $B_{j+p} = B_j$, $C_{j+p} = C_j$, $R_{j+p} = R_j$, $Q_{j+p} = Q_j$. For Hermitian matrices M and N, let $M \geq N$ (respectively, $M > N$) denote that the matrix difference $M - N$ is Hermitian positive semidefinite (respectively, Hermitian positive definite), $Q^{1/2}$ denote the Hermitian positive semidefinite square root of a Hermitian positive semidefinite matrix Q, and throughout assume that $R_j > 0$ for $j = 1, \cdots, p$.

As is well known, the PDRE (1.1) arises when solving the periodic linear-quadratic optimal control problem [4] for the linear discrete-time periodic system

$$\begin{aligned} x_{j+1} &= A_j x_j + B_j u_j\,, \\ y_j &= S_j x_j\,, \end{aligned} \tag{1.2}$$

by minimising the quadratic cost functional

$$J = \frac{1}{2} \sum_{j=0}^{\infty} [x_j^*,\, u_j^*] \begin{bmatrix} Q_j & C_j^* \\ C_j & R_j \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}, \tag{1.3}$$

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where $Q_j = S_j^* S_j$, and it is usually assumed that

$$\begin{bmatrix} Q_j & C_j^* \\ C_j & R_j \end{bmatrix} \geq 0 .$$

The periodic optimal feedback control u_j is given by

$$u_j^{\#} = -(R_j + B_j^* X_j B_j)^{-1}(B_j^* X_j A_j + C_j)x_j , \quad j = 1, \cdots, p ,$$

where $\{X_j\}_{j=1}^p$ is the Hermitian positive semidefinite solution to the PDRE (1.1). The state transition matrix of the periodic system (1.2) is defined as the $n_j \times n_i$ matrix $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) = I_{n_i}$ — i.e. the identity matrix of order n_i. The state transition matrix over one whole period $\Phi_A(j+p,j) = A_{j-1}\cdots A_1 A_p \cdots A_j \in \mathbb{C}^{n_j \times n_j}$ is called the monodromy matrix of the system (1.2) at time j, and its eigenvalues are called the characteristic multipliers at time j. From Theorem 1.3.20 in Ref. [8], matrices $\Phi_A(j+p,j)$ have the same nonzero characteristic multipliers for all j. The system (1.2) or $\Phi_A(1+p,1)$ is called asymptotically stable if and only if its characteristic multipliers belong to the open unit disc.

An Hermitian periodic solution $\{X_j^+\}_{j=1}^p$ of the PDRE (1.1) is said to be maximal if $X_j^+ \geq X_j, j = 1, \cdots, p$ for any Hermitian periodic solution $\{X_j\}_{j=1}^p$ of (1.1). An Hermitian periodic solution $\{X_j\}_{j=1}^p$ of the PDRE (1.1) is called stabilising (respectively, strong) if all eigenvalues of $\Phi_{\widehat{A}}(1+p,1)$ are in the open (respectively, closed) unit disc, where $\widehat{A}_j = A_j - B_j(R_j + B_j^* X_j B_j)^{-1}(B_j^* X_j A_j + C_j)$.

There are many articles in the literature on the perturbation theory and numerical methods of the PDRE (1.1) — e.g. see Refs. [10, 12, 14, 18, 21]. The study of periodic Riccati difference equations can be traced back to Ref. [16], which provided existence and uniqueness conditions of periodic stabilising and strong solutions. Bittanti *et al.* [3] and Souza [19,20] provided several theorems on the existence, uniqueness and stability properties of Hermitian periodic positive semidefinite solutions of the PDRE. The maximal solution and comparison theorems for algebraic Riccati equations have also been considered — e.g. see Refs. [9,17,22]). To date there seems to have been no systematic study on the maximal periodic solution and comparison theorems for the PDRE (1.1). Attention is focused on two problems here — viz. the existence and properties of the maximal periodic solution of the PRDE (1.1) and comparison theorems between two different PDRE, leading to extensions of corresponding results in Refs. [9,17,22]. Section 2 is devoted to basic concepts for periodic control systems and eigenproblems of periodic matrix pairs together with other results to be used in Section 3, where the existence of the maximal periodic solution of the PDRE (1.1) and its properties are discussed. Comparison theorems between two different PDRE are presented in Section 4, and conclusions are in Section 5.

2. Preliminaries

Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{m\times n}$. The matrix pair (C,A) is said to be observable if $Cx = 0$ and $Ax = \lambda x$ for any number λ imply $x = 0$. The matrix pair (A,B) is d-stabilisable

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if $y^*B = 0$ and $y^*A = \lambda y^*$ for some constant λ imply $|\lambda| < 1$ or $y = 0$, and the matrix pair (C,A) is *d-detectable* if (A^*, C^*) is d-stabilisable [9].

The definitions of observability, stabilisability and detectability for periodic systems are as follows.

Definition 2.1. (cf. Ref. [4].) Let $A_j \in \mathbb{C}^{n_{j+1}\times n_j}$, $B_j \in \mathbb{C}^{n_{j+1}\times m_j}$ and $C_j \in \mathbb{C}^{m_j\times n_j}$ for $j = 1,\cdots,p$. The matrix pairs $\{(A_j,B_j)\}_{j=1}^p$ are said to be pd-stabilisable if the matrix pair $(\mathscr{A}_j, \mathscr{B}_j)$ are d-stabilisable for $j = 1,\cdots,p$; The matrix pairs $\{(C_j,A_j)\}_{j=1}^p$ are said to be p-observable (respectively, pd-detectable) if the matrix pair $(\mathscr{C}_j, \mathscr{A}_j)$ is observable (respectively, d-detectable) for $j = 1,\cdots,p$, where $\mathscr{A}_j = A_{\pi_j(p)}\cdots A_{\pi_j(1)}$ and

$$\mathscr{B}_j = \left[A_{\pi_j(p)}\cdots A_{\pi_j(2)}B_{\pi_j(1)}|A_{\pi_j(p)}\cdots A_{\pi_j(3)}B_{\pi_j(2)}|\cdots|A_{\pi_j(p)}B_{\pi_j(p-1)}|B_{\pi_j(p)}\right],$$
$$\mathscr{C}_j = \left[C^T_{\pi_j(1)}|A^T_{\pi_j(1)}C^T_{\pi_j(2)}|A^T_{\pi_j(1)}A^T_{\pi_j(2)}C^T_{\pi_j(3)}|\cdots|A^T_{\pi_j(1)}\cdots A^T_{\pi_j(p-1)}C^T_{\pi_j(p)}\right]^T,$$

with the permutation π_j defined by

$$\pi_j(k) = \begin{cases} k-j+1+p\,, & \text{for } k = 1,\cdots,j-1\,,\\ k-j+1\,, & \text{for } k = j,\cdots,p\,. \end{cases}$$

The following two results give necessary and sufficient conditions for p-observability, pd-detectability and pd-stabilisability, respectively. The proofs in Ref. [5] for similar results on systems of constant dimension can readily be adapted for time-variant dimensions and so omitted here — cf. also Ref. [23].

Proposition 2.1. *Let $A_j \in \mathbb{C}^{n_{j+1}\times n_j}$, $B_j \in \mathbb{C}^{n_{j+1}\times m_j}$, $j = 1,\cdots,p$. The matrix pairs $\{(A_j,B_j)\}_{j=1}^p$ are pd-stabilisable if and only if the matrix*

$$\left(\begin{array}{ccccc|ccccc} -\lambda_1 I & & & & A_1 & B_1 & & & & \\ A_2 & -\lambda_2 I & & & & & B_2 & & & \\ & \ddots & \ddots & & & & & \ddots & & \\ & & \ddots & \ddots & & & & & \ddots & \\ & & & A_p & -\lambda_p I & & & & & B_p \end{array}\right)$$

has full row rank or $y_j^*A_j = \lambda_{j-1}y_{j-1}^*$, $y_j^*B_j = 0$, $j = 1,\cdots,p$ imply that $y_j = 0$, $j = 1,\cdots,p$ for any constant $\lambda_j \in \mathbb{C}$, $j = 1,\cdots,p$ such that $|\prod_{j=1}^p \lambda_j| \geq 1$, where $\lambda_0 = \lambda_p$ and $y_0 = y_p$.

Proposition 2.2. *Let $A_j \in \mathbb{C}^{n_{j+1}\times n_j}$, $C_j \in \mathbb{C}^{m_j\times n_j}$, $j = 1,\cdots,p$. The matrix pairs $\{(C_j,A_j)\}_{j=1}^p$*

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are p-observable (respectively, pd-detectable) if and only if the matrix

$$\left(\begin{array}{ccccc} -\lambda_1 I & & & & A_p \\ A_1 & -\lambda_2 I & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & A_{p-1} & -\lambda_p I \\ \hline C_1 & & & & \\ & C_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & C_p \end{array}\right)$$

has full column rank or $A_j x_j = \lambda_{j+1} x_{j+1}$, $C_j x_j = 0$, $j = 1, \cdots, p$ *imply that* $x_j = 0$, $j = 1, \cdots, p$ *for any constant* $\lambda_j \in \mathbb{C}$, $j = 1, \cdots, p$ *(respectively, for any constant* $\lambda_j \in \mathbb{C}$, $j = 1, \cdots, p$ *such that* $|\prod_{j=1}^{p} \lambda_j| \geq 1$*), where* $\lambda_{p+1} = \lambda_1$ *and* $x_{p+1} = x_1$.

Proposition 2.3. *Let* $W_j \in \mathbb{C}^{n_{j+1} \times n_j}, Z_j \in \mathbb{C}^{m_j \times n_j}$, $j = 1, \cdots, p$ *and matrix pairs* $\{(Z_j, W_j)\}_{j=1}^{p}$ *be p-observable, let* $Z_{j1} \in \mathbb{C}^{m_j \times n_j}$*, and define* $Z_{j0} \in \mathbb{C}^{m_j \times n_j}$ *so that* $Z_j^* Z_j + Z_{j1}^* Z_{j1} = Z_{j0}^* Z_{j0}$. *Then the matrix pairs* $\{(Z_{j0}, W_j + Y_j Z_{j1})\}_{j=1}^{p}$ *are p-observable for any* $Y_j \in \mathbb{C}^{n_{j+1} \times m_j}$, $j = 1, \cdots, p$.

Proof. Let $\widetilde{W}_j = W_j + Y_j Z_{j1}$ and

$$\widetilde{W}_j x_j = \lambda_{j+1} x_{j+1}\,, \quad Z_{j0} x_j = 0\,, \qquad j = 1, \cdots, p\,, \tag{2.1}$$

for any constant $\lambda_j \in \mathbb{C}$ where $\lambda_{p+1} = \lambda_1$ and $x_{p+1} = x_1$. From the second equality in (2.1),

$$x_j^* Z_j^* Z_j x_j + x_j^* Z_{j1}^* Z_{j1} x_j = x_j^* Z_{j0}^* Z_{j0} x_j = 0\,, \quad j = 1, \cdots, p\,. \tag{2.2}$$

It is notable that $x_j^* Z_j^* Z_j x_j \geq 0$ and $x_j^* Z_{j1}^* Z_{j1} x_j \geq 0$. From Eq. (2.2) we therefore have

$$x_j^* Z_j^* Z_j x_j = x_j^* Z_{j1}^* Z_{j1} x_j = 0\,, \quad j = 1, \cdots, p\,, \tag{2.3}$$

such that from Eq. (2.3)

$$Z_j x_j = 0, \quad Z_{j1} x_j = 0\,, \quad j = 1, \cdots, p\,. \tag{2.4}$$

From Eq. (2.3) and the second equality of (2.4),

$$W_j x_j = \lambda_{j+1} x_{j+1}, \qquad j = 1, \cdots, p\,. \tag{2.5}$$

Since the matrix pairs $\{(Z_j, W_j)\}_{j=1}^{p}$ are p-observable, Proposition 2.2 applied to the first equalities of (2.4) and (2.5) imply $x_j = 0$, $j = 1, \cdots, p$, so the matrix pairs $\{(Z_{j0}, W_j + Y_j Z_{j1})\}_{j=1}^{p}$ are p-observable (using Proposition 2.2 again). □

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Proposition 2.4. *Let $A_j \in \mathbb{C}^{n_{j+1}\times n_j}$, $B_j \in \mathbb{C}^{n_{j+1}\times m_j}$, $C_j \in \mathbb{C}^{m_j\times n_j}$, $R_j \in \mathbb{C}^{m_j\times m_j}$, $j = 1,\cdots,p$ with $R_j > 0$. Then the matrix pairs $\{(A_j, B_j)\}_{j=1}^p$ are pd-stabilisable if and only if the matrix pairs $\{(A_j - B_jR_j^{-1}C_j, B_jR_j^{-1}B_j^*)\}_{j=1}^p$ are pd-stabilisable.*

Proof. The following proof for sufficiency can be extended to similarly prove necessity. For any $\lambda_j \in \mathbb{C}$, $j = 1,\cdots,p$ with $|\prod_{j=1}^p \lambda_j| \geq 1$, let the column vectors $y_j \in \mathbb{C}^{n_{j+1}}$, $j = 1,\cdots,p$ satisfy

$$y_j^*(A_j - B_jR_j^{-1}C_j) = \lambda_{j-1}y_{j-1}^*, \quad y_j^*B_jR_j^{-1}B_j^* = 0\,, \quad j = 1,\cdots,p\,, \tag{2.6}$$

where $\lambda_0 = \lambda_p$ and $y_0 = y_p$. From the second equality of (2.6),

$$y_j^*B_jR_j^{-1}B_j^*y_j = 0, \quad j = 1,\cdots,p\,,$$

which implies that

$$y_j^*B_j = 0\,, \quad j = 1,\cdots,p\,. \tag{2.7}$$

Combining the first equality of (2.6) with (2.7) yields

$$y_j^*A_j = \lambda_{j-1}y_{j-1}^*\,, \quad j = 1,\cdots,p\,. \tag{2.8}$$

Since $\{(A_j, B_j)\}_{j=1}^p$ are pd-stabilisable, from Eqs. (2.7) and (2.8) an application of Proposition 2.1 yields

$$y_j = 0\,, \quad j = 1,\cdots p\,,$$

so $\{(A_j - B_jR_j^{-1}C_j,\ B_jR_j^{-1}B_j^*)\}_{j=1}^p$ are pd-stabilisable. □

Let $\{(G_j, F_j)\}_{j=1}^p$ be a set of matrix pairs with $F_j \in \mathbb{C}^{n_{j+1}\times m_{j+1}}$, $G_j \in \mathbb{C}^{n_{j+1}\times m_j}$ and $\sum_{j=1}^p n_j = \sum_{j=1}^p m_j = N$. The matrix pairs $\{(G_j, F_j)\}_{j=1}^p$ are said to be regular when $\det[C((\alpha_j,\beta_j)_{j=1}^p)] \not\equiv 0$, where

$$C((\alpha_j,\beta_j)_{j=1}^p) \equiv \begin{bmatrix} \alpha_1F_1 & 0 & \cdots & 0 & -\beta_1G_1 \\ -\beta_2G_2 & \alpha_2F_2 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & 0 & -\beta_pG_p & \alpha_pF_p \end{bmatrix} \in \mathbb{C}^{N\times N}\,,$$

in which α_j, β_j are complex variables for $j = 1,\cdots,p$. If there are complex numbers $\alpha_1,\cdots,\alpha_p,\beta_1,\cdots,\beta_p$ with $(\prod_{j=1}^p \alpha_j, \prod_{j=1}^p \beta_j) \equiv (\pi_\alpha, \pi_\beta) \neq (0,0)$ satisfying

$$\det\Big[C((\alpha_j,\beta_j)_{j=1}^p)\Big] = 0\,,$$

then (π_α, π_β) is an eigenvalue of $\{(G_j, F_j)\}_{j=1}^p$. If $\pi_\beta \neq 0$, then $\lambda = \pi_\alpha/\pi_\beta$ is a finite eigenvalue; otherwise, (π_α, π_β) is an infinite eigenvalue — cf. Refs. [1,6,13,14] for periodic eigenproblems.

Let $\mathbf{\Gamma} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle. The following result is a generalisation of Theorem 15.1.2 in Ref. [9].

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Proposition 2.5. *Let $M_j \in \mathbb{C}^{n_{j+1}\times n_j}$, $D_j \in \mathbb{C}^{n_{j+1}\times n_{j+1}}$ and $N_j \in \mathbb{C}^{n_j\times n_j}$, $j = 1,\cdots,p$ with $D_j \geq 0$, $N_j \geq 0$, $j = 1,\cdots,p$, and let the matrix pairs $\{(M_j, D_j)\}_{j=1}^p$ be pd-stabilisable and the matrix pairs $\{(N_j, M_j)\}_{j=1}^p$ be pd-detectable. Then the periodic matrix pairs $\{(G_j, F_j)\}_{j=1}^p$ are regular and have no eigenvalues on the unit circle $\mathbf{\Gamma}$, where*

$$G_j = \begin{bmatrix} M_j & 0 \\ -N_j & I_{n_j} \end{bmatrix} \in \mathbb{C}^{(n_j+n_{j+1})\times 2n_j}\,, \qquad F_j = \begin{bmatrix} I_{n_{j+1}} & D_j \\ 0 & M_j^* \end{bmatrix} \in \mathbb{C}^{(n_j+n_{j+1})\times 2n_{j+1}}\,,$$

for $j = 1,\cdots,p$, where $n_{p+1} = n_1$.

Proof. Without loss of generality, we consider $p = 3$ for simplicity. Assume that there is an eigenvalue λ of $\{(G_j, F_j)\}_{j=1}^p$ on the unit circle $\mathbf{\Gamma}$ — i.e. $|\lambda| = 1$. Then there exist nonzero numbers α_i and $\beta_i (i = 1,2,3)$ such that

$$\det\left(\begin{bmatrix} \beta_1 F_1 & 0 & -\alpha_1 G_1 \\ -\alpha_2 G_2 & \beta_2 F_2 & 0 \\ 0 & -\alpha_3 G_3 & \beta_3 F_3 \end{bmatrix}\right) = 0\,, \tag{2.9}$$

and $\lambda = \alpha_1\alpha_2\alpha_3/(\beta_1\beta_2\beta_3)$. From Eq. (2.9) there exist column vectors $z_j = (x_j^T, y_j^T)^T$ where $x_j, y_j \in \mathbb{C}^{n_{j+1}}$ for $j = 1,2,3$, which are not all the zero vector, such that

$$\begin{bmatrix} \beta_1 F_1 & 0 & -\alpha_1 G_1 \\ -\alpha_2 G_2 & \beta_2 F_2 & 0 \\ 0 & -\alpha_3 G_3 & \beta_3 F_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0\,. \tag{2.10}$$

Letting

$$w_1 = \beta_1 z_1\,, \qquad w_2 = \frac{\beta_1\beta_2}{\alpha_2} z_2\,, \qquad w_3 = \alpha_1 z_3\,,$$

we can rewrite Eq. (2.10) as

$$\begin{bmatrix} F_1 & 0 & -G_1 \\ -G_2 & F_2 & 0 \\ 0 & -G_3 & \lambda F_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0\,.$$

Since $w_i = 0$ if and only if $z_i = 0$ for $i = 1,2,3$, without loss of generality we may assume that $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 1$ and $\beta_3 = \lambda$. Then from Eq. (2.10) we obtain

$$M_1 x_3 = x_1 + D_1 y_1\,, \qquad -N_1 x_3 + y_3 = M_1^* y_1\,, \tag{2.11}$$

$$M_2 x_1 = x_2 + D_2 y_2\,, \qquad -N_2 x_1 + y_1 = M_2^* y_2\,, \tag{2.12}$$

$$M_3 x_2 = \lambda x_3 + \lambda D_3 y_3\,, \qquad -N_3 x_2 + y_2 = \lambda M_3^* y_3\,. \tag{2.13}$$

Multiplying the first equation of (2.11) on the left by y_1^* and taking the conjugate transpose on the both sides, we obtain

$$x_3^* M_1^* y_1 = x_1^* y_1 + y_1^* D_1 y_1\,, \tag{2.14}$$

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Multiplying the second equation of (2.11) on the left by x_3^*, we get

$$-x_3^* N_1 x_3 + x_3^* y_3 = x_3^* M_1^* y_1 \,, \tag{2.15}$$

so on combining Eqs. (2.14) and (2.15) we have

$$y_1^* D_1 y_1 + x_3^* N_1 x_3 = x_3^* y_3 - x_1^* y_1. \tag{2.16}$$

Similarly, from the equations in (2.12) we obtain

$$y_2^* D_2 y_2 + x_1^* N_2 x_1 = x_1^* y_1 - x_2^* y_2. \tag{2.17}$$

Now multiplying the first equation of (2.13) on the left by y_3^*, we obtain

$$y_3^* M_3 x_2 = \lambda y_3^* x_3 + \lambda y_3^* D_3 y_3 \,,$$

and since $|\lambda| = 1$ implies that $\lambda^{-1} = \overline{\lambda}$ it follows that

$$\lambda x_2^* M_3^* y_3 = x_3^* y_3 + y_3^* D_3 y_3. \tag{2.18}$$

Furthermore, multiplying the second equation of (2.13) on the left by x_2^* we obtain

$$-x_2^* N_3 x_2 + x_2^* y_2 = \lambda x_2^* M_3^* y_3 \,, \tag{2.19}$$

and hence from Eqs. (2.18) and (2.19)

$$y_3^* D_3 y_3 + x_2^* N_3 x_2 = x_2^* y_2 - x_3^* y_3 \,. \tag{2.20}$$

Combining Eqs. (2.16), (2.17) and (2.20) yields

$$y_1^* D_1 y_1 + x_3^* N_1 x_3 + y_2^* D_2 y_2 + x_1^* N_2 x_1 + y_3^* D_3 y_3 + x_2^* N_3 x_2 = 0 \,.$$

Since $N_j \geq 0 \,, D_j \geq 0 \,, j = 1, 2, 3 \,,$

$$\begin{aligned} &y_1^* D_1 y_1 = 0 \,, \qquad y_2^* D_2 y_2 = 0 \,, \qquad y_3^* D_3 y_3 = 0 \,, \\ &x_3^* N_1 x_3 = 0 \,, \qquad x_1^* N_2 x_1 = 0 \,, \qquad x_2^* N_3 x_2 = 0 \,, \end{aligned}$$

so we have

$$y_1^* D_1 = 0 \,, \qquad y_2^* D_2 = 0 \,, \qquad y_3^* D_3 = 0 \,, \tag{2.21}$$

and

$$N_1 x_3 = 0 \,, \qquad N_2 x_1 = 0 \,, \qquad N_3 x_2 = 0 \,. \tag{2.22}$$

Now from Eqs. (2.11)-(2.13) and (2.21)-(2.22) it follows that

$$y_1^* M_1 = y_3^* \,, \qquad y_2^* M_2 = y_1^* \,, \qquad y_3^* M_3 = \lambda y_2^* \,, \tag{2.23}$$

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and

$$M_1 x_3 = x_1\,, \qquad M_2 x_1 = x_2\,, \qquad M_3 x_2 = \lambda x_3\,. \tag{2.24}$$

Since the matrix pairs $\{(N_j, M_j)\}_{j=1}^3$ are pd-detectable and $|\lambda| = 1$, we apply Proposition 2.2 to (2.22) and (2.24) to get $x_i = 0, i = 1,2,3$. Similarly, since the matrix pairs $\{(M_j, D_j)\}_{j=1}^3$ are pd-stabilisable and $|\lambda| = 1$, on applying Proposition 2.1 to Eqs. (2.21) and (2.23) we obtain $y_i = 0$, $i = 1,2,3$, so we have $z_1 = z_2 = z_3 = 0$, which is a contradiction with the premise that not all of z_i $(i = 1,2,3)$ are zero. Hence λ on the unit circle is not an eigenvalue of $\{(G_j, F_j)\}_{j=1}^3$ — i.e.

$$\det\left(\begin{bmatrix} F_1 & 0 & -G_1 \\ -G_2 & F_2 & 0 \\ 0 & -G_3 & \lambda F_3 \end{bmatrix}\right) \neq 0\,,$$

which implies that the matrix pairs $\{(G_j, F_j)\}_{j=1}^3$ are regular. □

We end this section by stating the following result with respect to solutions to the discrete-time periodic Lyapunov equation.

Proposition 2.6 (cf. Ref. [2].). *Let $A_j \in \mathbb{C}^{n_{j+1}\times n_j}$, $D_j \in \mathbb{C}^{m_j \times n_j}$, $j = 1,\cdots,p$ and assume that $\Phi_A(1+p,1)$ is asymptotically stable and let $\{X_j\}_{j=1}^p$ be the unique Hermitian solution of the discrete-time periodic Lyapunov equation*

$$X_{j-1} = A_j^* X_j A_j + D_j D_j^*\,, \quad j = 1,\cdots,p\,.$$

Then $X_j \geq 0$, $j = 1,\cdots,p$. If furthermore, the periodic matrix pairs $\{(D_j, A_j)\}_{j=1}^p$ are p-observable, then $X_j > 0$, $j = 1,\cdots,p$.

3. The Maximal Periodic Solution of the PDRE

A sufficient condition of existence is now obtained, together with some properties of the maximal periodic solution of the PDRE (1.1).

For the PDRE (1.1), we define

$$\widehat{Q}_j = Q_j - C_j^* R_j^{-1} C_j\,, \tag{3.1}$$

and

$$\mathscr{R}(X_j) = -X_{j-1} + A_j^* X_j A_j + Q_j - (B_j^* X_j A_j + C_j)^*(R_j + B_j^* X_j B_j)^{-1}(B_j^* X_j A_j + C_j)\,,$$

for $j = 1,\cdots,p$, where X_j satisfies $R_j + B_j^* X_j B_j > 0$, and

$$\mathbf{M} = \left\{\{X_j\}_{j=1}^p | X_j = X_j^*,\, R_j + B_j^* X_j B_j > 0, \mathscr{R}(X_j) \geq 0, j = 1,\cdots,p\right\}.$$

The following theorem extends the existence of the maximal solution to the PDRE from the previous result for the discrete-time Riccati equation — cf. Ref. [17] Theorem 2.1 or Ref. [9] Theorem 13.1.1.

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Theorem 3.1. *Assume that the matrix pairs* $\{(A_j, B_j)\}_{j=1}^p$ *are pd-stabilisable and* $\mathbf{M} \neq \emptyset$. *Then there exists a unique maximal periodic solution* $\{X_j^+\}_{j=1}^p$ *of the PDRE* (1.1). *Moreover, all the eigenvalues of* $\Phi_{A^+}(1+p,1)$ *are in the closed unit disc, where*

$$A_j^+ = A_j - B_j(R_j + B_j^* X_j^+ B_j)^{-1}(B_j^* X_j^+ A_j + C_j)\,. \tag{3.2}$$

Proof. The uniqueness of $\{X_j^+\}_{j=1}^p$ follows readily from the maximality property, so the proof largely involves showing the existence of $\{X_j^+\}_{j=1}^p$. Let $\{X_j\}_{j=1}^p \in \mathbf{M}$. Since the matrix pairs $\{(A_j, B_j)\}_{j=1}^p$ are pd-stabilisable, there exist $L_j^{(0)} \in \mathbb{C}^{m_j \times n_j}, j = 1, \cdots, p$ such that $\Phi_{A^{(0)}}(1+p,1)$ is asymptotically stable, where $A_j^{(0)} = A_j - B_j L_j^{(0)}$ (see [3]). Let $\{X_j^{(0)}\}_{j=1}^p$ be the unique Hermitian periodic solution of the periodic discrete-time Lyapunov equation

$$X_{j-1}^{(0)} - (A_j^{(0)})^* X_j^{(0)} A_j^{(0)} = \widehat{Q}_j + (L_j^{(0)} - R_j^{-1} C_j)^* R_j (L_j^{(0)} - R_j^{-1} C_j)\,, \quad j = 1, \cdots, p\,, \tag{3.3}$$

where $\widehat{Q}_j$ is defined by (3.1). Thus from Eq. (3.3), straightforward calculations give

$$\begin{aligned} &(X_{j-1}^{(0)} - X_{j-1}) - (A_j^{(0)})^* (X_j^{(0)} - X_j) A_j^{(0)} \\ =&\mathscr{R}(X_j) + (H_j L_j^{(0)} - E_j)^* H_j^{-1} (H_j L_j^{(0)} - E_j) \geq 0\,, \quad j = 1, \cdots, p\,, \end{aligned} \tag{3.4}$$

where

$$H_j = R_j + B_j^* X_j B_j\,, \qquad E_j = C_j + B_j^* X_j A_j\,. \tag{3.5}$$

Since $\Phi_{A^{(0)}}(1+p,1)$ is asymptotically stable, on applying Proposition 2.6 to Eq. (3.4) we get

$$X_j^{(0)} \geq X_j, \quad j = 1, \cdots, p\,, \tag{3.6}$$

hence $R_j + B_j^* X_j^{(0)} B_j > 0$, $j = 1, \cdots, p$. Now assume that inductively we have already determined the matrices $X_j^{(k)}, A_j^{(k)}, L_j^{(k)}$, $j = 1, \cdots, p$, $k = 1, \cdots, n-1$, with

$$X_j^{(k)} = \left(X_j^{(k)}\right)^*\,, \qquad X_j^{(0)} \geq X_j^{(1)} \geq \cdots \geq X_j^{(n-1)} \geq X_j\,, \qquad A_j^{(k)} = A_j - B_j L_j^{(k)}\,,$$

where

$$L_j^{(k)} = (R_j + B_j^* X_j^{(k-1)} B_j)^{-1} (C_j + B_j^* X_j^{(k-1)} A_j)$$

and

$$X_{j-1}^{(k)} - (A_j^{(k)})^* X_j^{(k)} A_j^{(k)} = \hat{Q}_j + (L_j^{(k)} - R_j^{-1} C_j)^* R_j (L_j^{(k)} - R_j^{-1} C_j)\,, \tag{3.7}$$

and that $\Phi_{A^{(k)}}(1+p,1)$ is asymptotically stable for $k = 0, 1, \cdots, n-1$. From the induction assumptions $X_j^{(n-1)} \geq X_j$, $j = 1, \cdots, p$, we have

$$R_j + B_j^* X_j^{(n-1)} B_j \geq R_j + B_j^* X_j B_j > 0, \quad j = 1, \cdots, p.$$

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Let us define

$$H_j^{(n-1)} = R_j + B_j^* X_j^{(n-1)} B_j\,, \qquad E_j^{(n-1)} = C_j + B_j^* X_j^{(n-1)} A_j\,,$$
$$L_j^{(n)} = \left(H_j^{(n-1)}\right)^{-1} E_j^{(n-1)}\,, \qquad A_j^{(n)} = A_j - B_j L_j^{(n)}\,,$$

for $j = 1, \cdots, p$. It then follows from Eq (3.7) that

$$\left(X_{j-1}^{(n-1)} - X_{j-1}\right) - \left(A_j^{(n)}\right)^* \left(X_j^{(n-1)} - X_j\right) A_j^{(n)} = W_j, \tag{3.8}$$

for $j = 1, \cdots, p$, where

$$W_j = \left(L_j^{(n)} - L_j^{(n-1)}\right)^* H_j^{(n-1)} (L_j^{(n)} - L_j^{(n-1)}) + \mathscr{R}(X_j) + \left(H_j L_j^{(n)} - E_j\right)^* H_j^{-1} \left(H_j L_j^{(n)} - E_j\right) \geq 0\,,$$

and H_j, E_j are given by Eq. (3.5). Letting

$$\Phi_{A^{(n)}}(j,i) = A_{j-1}^{(n)} A_{j-1}^{(n)} \cdots A_i^{(n)} \quad \text{and} \quad \Phi_{A^{(n)}}(i,i) = I_{n_i}\,,$$

from Eq. (3.8) we have

$$\left(X_{j-1}^{(n-1)} - X_{j-1}\right) - (\Phi_{A^{(n)}}(j+p,j))^* \left(X_{j-1}^{(n-1)} - X_{j-1}\right) \Phi_{A^{(n)}}(j+p,j)$$
$$= \sum_{i=j}^{j+p} (\Phi_{A^{(n)}}(i,j))^* W_i \Phi_{A^{(n)}}(i,j)\,, \quad j = 1, \cdots, p\,.$$

Assume that $\Phi_{A^{(n)}}(1+p,1)$ is not asymptotically stable. Then there exists an eigenvalue λ of $\Phi_{A^{(n)}}(1+p,1)$ with $|\lambda| \geq 1$. From the periodic Schur decomposition [7], we know that there exist nonzero column vectors $x_j \in \mathbb{C}^{n_j}, j = 1, \cdots, p$ such that

$$A_1^{(n)} x_1 = \lambda x_2, \quad A_2^{(n)} x_2 = x_3, \quad \cdots, \quad A_{p-1}^{(n)} x_p = x_p, \quad A_p^{(n)} x_p = x_1\,,$$

so we have $\Phi_{A^{(n)}}(j+p,j) x_j = (A_{j-1}^{(n)} \cdots A_1^{(n)} A_p^{(n)} A_{p-1}^{(n)} \cdots A_j^{(n)}) x_j = \lambda x_j$ for $j = 1, \cdots, p$ such that

$$0 \geq \left(1 - |\lambda|^2\right) x_j^* \left(X_{j-1}^{(n-1)} - X_{j-1}\right) x_j = \sum_{i=j}^{j+p} x_j^* (\Phi_{A^{(n)}}(i,j))^* W_j \Phi_{A^{(n)}}(i,j) x_j \geq 0\,,$$

for $j = 1, \cdots, p$. Since $\mathscr{R}(X_j) \geq 0$, $H_j^{(n-1)} > 0$, $H_j > 0$, and hence

$$L_j^{(n)} x_j = L_j^{(n-1)} x_j\,, \quad j = 1, \cdots, p\,,$$

so that

$$\left(A_{j-1}^{(n-1)} \cdots A_1^{(n-1)} A_p^{(n-1)} A_{p-1}^{(n-1)} \cdots A_j^{(n-1)}\right) x_j = \left(A_{j-1}^{(n)} \cdots A_1^{(n)} A_p^{(n)} A_{p-1}^{(n)} \cdots A_j^{(n)}\right) x_j = \lambda x_j\,,$$

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which is a contradiction with the asymptotical stability of $\Phi_{A^{(n-1)}}(1+p,1)$. Thus we have that $\Phi_{A^{(n)}}(1+p,1)$ is also asymptotically stable.

Next we define $\{X_j^{(n)}\}_{j=1}^p$ as the unique solution of the periodic discrete-time Lyapunov equation

$$X_{j-1}^{(n)} - \left(A_j^{(n)}\right)^* X_j^{(n)} A_j^{(n)} = \widehat{Q}_j + \left(L_j^{(n)} - R_j^{-1} C_j\right)^* R_j \left(L_j^{(n)} - R_j^{-1} C_j\right), \quad j = 1, \cdots, p\,, \tag{3.9}$$

so all the $X_j^{(n)}$, $j = 1, \cdots, p$, are Hermitian. Straightforward calculations then yield

$$\begin{aligned} &\left(X_{j-1}^{(n)} - X_{j-1}\right) - \left(A_j^{(n)}\right)^* \left(X_j^{(n)} - X_j\right) A_j^{(n)} \\ =&\mathscr{R}(X_j) + \left(H_j L_j^{(n)} - E_j\right)^* H_j^{-1} \left(H_j L_j^{(n)} - E_j\right) \geq 0\,, \quad j = 1, \cdots, p \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\left(X_{j-1}^{(n-1)} - X_{j-1}^{(n)}\right) - \left(A_j^{(n)}\right)^* \left(X_j^{(n-1)} - X_j^{(n)}\right) A_j^{(n)} \\ =&\left(L_j^{(n)} - L_j^{(n-1)}\right)^* H_j^{(n-1)} \left(L_j^{(n)} - L_j^{(n-1)}\right) \geq 0\,, \quad j = 1, \cdots, p\,. \end{aligned} \tag{3.11}$$

Since $\Phi_{A^{(n)}}(1+p,1)$ is asymptotically stable, we respectively apply Proposition 2.6 to Eqs. (3.10) and (3.11) to get

$$X_j^{(n-1)} \geq X_j^{(n)} \geq X_j\,, \quad j = 1, \cdots, p\,,$$

so we obtain a non-increasing sequence $\{X_j^{(0)}, X_j^{(1)} \cdots, X_j^{(n)}, \cdots\}$ that is bounded below by X_j. Hence we know that

$$X_j^+ = \lim_{n\to\infty} X_j^{(n)}\,, \quad j = 1, \cdots, p$$

exist and are Hermitian, and $X_j^+ \geq X_j$, $j = 1, \cdots, p$. Since $\{X_j\}_{j=1}^p$ is an arbitrary element in $\mathbf{M}$ and $\{X_j^+\}_{j=1}^p$ is independent of the choice of $\{X_j\}_{j=1}^p$, we have

$$X_j^+ \geq X_j, \quad j = 1, \cdots, p\,, \quad \text{for all } \{X_j\}_{j=1}^p \in \mathbf{M}\,.$$

Hence $\{X_j^+\}_{j=1}^p$ is the maximal element of $\mathbf{M}$. The limit $n \to \infty$ in Eq. (3.9) then yields $\mathscr{R}(X_j^+) = 0, j = 1, \cdots, p$, and $\{X_j^+\}_{j=1}^p$ is the maximal solution of the PDRE (1.1). Finally, since $\Phi_{A^{(k)}}(1+p,1)$ is asymptotically stable for all $k \geq 0$ and

$$\lim_{k\to\infty} A_j^{(k)} = A_j - B_j (R_j + B_j^* X_j^+ B)^{-1} (C_j + B_j^* X_j^+ A_j) = A_j^+\,,$$

all the eigenvalues of $\Phi_{A^+}(1+p,1)$ lie in the closed unit disc. □

Corollary 3.1. *Assume that $R_j > 0$, the matrix pairs $\{(A_j, B_j)\}_{j=1}^p$ are pd-stabilisable and*

$$\begin{bmatrix} Q_j & C_j^* \\ C_j & R_j \end{bmatrix} \geq 0\,, \quad j = 1, \cdots, p\,. \tag{3.12}$$

Then the maximal solution $\{X_j^+\}_{j=1}^p$ of the PDRE (1.1) is unique, and $X_j^+ \geq 0$, $j = 1, \cdots, p$. Moreover, all the eigenvalues of $\Phi_{A^+}(1+p,1)$ lie in the closed unit disc, where A_j^+ is defined by (3.2).

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Proof. Because of the equality

$$\begin{bmatrix} I_{n_j} & -C_j^* R_j^{-1} \\ 0 & I_{m_j} \end{bmatrix} \begin{bmatrix} Q_j & C_j^* \\ C_j & R_j \end{bmatrix} \begin{bmatrix} I_{n_j} & 0 \\ -R_j^{-1} C_j & I_{m_j} \end{bmatrix} = \begin{bmatrix} Q_j - C_j^* R_j^{-1} C_j & 0 \\ 0 & R_j \end{bmatrix},$$

for $j = 1, \cdots, p$, the condition (3.12) is equivalent to $Q_j - C_j^* R_j^{-1} C_j \geq 0$. Consequently, $X_j = 0$, $j = 1, \cdots, p$ satisfy the PDRE (1.1) — i.e. $\mathbf{M} \neq \emptyset$. An application of Theorem 3.1 yields the corollary. □

The following result is a generalisation of Theorem 13.1.3 in Ref. [9] with respect to the discrete-time Riccati equation to the PDRE.

Theorem 3.2. *In addition to the hypotheses of Corollary* 3.1. *assume that the matrix pairs* $\{(Q_j - C_j^* R_j^{-1} C_j,\ A_j - B_j R_j^{-1} C_j)\}$ *are pd-detectable. Then the maximal solution* $\{X_j^+\}_{j=1}^p$ *of the PDRE* (1.1) *exists, and* $X_j^+ \geq 0$, $j = 1, \cdots, p$ *and* $\Phi_{A^+}(1+p, 1)$ *is asymptotically stable, where* A_j^+ *is defined by* (3.2). *Furthermore, if the matrix pairs* $\{(Q_j - C_j^* R_j^{-1} C_j,\ A_j - B_j R_j^{-1} C_j)\}_{j=1}^p$ *are p-observable, then we have* $X_j^+ > 0$, $j = 1, \cdots, p$.

Proof. We know from Corollary 3.1 that $\{X_j^+\}_{j=1}^p$ exists, $X_j^+ \geq 0$, $j = 1, \cdots, p$, and all the eigenvalues of $\Phi_{A^+}(1+p, 1)$ are in the closed unit disc. Since $\{(A_j, B_j)\}_{j=1}^p$ are pd-stabilisable, from Proposition 2.4 we have that $\{(A_j - B_j R_j^{-1} C_j,\ B_j R_j^{-1} B_j^*)\}$ are pd-stabilisable. So applying Proposition 2.5 we know the periodic matrix pairs $\{(G_j, F_j)\}_{j=1}^p$ have no eigenvalues on the unit circle, where

$$G_j = \begin{bmatrix} A_j - B_j R_j^{-1} C_j & 0 \\ -Q_j + C_j^* R_j^{-1} C_j & I_{n_j} \end{bmatrix}, \qquad F_j = \begin{bmatrix} I_{n_{j+1}} & B_j R_j^{-1} B_j^* \\ 0 & (A_j - B_j R_j^{-1} C_j)^* \end{bmatrix}.$$

Then it is readily verified from Proposition 12.1.1 in Ref. [9] that

$$G_j \begin{bmatrix} I_{n_j} \\ X_{j-1}^+ \end{bmatrix} = F_j \begin{bmatrix} I_{n_{j+1}} \\ X_j^+ \end{bmatrix} K_j\,, \quad j = 1, \cdots, p\,, \tag{3.13}$$

where

$$K_j = (I_{n_{j+1}} + B_j R_j^{-1} B_j^* X_j^+)^{-1} (A_j - B_j R_j^{-1} C_j)\,.$$

From the formula $(I_{n_{j+1}} + B_j R_j^{-1} B_j^* X_j^+)^{-1} = I_{n_{j+1}} - B_j (R_j + B_j^* X_j^+ B_j)^{-1} B_j^* X_j^+$,

$$\begin{aligned} K_j &= [I_{n_{j+1}} - B_j (R_j + B_j^* X_j^+ B_j)^{-1} B_j^* X_j^+](A_j - B_j R_j^{-1} C_j) \\ &= A_j - B_j (R_j + B_j^* X_j^+ B_j)^{-1} B_j^* X_j^+ A_j - [I_{n_{j+1}} - B_j (R_j + B_j^* X_j^+ B_j)^{-1} B_j^* X_j^+] B_j R_j^{-1} C_j \\ &= A_j - B_j (R_j + B_j^* X_j^+ B_j)^{-1} B_j^* X_j^+ A_j - B_j [I_{m_j} - (R_j + B_j^* X_j^+ B_j)^{-1} B_j^* X_j^+ B_j] R_j^{-1} C_j \\ &= A_j - B_j (R_j + B_j^* X_j^+ B)^{-1} (C_j + B_j^* X_j^+ A_j) = A_j^+\,. \end{aligned}$$

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Thus from Theorem 5.2 in Ref. [13] and Eq. (3.13), the eigenvalues of $\Phi_K(1+p,1)$ are those of the periodic matrix pairs $\{(G_j,F_j)\}_{j=1}^p$, so $\Phi_{A^+}(1+p,1)$ has no eigenvalues on the unit circle and is therefore asymptotically stable.

For the last statement of the theorem, we introduce the abbreviations

$$F_j=(R_j+B_j^*X_j^+B_j)^{-1}(C_j+B_j^*X_j^+A_j),\quad j=1,\cdots,p$$

for the feedback matrix on $\{X_j^+\}_{j=1}^p$, so that $A_j^+=A_j-B_jF_j$ in (3.2). A lengthy computation then shows that PDRE (1.1) with $X_j=X_j^+$ is equivalent to

$$X_{j-1}^+-(A_j^+)^*X_j^+A_j^+=\widehat{Q}_j+(C_j-R_jF_j)^*R_j^{-1}(C_j-R_jF_j)\,,\quad j=1,\cdots,p\,, \tag{3.14}$$

where $\widehat{Q}_j$ is defined by (3.1). From the first part of the theorem, $\Phi_{A^+}(1+p,1)$ is asymptotically stable. Letting

$$W_j=A_j-B_jR_j^{-1}C_j,\quad Z_j=\widehat{Q}_j^{1/2}\,,\quad Z_{j1}=R_j^{-1/2}(C_j-R_jF_j)\,,$$
$$Z_{j0}=[\widehat{Q}_j+(C_j-R_jF_j)^*R_j^{-1}(C_j-R_jF_j)]^{1/2}\,,\quad Y_j=B_jR_j^{-1/2}$$

for $j=1,\cdots,p$, we have

$$Z_j^*Z_j+Z_{j1}^*Z_{j1}=Z_{j0}^*Z_{j0} \tag{3.15}$$

and

$$\begin{aligned}W_j+Y_jZ_{j1}&=A_j-B_jR_j^{-1}C_j+B_jR_j^{-1/2}R_j^{-1/2}(C_j-R_jF_j)\\&=A_j-B_jF_j=A_j^+\,,\end{aligned} \tag{3.16}$$

F for $j=1,\cdots,p$. Since $\{(\widehat{Q}_j,A_j-B_jR_j^{-1}C_j)\}_{j=1}^p$ are p-observable, it is easy to verify that $\{(\widehat{Q}_j^{1/2},A_j-B_jR_j^{-1}C_j)\}_{j=1}^p$ are p-observable. Applying Proposition 2.3 to Eqs. (3.15) and (3.16) yields that $\{([\widehat{Q}_j+(C_j-R_jF_j)^*R_j^{-1}(C_j-R_jF_j)]^{1/2},A_j^+)\}_{j=1}^p$ are p-observable. Finally, Proposition 2.6 can be applied to (3.14) so that $X_j^+>0$, $j=1,\cdots,p$. □

Remark 3.1. When $n_j=n$, $m_j=m$, $R_j=I_m$, $C_j=0$ and $Q_j=D_jD_j^*$ for $j=1,\cdots,p$, the PDRE (1.1) is reduced to

$$X_{j-1}=A_j^*X_jA_j-A_j^*X_jB_j(I_m+B_j^*X_jB_j)^{-1}B_j^*X_jA_j+D_jD_j^*,\ j=1,\cdots,p. \tag{3.17}$$

1) From Corollary 3.1, the matrix pairs $\{(A_j,B_j)\}_{j=1}^p$ are pd-stabilisable, so there exists the maximal (or strong) solution $\{X_j^+\}_{j=1}^p$ of the PDRE (3.17) such that $X_j^+\geq 0$, $j=1,\cdots,p$.
2) From Theorem 3.2, if the matrix pairs $\{(A_j,B_j)\}_{j=1}^p$ are pd-stabilisable and $\{(D_j,A_j)\}_{j=1}^p$ are pd-detectable, then there exists the maximal (or stabilising) solution $\{X_j^+\}_{j=1}^p$ of the PDRE (3.17) such that $X_j^+\geq 0$, $j=1,\cdots,p$. Furthermore, if $\{(D_j,A_j)\}_{j=1}^p$ are p-observable, there exists the maximal (or stabilising) solution $\{X_j^+\}_{j=1}^p$ of (3.17) such that $X_j^+>0$ for $j=1,\cdots,p$.
These results were proved by Bittanti *et al.* [3] — cf. also Souza [19,20].

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4. Comparison Theorems

Two comparison theorems between different periodic discrete-time Riccati equaitons are now presented, with at first the same matrices A_j and B_j in both equations. Thus together with the PDRE (1.1) we consider the second PDRE

$$X_{j-1} = A_j^* X_j A_j + \widetilde{Q}_j - (B_j^* X_j A_j + \widetilde{C}_j)^* (\widetilde{R}_j + B_j^* X_j B_j)^{-1} (B_j^* X_j A_j + \widetilde{C}_j)\,, \quad j = 1, \cdots, p\,, \quad (4.1)$$

where $\widetilde{Q}_j^* = \widetilde{Q}_j$ and $\widetilde{R}_j > 0$.

Lemma 4.1. *Let $\{X_j\}_{j=1}^p$ be the Hermitian solution of PDRE (4.1) such that $R_j + B_j^* X_j B_j > 0$, $j = 1, \cdots, p$. Then we have*

$$\mathscr{R}(X_j) = \begin{bmatrix} I_{n_j} & -L_j^* \end{bmatrix} \begin{bmatrix} Q_j - \widetilde{Q}_j & (C_j - \widetilde{C}_j)^* \\ C_j - \widetilde{C}_j & R_j - \widetilde{R}_j \end{bmatrix} \begin{bmatrix} I_{n_j} \\ -L_j \end{bmatrix} + G_j^* (\widetilde{R}_j + B_j^* X_j B_j) G_j\,,$$

for $j = 1, \cdots, p$, where

$$L_j = (R_j + B_j^* X_j B_j)^{-1} (C_j + B_j^* X_j A_j)\,, \quad G_j = L_j - (\widetilde{R}_j + B_j^* X_j B_j)^{-1} (\widetilde{C}_j + B_j^* X_j A_j)\,.$$

Proof. Define

$$H_j = R_j + B_j^* X_j B_j\,, \qquad \widetilde{H}_j = \widetilde{R}_j + B_j^* X_j B_j\,, \qquad E_j = C_j + B_j^* X_j A_j\,, \qquad \widetilde{E}_j = \widetilde{C}_j + B_j^* X_j A_j\,.$$

Then we have

$$L_j = H_j^{-1} E_j\,, \qquad G_j = H_j^{-1} E_j - \widetilde{H}_j^{-1} \widetilde{E}_j\,, \qquad H_j - \widetilde{H}_j = R_j - \widetilde{R}_j\,, \qquad C_j - \widetilde{C}_j = E_j - \widetilde{E}_j\,,$$

hence from the hypothesis that $\{X_j\}_{j=1}^p$ solves (4.1) we obtain

$$\begin{aligned}
&\mathscr{R}(X_j) - \begin{bmatrix} I_{n_j} & -L_j^* \end{bmatrix} \begin{bmatrix} Q_j - \widetilde{Q}_j & (C_j - \widetilde{C}_j)^* \\ C_j - \widetilde{C}_j & R_j - \widetilde{R}_j \end{bmatrix} \begin{bmatrix} I_{n_j} \\ -L_j \end{bmatrix} + G_j^* \widetilde{H}_j G_j \\
&= (-X_{j-1} + A_j^* X_j A_j + Q_j - E_j^* H_j^{-1} E_j) + (X_{j-1} - A_j^* X_j A_j - \widetilde{Q}_j + \widetilde{E}_j^* \widetilde{H}_j^{-1} \widetilde{E}_j) \\
&\quad - (Q_j - \widetilde{Q}_j) + L_j^* (C_j - \widetilde{C}_j) + (C_j^* - \widetilde{C}_j^*) L_j - L_j^* (R_j - \widetilde{R}_j) L_j \\
&\quad - (H_j^{-1} E_j - \widetilde{H}_j^{-1} \widetilde{E}_j)^* \widetilde{H}_j (H_j^{-1} E_j - \widetilde{H}_j^{-1} \widetilde{E}_j) \\
&= - E_j^* H_j^{-1} E_j + \widetilde{E}_j^* \widetilde{H}_j^{-1} \widetilde{E}_j + E_j^* H_j^{-1} (E_j - \widetilde{E}_j) + (E_j^* - \widetilde{E}_j^*) H_j^{-1} E_j \\
&\quad - E_j^* H_j^{-1} (H_j - \widetilde{H}_j) H_j^{-1} E_j - (H_j^{-1} E_j - \widetilde{H}_j^{-1} \widetilde{E}_j)^* \widetilde{H}_j (H_j^{-1} E_j - \widetilde{H}_j^{-1} \widetilde{E}_j) \\
&= 0\,,
\end{aligned}$$

to complete the proof. □

The following result generalises Theorem 3.2(ii) in Ref. [17] to the PDRE.

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Theorem 4.1. *Assume that the matrix pairs* $\{(A_j, B_j)\}_{j=1}^p$ *are pd-stabilisable, and that* $R_j > 0$ *for* $j = 1, \cdots, p$ *and the inequalities*

$$\begin{bmatrix} Q_j & C_j^* \\ C_j & R_j \end{bmatrix} \geq \begin{bmatrix} \widetilde{Q}_j & \widetilde{C}_j^* \\ \widetilde{C}_j & \widetilde{R}_j \end{bmatrix}, \quad j = 1, \cdots, p\,, \tag{4.2}$$

hold. If the PDRE (4.1) *admits a Hermitian solution* $\{\widetilde{X}_j\}_{j=1}^p$ *such that* $\widetilde{R}_j + B_j^* \widetilde{X}_j B_j > 0$ *for* $j = 1, \cdots, p$*, then both PDREs* (1.1) *and* (4.1) *admit maximal solutions* $\{X_j^+\}_{j=1}^p$ *and* $\{\widetilde{X}_j^+\}_{j=1}^p$ *respectively, and*

$$X_j^+ \geq \widetilde{X}_j^+, \quad j = 1, \cdots, p\,.$$

Proof. Since $\{\widetilde{X}_j\}_{j=1}^p$ is the Hermitian solution of PDRE (4.1) such that $\widetilde{R}_j + B_j^* \widetilde{X}_j B_j > 0$, $j = 1, \cdots, p$, from Lemma 4.1 we have $\mathscr{R}(\widetilde{X}_j) \geq 0$, $j = 1, \cdots, p$. The inequalities (4.2) imply that $R_j \geq \widetilde{R}_j$, $j = 1, \cdots, p$, and consequently

$$R_j + B_j^* \widetilde{X}_j B_j \geq \widetilde{R}_j + B_j^* \widetilde{X}_j B_j > 0\,, \quad j = 1, \cdots, p\,.$$

Hence Theorem 3.1 is applicable, so there exists the maximal solution $\{X_j^+\}_{j=1}^p$ of the PDRE (1.1) satisfying $X_j^+ \geq \widetilde{X}_j$, $j = 1, \cdots, p$. Using Theorem 3.1 again, we know there exists the maximal solution $\{\widetilde{X}_j^+\}_{j=1}^p$ of (4.1). As in the first part of the proof, we have

$$X_j^+ \geq \widetilde{X}_j^+, \quad j = 1, \cdots, p\,,$$

to complete the proof. □

Let us now consider the new PDRE

$$X_{j-1} = \widetilde{A}_j^* X_j \widetilde{A}_j + \widetilde{Q}_j - (\widetilde{B}_j^* X_j \widetilde{A}_j + \widetilde{C}_j)^* (\widetilde{R}_j + \widetilde{B}_j^* X_j \widetilde{B}_j)^{-1} (\widetilde{B}_j^* X_j \widetilde{A}_j + \widetilde{C}_j)\,, \quad j = 1, \cdots, p\,, \tag{4.3}$$

where all of the matrices A_j, B_j, C_j, Q_j and R_j of (1.1) are subject to change to result in new coefficients $\widetilde{A}_j$, $\widetilde{B}_j$, $\widetilde{C}_j$, $\widetilde{Q}_j$ and $\widetilde{R}_j$, and $\widetilde{Q}_j = \widetilde{Q}_j^*$, $\widetilde{R}_j = \widetilde{R}_j^*$, and the associated function

$$\widetilde{\mathscr{R}}(X_j) = -X_{j-1} + \widetilde{A}_j^* X_j \widetilde{A}_j + \widetilde{Q}_j - (\widetilde{B}_j^* X_j \widetilde{A}_j + \widetilde{C}_j)^* (\widetilde{R}_j + \widetilde{B}_j^* X_j \widetilde{B}_j)^{-1} (\widetilde{B}_j^* X_j \widetilde{A}_j + \widetilde{C}_j)\,, \quad j = 1, \cdots, p\,.$$

From Theorem 13.3.3 in Ref. [9] (cf. also Ref. [22]), it is easy to obtain the following lemma, so its proof is omitted.

Lemma 4.2. *Let us assume that* X_j, $j = 1, \cdots, p$ *are Hermitian matrices such that* $\widetilde{R}_j + \widetilde{B}_j^* X_j \widetilde{B}_j > 0$, $j = 1, \cdots, p$. *If* R_j *and* $\widetilde{R}_j$ *are Hermitian positive definite and*

$$\begin{bmatrix} Q_j - C_j^* R_j^{-1} C_j & A_j^* - C_j^* R_j^{-1} B_j^* \\ A_j - B_j R_j^{-1} C_j & -B_j R_j^{-1} B_j^* \end{bmatrix} \geq \begin{bmatrix} \widetilde{Q}_j - \widetilde{C}_j^* \widetilde{R}_j^{-1} \widetilde{C}_j & \widetilde{A}_j^* - \widetilde{C}_j^* \widetilde{R}_j^{-1} \widetilde{B}_j^* \\ \widetilde{A}_j - \widetilde{B}_j \widetilde{R}_j^{-1} \widetilde{C}_j & -\widetilde{B}_j \widetilde{R}_j^{-1} \widetilde{B}_j^* \end{bmatrix}, \tag{4.4}$$

for $j = 1, \cdots, p$*, then we have*

$$R_j + B_j^* X_j B_j > 0\,, \quad \mathscr{R}(X_j) \geq \widetilde{\mathscr{R}}(X_j)\,, \quad j = 1, \cdots, p\,.$$

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We also have the following lemma.

Lemma 4.3. *If the matrix pairs* $\{(A_j, B_j)\}_{j=1}^p$ *are pd-stabilisable and the inequalities in* (4.4) *hold, then the matrix pairs* $\{(\widetilde{A}_j, \widetilde{B}_j)\}_{j=1}^p$ *are also pd-stabilisable.*

Proof. Let

$$y_j^* \widetilde{A}_j = \lambda_{j-1} y_{j-1}^* , \quad y_j^* \widetilde{B}_j = 0 , \quad j = 1, \cdots, p , \tag{4.5}$$

for all $\lambda_j \in \mathbb{C}$ such that $|\prod_{j=1}^p \lambda_j| \geq 1$, where $\lambda_0 = \lambda_p, y_0 = y_p$. In terms of (4.4) we get

$$0 \leq B_j R_j^{-1} B_j^* \leq \widetilde{B}_j \widetilde{R}_j^{-1} \widetilde{B}_j^* , \quad j = 1, \cdots, p . \tag{4.6}$$

It follows from the second inequality of (4.5) and (4.6) that

$$y_j^* B_j = 0, \quad j = 1, \cdots, p. \tag{4.7}$$

If the difference between the left-hand side and the right-hand side of (4.4) is denoted by Δ_j, then it is easy to verify from (4.7) that

$$[0, y_j^*] \Delta_j \begin{bmatrix} 0 \\ y_j \end{bmatrix} = 0 , \quad j = 1, \cdots, p .$$

Since $\Delta_j \geq 0$, we obtain $[0, y_j^*]\Delta_j = 0$, which implies

$$y_j^* \left[(A_j - B_j R_j^{-1} C_j) - (\widetilde{A}_j - \widetilde{B}_j \widetilde{R}_j^{-1} \widetilde{C}_j) \right] = 0 , \quad j = 1, \cdots, p , \tag{4.8}$$

so from (4.5), (4.7) and (4.8) we have

$$y_j^* A_j = \lambda_{j-1} y_{j-1}^* , \quad j = 1, \cdots, p .$$

Since the matrix pairs $\{(A_j, B_j)\}_{j=1}^p$ are pd-stabilisable, from Proposition 2.1 we obtain $y_j = 0$, $j = 1, \cdots, p$. Consequently, from Proposition 2.1 the matrix pairs $\{(\widetilde{A}_j, \widetilde{B}_j)\}_{j=1}^p$ are pd-stabilisable. □

Now we are ready to state and prove the main comparison theorem between the PDRE (1.1) and (4.3), which is a generalisation of Theorem 13.3.5 in Ref. [9] (cf. also Ref. [22]).

Theorem 4.2. *Assume that the matrix pairs* $\{(A_j, B_j)\}_{j=1}^p$ *are pd-stabilisable, that* $R_j > 0, \widetilde{R}_j > 0$ *for* $j = 1, \cdots, p$, *and that the inequalities in* (4.4) *hold. If there is an Hermitian matrix set* $\{X_j\}_{j=1}^p$ *such that* $\widetilde{\mathscr{R}}(X_j) \geq 0, \widetilde{R}_j + \widetilde{B}_j^* X_j \widetilde{B}_j > 0$ *for* $j = 1, \cdots, p$, *then the PDRE* (1.1) *and* (4.3) *have maximal solutions* $\{X_j^+\}_{j=1}^p$ *and* $\{\widetilde{X}_j^+\}_{j=1}^p$ *respectively, and*

$$X_j^+ \geq \widetilde{X}_j^+ , \quad j = 1, \cdots, p . \tag{4.9}$$

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Proof. From Lemma 4.3, the matrix pairs $\{(\widetilde{A}_j, \widetilde{B}_j)\}_{j=1}^p$ are pd-stabilisable. We can now apply Theorem 3.1, and there exists the maximal solution $\{\widetilde{X}_j^+\}_{j=1}^p$ of the PDRE (4.3). Since $\widetilde{X}_j^+ \geq X_j$, we get $\widetilde{R}_j + \widetilde{B}_j^* \widetilde{X}_j^+ \widetilde{B}_j \geq \widetilde{R}_j + \widetilde{B}_j^* X_j \widetilde{B}_j > 0$ for $j = 1, \cdots, p$. From Lemma 4.2, if $\widetilde{\mathscr{R}}(\widetilde{X}_j^+) = 0$ and $\widetilde{R}_j + \widetilde{B}_j^* \widetilde{X}_j^+ \widetilde{B}_j > 0$, then also $\mathscr{R}(\widetilde{X}_j^+) \geq \widetilde{\mathscr{R}}(\widetilde{X}_j^+) = 0$ and $R_j + B_j^* \widetilde{X}_j^+ B_j > 0$. Applying Theorem 3.1 again, the PDRE (1.1) has the maximal solution $\{X_j^+\}_{j=1}^p$ and the inequalities in (4.9) hold. □

Remark 4.1. The statement of Theorem 4.1 is the case of one periodic system (1.2) with two different cost functionals (1.3). The statement of Theorem 4.2 is the case when two different periodic systems are associated with two cost functionals, respectively.

Remark 4.2. It is notable that Theorem 1 of Souza [19] can be derived from Corollary 3.1 and Theorem 4.1. When $n_j = n$, $m_j = m$, $C_j = 0$ and $R_j = I_m$ for $j = 1, \cdots, p$, the PDRE (1.1) can be written as

$$X_{j-1} = A_j^* X_j A_j - A_j^* X_j B_j (I_m + B_j^* X_j B_j)^{-1} B_j^* X_j A_j + Q_j \,, \quad j = 1, \cdots, p \,. \tag{4.10}$$

Furthermore, the other PDRE has the form

$$X_{j-1} = A_j^* X_j A_j - A_j^* X_j B_j (I_m + B_j^* X_j B_j)^{-1} B_j^* X_j A_j + \widetilde{Q}_j \,, \quad j = 1, \cdots, p \,. \tag{4.11}$$

If the matrix pairs $\{(A_j, B_j)\}_{j=1}^p$ are pd-stabilisable and $Q_j \geq \widetilde{Q}_j \geq 0$, $j = 1, \cdots, p$. then from Corollary 3.1 and Theorem 4.1 there exist maximal solutions $\{X_j^+\}_{j=1}^p$ and $\{\widetilde{X}_j^+\}_{j=1}^p$ of (4.10) and (4.11) respectively, and we have

$$X_j^+ \geq \widetilde{X}_j^+, \quad j = 1, \cdots, p \,,$$

which was proved by Souza [19].

5. Conclusions

The existence and properties of the maximal solution for the PDRE (1.1) were discussed and two comparison theorems for two different PDREs presented, which are extensions to the PDRE of corresponding results for the discrete-time Riccati equation [9, 17, 22]. Our analysis is based on periodic eigenproblems [1, 6, 7, 14] and the necessary and sufficient conditions for pd-stabilisability, pd-detectability and p-observability [5, 6, 23].

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