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Vector bundles and complex polarizations

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Abstract. This paper discusses the local and global geometry of coisotropic foliations and complex polarizations on symplectic manifolds and draws attention to an analogy between coisotropic foliations and Hermitian vector bundles, in which connections and characteristic classes are modelled by objects in symplectic geometry.

1. *Introduction*

In a number of mathematical and physical problems, it is useful to construct 'classical' or symplectic geometric analogues. For example in group theory one can obtain representations of a Lie group from coadjoint orbits in the dual of its Lie algebra (**1**, **5**, **6**). The orbits, which are homogeneous symplectic manifolds, are the classical analogues of the irreducible unitary representations of the group. And, in the theory of Fourier integral operators, one uses the characteristics of a hyperbolic partial differential equation on a manifold to obtain global information about the evolution generated by the equation. The characteristics are generated by a Hamiltonian on the cotangent bundle of the manifold and the flow of this Hamiltonian is the classical analogue of the evolution (**4**, **12**).

The symplectic analogues are used both as technical tools and also, more significantly, as guides in exploring new territory and formulating unifying principles.

The idea of a 'classical analogy' originates in quantum theory: to each quantum system consisting of a Hilbert space and a Hamiltonian operator, there corresponds a classical system consisting of a symplectic phase space and a Hamiltonian function. The classical system provides both a method for constructing the quantum system, by canonical or geometric quantization (**6**, **9**, **10**), and a framework for interpreting it. Both the examples above are suggested by the correspondence between classical and quantum systems: the first by the quantization of systems with symmetry, the second by the WKB approximation.

In this paper, I shall explore a further aspect of the general scheme of classical analogy, namely the analogy between a coisotropic foliation E of a symplectic manifold M and a Hermitian vector bundle over M/E. In particular, I shall show that

(1) a section C of $D = E^0$ (that is, a submanifold $C \subset M$ intersecting each leaf of D transversally in a single point) is analogous to a connection on the vector bundle, and a coisotropic section is analogous to a flat connection;

(2) E is essentially trivial if D admits a coisotropic section. However, there are a number of obstructions on M/E (in the real cohomology) to the existence of such a section: these are analogous to the Chern classes of the vector bundle.

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The analogies become exact on quantization: if E is the real part of $P+\bar{P}$, where P is a polarization, and if M is quantizable, then the polarized wave functions on M correspond to sections of a Hermitian vector bundle over M/E. Every Hermitian vector bundle over a manifold Q arises from the quantization of a polarized symplectic manifold M for which $Q = M/E$. The connections on the vector bundle are associated with certain sections of D and the Chern classes are related to the obstructions by a simple formula.

The construction of M can be thought of as an extension to a general gauge field of the familiar technique for analysing the charged particle orbits of an electromagnetic field on Q of adding the electromagnetic field to the canonical 2-form on T^*Q.

Apart from extending the general scheme of symplectic analogy in a way which may, for example, have applications in the theory of Fourier integral operators on vector bundles, the methods I shall use also lead to some new results on the local and global geometry of complex polarizations.

2. *Coisotropic foliations*

Let (M, ω) be a $2n$-dimensional symplectic manifold and let E be a coisotropic foliation of M of codimension k. That is, E is an integrable distribution on M and, at each $m \in M$,

$$D_m = E_m^0 = \{X \in T_m M \mid \omega(X, Y) = 0 \,\forall\, Y \in E_m\} \subset E_m. \tag{2·1}$$

It is easy to see that D is also integrable and that D_m is an isotropic subspace of $T_m M$ at each point m.

The symplectic structure ω induces a flat partial connection on M, in which vector fields tangent to D can be differentiated along vector fields tangent to E. If

$$X \in V(M; E) \quad \text{and} \quad Y \in V(M; D),$$

then $\nabla_X Y$ is the vector field on M given by

$$\nabla_X Y \lrcorner \omega = X \lrcorner d(Y \lrcorner \omega). \tag{2·2}$$

The properties of ∇ (it is essentially the connection introduced by Bott) are listed in Woodhouse (14). For the present purpose, we shall only need that $\nabla_X Y$ is tangent to D and that ∇ restricts to a flat, torsion-free affine connection on each leaf of D. The covariantly constant vector fields on the leaves of D are the (restrictions of) the Hamiltonian vector fields X_h on M where $h \in C^\infty(M)$ is constant along E: such vector fields are necessarily tangent to D. (It is worth remarking that, when $X \in V(M; D)$ and $Y \in V(M; E)$, equation (2·2) also serves to define $\nabla_X Y \in V(M; E)$: this gives a dual form of the partial connection.)

Example. Suppose that $M = M_1 \times M_2$ (as a symplectic manifold), where $M_1 = T^*Q$ is the cotangent bundle of a k-dimensional manifold Q with the natural symplectic structure and M_2 is some other symplectic manifold. Then M carries a natural coisotropic foliation: the leaves of E are of the form $T_q^* Q \times M_2$ where q is a fixed point of Q and the leaves of D are of the form $T_q^* Q \times m_2$ where $q \in Q$ and $m_2 \in M_2$. The flat affine connection on the leaves of D coincides with that associated with their vector space structure.

A significant feature of this example is that D admits a natural section C which is

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also a coisotropic submanifold of M: $C = Q_0 \times M_2$ where Q_0 is the zero section in T^*Q. ∎

The following proposition shows that the example typifies the local structure of coisotropic foliations.

PROPOSITION 1. *Let E be a reducible coisotropic foliation of a symplectic manifold (M, ω) of codimension k and let D be the corresponding isotropic foliation. Suppose that the leaves of D are simply connected and convex; that D admits a coisotropic section C; and that each leaf Λ of E intersects C in a section of the characteristic foliation K of C. Then (M, ω) can be identified with an open subset of the product of $M_1 = T^*(M/E)$ and $M_2 = C/K$, with C coinciding with $Q_0 \times M_2$ where Q_0 is the zero section in M_1.*

By saying that a foliation is reducible, it is meant that the space of leaves is a Hausdorff manifold. The characteristic (or kernel) foliation of a coisotropic submanifold $C \subset M$ is the distribution K defined by

$$K_m = \{X \in T_m C | \omega(X, Y) = 0 \ \forall Y \in T_m C\} = (T_m C)^0. \tag{2·3}$$

This has dimension equal to the codimension of C and is automatically integrable.

The symplectic form ω_2 on M_2 is that for which

$$\omega|_C = \sigma^*(\omega_2), \tag{2·4}$$

where $\sigma: C \to C/K$ is the projection.

One proof of the proposition is given in Woodhouse ((14), pp. 104–106). An alternative is the following.

Proof. Let $\{x^a\}$ $(a = 1, 2, \ldots, k)$ be a local coordinate system on some open set $U \subset Q = M/E$ and let $q^a = x^a \circ \rho$ where $\rho: M \to Q$ is the projection.

The Hamiltonian vector fields X_{q^a} generated by the q^a's are tangent to and span the leaves of D. Moreover, on each leaf of D, they are constant with respect to the natural flat affine connection.

Let p_a $(a = 1, 2, \ldots, k)$ be smooth functions on M satisfying

$$X_{q^a}(p_b) = \delta^a_b \quad \text{and} \quad p_a = 0 \quad \text{on } C. \tag{2·5}$$

These conditions determine the p_a's uniquely in $\rho^{-1}(U) \subset M$ since, by hypothesis, the leaves of D are simply connected: on each leaf of D, the p_a's are affine linear coordinates taking values in some convex neighbourhood of the origin in $\mathbb{R}^k$.

Now consider the projections

$$\tau: M \to C = M/D, \quad \sigma: C \to M_2 = C/K, \tag{2·6}$$

and their composition $pr_2: M \to M_2$. If Λ is a leaf of E, then Λ/D is a symplectic manifold, canonically diffeomorphic to the symplectic submanifold $\Lambda \cap C \subset M$ under the restricted projection $\tau_\Lambda = \tau|_\Lambda: \Lambda \to C \cap \Lambda = \Lambda/D$. Hence

$$\omega|_\Lambda = \tau_\Lambda^*(\omega|_{\Lambda \cap C}). \tag{2·7}$$

But

$$\omega|_{\Lambda \cap C} = \sigma^*(\omega_2)|_{\Lambda \cap C}, \tag{2·8}$$

by the definition of ω_2. It follows therefore that

$$pr_2^*(\omega_2)|_\Lambda = \omega|_\Lambda, \tag{2·9}$$

for every leaf Λ of E.

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Let y^α $(\alpha = 1, 2, \ldots, 2(n-k))$ be coordinates on some open set $V \subset M_2$ and let $r^\alpha = y^\alpha \circ pr_2$. Then together p_a, q^b, r^α form a coordinate system on

$$\rho^{-1}(U) \cap pr_2^{-1}(V) \subset M,$$

in which C is given by $p_a = 0$ and

$$\left.\begin{aligned} \rho &: (p_a, q^b, r^\alpha) \to x^b = q^b, \\ pr_2 &: (p_a, q^b, r^\alpha) \to y^\alpha = r^\alpha, \\ \tau &: (p_a, q^b, r^\alpha) \to (0, q^b, r^\alpha), \\ \sigma &: (0, q^b, r^\alpha) \to x^b = q^b. \end{aligned}\right\} \tag{2·10}$$

Moreover, in these coordinates, ω must be of the form

$$\omega = dp_a \wedge dq^a + \xi_{ab}\, dq^a \wedge dq^b + \zeta_{a\alpha}\, dq^a \wedge dr^\alpha + \omega_{\alpha\beta}\, dr^\alpha \wedge dr^\beta, \tag{2·11}$$

with appropriate summation conventions. There are no $dp \wedge dr$ nor $dp \wedge dp$ terms since

$$E \quad \text{is spanned by} \quad \frac{\partial}{\partial r^\alpha} \quad \text{and} \quad \frac{\partial}{\partial p_a}, \tag{2·12}$$

and

$$D = E^0 \quad \text{is spanned by} \quad \frac{\partial}{\partial p_a}.$$

The form of the $dp \wedge dq$ terms follows from (2·5).

Equation (2·9) implies that

$$\omega_2 = \omega_{\alpha\beta}\, dy^\alpha \wedge dy^\beta, \tag{2·13}$$

and hence that the coefficients $\omega_{\alpha\beta}$ are independent of the p_a's and q^a's. The fact that C is coisotropic with characteristic foliation spanned by the vector fields $\partial/\partial q^a$ implies that the coefficients ξ_{ab} and $\zeta_{a\alpha}$ vanish when $p_a = 0$. However, ω is closed. Therefore ξ_{ab} and $\zeta_{a\alpha}$ must be independent of p_a and hence they must be identically zero. It follows that

$$\omega = \omega_{\alpha\beta}(r)\, dr^\alpha \wedge dr^\beta + dp_a \wedge dq^a. \tag{2·14}$$

The proof is now completed by noting that, if the coordinates on M_2 or on Q are changed, then the foliations with leaves given by $r^\alpha = \text{const.}$ and by $p_a = \text{const.}$, $q^a = \text{const.}$ are unaltered. Thus M has a natural product structure. Furthermore, the coordinates p_a, q^b transform in the same way as canonical coordinates on $M_1 = T^*Q$. Thus the local identification of M with $M_1 \times M_2$ is natural and respects symplectic structure. ∎

The advantage of this proof is that it also gives some advance insight into the relationship between connections and coisotropic sections. For suppose that $\tilde{C}$ is a second coisotropic section of D which is used to construct a second coordinate system $\tilde{p}_a$, $\tilde{q}^b$, $\tilde{r}^\alpha$ on M. There is no loss of generality in taking $\tilde{q}^a = q^a$ since the projection onto Q is the same in both cases. We then have

$$\tilde{r}^\alpha = r^\alpha(r, q), \quad \tilde{q}^a = q^a, \quad \tilde{p}_a = p_a - A_a(q, r), \tag{2·15}$$

for some functions A_a. The relationship between the $\tilde{p}_a$'s and the p_a's follows from the fact that both restrict to affine linear coordinate systems on the leaves of D. The lack

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of dependence of the $\tilde{r}^\alpha$'s on the p_a's follows from the fact that the leaves of D are given both by $\tilde{r}^\alpha = \text{const.}$, $\tilde{q}^a = \text{const.}$ and by $r^\alpha = \text{const.}$, $q^a = \text{const.}$

Substituting (2·15) into

$$\omega = d\tilde{p}_a \wedge d\tilde{q}^a + \tilde{\omega}_{\alpha\beta} d\tilde{r}^\alpha \wedge d\tilde{r}^\beta = dp_a \wedge dq^a + \omega_{\alpha\beta} dr^\alpha \wedge dr^\beta, \tag{2·16}$$

we obtain the relations

$$2\omega_{\alpha\beta} X^\alpha_a + \frac{\partial A_a}{\partial r^\beta} = 0 \quad \text{where} \quad X^\alpha_a = \frac{\partial r^\alpha}{\partial \tilde{r}^\gamma} \frac{\partial \tilde{r}^\gamma}{\partial q^a}, \tag{2·17}$$

$$2\omega_{\alpha\beta} X^\alpha_a X^\beta_b + \frac{\partial A_a}{\partial q^b} - \frac{\partial A_b}{\partial q^a} = 0. \tag{2·18}$$

These can be put into a more transparent form by regarding

$$A = A_a(q, r) dq^a \tag{2·19}$$

as a 1-form on Q taking values in $C^\infty(M_2)$. With this interpretation, (2·17) and (2·18) combine to give

$$\tfrac{1}{2}[X \lrcorner A,\ Y \lrcorner A]_2 - dA(X, Y) = 0 \quad \forall X, Y \in V(Q). \tag{2·20}$$

Here the square bracket denotes the Poisson bracket on M_2.

If we use C to identify M with a subset of $T^*Q \times M_2$, then *any* section S of D will determine a 1-form A on Q with values in $C^\infty(M_2)$, and conversely provided that the leaves of D are geodesically complete as well as being convex and simply connected. Reversing the above argument, it follows that (2·20) is both a necessary and sufficient condition for S to be coisotropic. The significant point for what follows is the formal similarity between (2·20) and the condition for the curvature of a connection to vanish.

3. *The obstructions*

Any coisotropic section of D will decompose M locally into the product of a cotangent bundle and another symplectic manifold. Thus, since there is no problem in finding local coisotropic sections of D, the local structure of a coisotropic foliation is always trivial: it is the same as that in the example. However, it may not be possible to find coisotropic sections globally.

Suppose that E is reducible, that, for each leaf Λ of E, Λ/D is a compact Hausdorff manifold, and that the leaves of D are simply connected and geodesically complete: each is then an affine space isomorphic to $\mathbb{R}^k$. Under these conditions, there is no obstruction to finding a global section S of D.

If S is coisotropic and satisfies the hypotheses of proposition 1, then the proof extends easily to give a global identification between M and $T^*(M/E) \times M_2$ where M_2 is a compact symplectic manifold, canonically diffeomorphic to each Λ/D. But in general it is not possible to make S coisotropic globally: there are a number of even-degree real cohomology classes on $Q = M/E$ which are naturally associated with E and which must vanish if D admits a coisotropic section. These classes – the 'classical analogues' of the Chern classes – are defined as follows.

Let S be a section of D (that is, S is a $(2n-k)$-dimensional submanifold of M which meets each leaf of D transversally in a single point). Let $m \in S$, let $q = \rho(m)$ be the corresponding point of Q, and let $K_m \subset T_m S$ be the space of tangent vectors to S defined by

$$K_m = \{Y \in T_m S \,|\, \omega(Y, Z) = 0 \ \forall Z \in T_m S \cap E_m\}; \tag{3·1}$$

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K_m is a k-dimensional subspace of T_mS which is mapped isomorphically onto T_qQ under $\rho_*: T_mS \to T_qQ$.

As m varies, the K_m's form a k-dimensional distribution on S. This is integrable when S is coisotropic, in which case K coincides with the characteristic distribution on S, but not in general.

Any vector field on Q lifts uniquely to a vector field Z on S tangent to K. Let $Y_1, Y_2, \ldots \in V(Q)$ and let $Z_1, Z_2, \ldots$ be their lifts to S. For each integer $r \leqslant \frac{1}{2}k$ and for each $q \in Q$, put

$$\gamma_r(Y_1, Y_2, \ldots, Y_{2r})_q = \frac{1}{(2\pi)^s (2r)!} \int_{\Lambda_q \cap S} (Z_{2r} \lrcorner (Z_{2r-1} \lrcorner \ldots \lrcorner (Z_1 \lrcorner \omega^s) \ldots), \tag{3·2}$$

where $s = r + n - k$ and $\Lambda_q = \rho^{-1}(q)$ is the leaf of E corresponding to q. The form $\omega^s = \omega \wedge \omega \wedge \ldots \wedge \omega$ restricts to a non-vanishing $2s$-form on S; thus the integrand on the right-hand side is a $2(n-k)$-form and so the integral makes sense since $\Lambda_q \cap S$ is a $2(n-k)$-dimensional compact symplectic manifold (sympletic manifolds have natural orientations).

It is clear that the γ_r's are skew-symmetric and linear in the Y's over $C^\infty(Q)$. For each r, therefore, γ_r is a $2r$-form on Q. We can also write $\gamma_r = (2\pi)^{-r} \xi_*(\omega^s|_S)$ where ξ_* is the *push-forward* operator associated with the projection $\xi: S \to Q$. For a general p-form α on S, $\xi_*(\alpha)$ is the $(p-N)$-form ($N = n-k$) on Q given at $q \in Q$ by

$$(\xi_* \alpha)(Y_1, \ldots, Y_{p-N}) = \left(\frac{1}{2\pi}\right)^N \int_{\Lambda_q \cap S} \frac{1}{p!} Z_{p-N} \lrcorner \ldots \lrcorner (Z_1 \lrcorner \alpha) \ldots) \tag{3.3}$$

(for $p \geqslant N$). The definition of $\xi_* \alpha$ does not depend on the particular choice of the lifts of $Y_1, \ldots, Y_p$ to S: any vector fields Z_i that project under ξ onto the Y_i's will give the same result.

PROPOSITION 2. *The forms γ_r are closed and their cohomology classes in $H^1(Q, \mathbb{R})$ are independent of S. Moreover, these classes vanish if D admits a coisotropic section.*

Proof. First note that, if S is coisotropic, then $\gamma_r = 0$ for all r since K then coincides with the characteristic foliation on S.

Now suppose that S and S' are two sections of D. Fix $r \leqslant \frac{1}{2}k$ and write $\gamma = \gamma_r$. Let γ' be the form defined in the same way as γ, but with S replaced by S'. We shall show that γ and γ' differ by an exact form by adapting an argument used to prove the homotopy invariance of de Rham cohomology classes and by using the fact that d commutes with ξ_* (that is, $d(\xi_* \theta) = \xi_* d\theta$).

Let $q \in Q$ and let $m \in S \cap \Lambda_q$; let $m' \in S' \cap \Lambda_q$ be the intersection point of S' with the leaf Δ_m of D through m and let $\lambda_m: [0, 1] \to \Delta_m$ be the geodesic in Δ_m joining m to m'. The tangent vector X_m to λ_m at m is the position vector of m' relative to m in the affine space Δ_m.

For each $t \in [0, 1]$, put

$$S_t = \{\lambda_m(t) | m \in S\}. \tag{3·4}$$

Then S_t is a family of $(2n-k)$-dimensional submanifolds of M with $S = S_0$ and $S' = S_1$. Let Σ be the union of the S_t's: Σ is a $(2n-k+1)$-dimensional submanifold of M with boundary made up of S and S', except for singularities where S and S' intersect. Let X be the vector field on Σ tangent to the geodesics λ_m.

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Since ω is closed, $\pounds_X \omega^{r+N} = d(X \lrcorner \omega^{r+N})$. Thus, putting $S = S_t$ by identifying along the geodesics λ_m,

$$\frac{d}{dt}(\omega_t^{r+N}) = d[(X \lrcorner \omega^{r+N})_t], \tag{3·5}$$

where the subscript t denotes restriction to S_t and the derivative with respect to t is taken at fixed points of $S = S_t$. Hence

$$\omega_1^{r+N} - \omega_0^{r+N} = \omega^{r+N}|_{S'} - \omega^{r+N}|_S = d\theta, \tag{3·6}$$

where

$$\theta = \int_0^1 [(X \lrcorner \omega^{r+N})_t]\, dt. \tag{3·7}$$

Now put $\alpha = -(2\pi)^{-r} \xi_*(\theta)$. That is, α is the $(2r-1)$-form on Q given by

$$\begin{aligned} \alpha_q(Y_1, \ldots, Y_{2r-1}) &= \frac{1}{(2\pi)^s (2r-1)!} \int_0^1 \left[\int_{S_t \cap \Lambda_q} X \lrcorner (Z_{2r-1} \lrcorner \ldots \lrcorner (Z_1 \lrcorner \omega^s) \ldots) \right] dt \\ &= \frac{1}{(2\pi)^s (2r-1)!} \int_{\Sigma \cap \Lambda_q} Z_{2r-1} \lrcorner (\ldots \lrcorner (Z_1 \lrcorner \omega^s) \ldots). \end{aligned} \tag{3·8}$$

It is clear that $d\alpha = \gamma - \gamma'$. The proposition now follows immediately since, by taking S' to be a local coisotropic section, we have that γ is locally exact, and hence closed; and, by taking S and S' to be general sections, we have that the cohomology class of γ is independent of S. ∎

The proof can be made more transparent by examining the forms α and γ in more detail in the case $r = 1$. Then α is the 1-form on Q given by

$$\alpha_q(Y) = (2\pi)^{-N-1} \int_{\Sigma \cap \Lambda_q} Z \lrcorner \omega^{N+1} \tag{3·9}$$

(dropping the subscripts on Y and Z). This can be re-expressed as

$$\begin{aligned} (2\pi)^{N+1} \alpha_q(Y) &= (N+1) \int_{\Sigma \cap \Lambda_q} (Z \lrcorner \omega) \wedge \omega^{n-k} \\ &= (N+1) \int_0^1 \left[\int_{S_t \cap \Lambda_q} X \lrcorner ((Z \lrcorner \omega) \wedge \omega^{n-k}) \right] dt \\ &= (N+1) \int_{S \cap \Lambda_q} 2\omega(Z, X)\, \omega^{n-k} \end{aligned} \tag{3.10}$$

since $X \lrcorner \omega|_{\Lambda_q} = 0$ and since $\omega(Z, X)$ is constant along X as a consequence of the geodetic condition $\nabla_X X = 0$.

Now suppose that $S' = C$ is a local coisotropic section of D and that C is used to construct coordinates p_a, q^b, r^α as in §2. Then S is given by k equations of the form

$$p_a - A_a(q, r) = 0, \tag{3·11}$$

where, as before, we can regard $A = A_a dq^a$ as a 1-form on Q taking values in $C^\infty(M_2)$. The vector field X is

$$X = -A_a \frac{\partial}{\partial p_a} \tag{3·12}$$

and if $Y = Y^a(q)\, \partial/\partial q^a$, then

$$Z = Y^a \left[\frac{\partial}{\partial q^a} + \left(\frac{\partial A_b}{\partial q^a} - 2\omega_{\alpha\beta} X_a^\alpha X_b^\beta \right) \frac{\partial}{\partial p_b} - X_a^\alpha \frac{\partial}{\partial r^\alpha} \right], \quad \text{where} \quad 2\omega_{\alpha\beta} X_a^\alpha + \frac{\partial A_a}{\partial r^\beta} = 0. \tag{3·13}$$

The symplectic form ω is given by (2·14).

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Substituting (3·12) and (3·13) into (3·10), we find that

$$(2\pi)^{N+1}\alpha_q(Y) = (N+1)\int_{M_2}(Y \lrcorner A)_q\,\omega_2^N. \tag{3·14}$$

Dealing with γ in the same way,

$$(2\pi)^{N+1}\gamma_q(Y_1, Y_2) = (N+1)\int_{S\cap\Lambda_q}\omega(Z_1, Z_2)\,\omega^N. \tag{3·15}$$

Using (2·14),

$$\begin{aligned}\omega(Z_1, Z_2) &= \tfrac{1}{2}Y_1^a\,Y_2^b\left(\frac{\partial A_b}{\partial q^a}-\frac{\partial A_a}{\partial q^b}\right)-Y_1^a\,Y_2^b\,X_a^\alpha\,X_b^\beta\,\omega_{\alpha\beta}\\ &= dA(Y_1, Y_2)-\tfrac{1}{2}[Y_1 \lrcorner A,\, Y_2 \lrcorner A]_2.\end{aligned} \tag{3·16}$$

But, if $f, g\in C^\infty(M_2)$, then

$$\int_{M_2}[f, g]_2\,\omega_2^N = \int_{M_2}\pounds_{X_f}(g\omega_2^N) = \int_{M_2}d(gX_f \lrcorner \omega_2^N) = 0, \tag{3·17}$$

since M_2 is compact. Therefore

$$(2\pi)^{N+1}\gamma(Y_1, Y_2) = (N+1)\int_{M_2}dA(Y_1, Y_2)\,\omega_2^N, \tag{3·18}$$

and it is obvious by differentiating under the integral sign (equation A 3) that $d\alpha = \gamma$, as required by the proof of the proposition.

One other useful fact can be extracted from these local coordinate expressions, namely a formula for the commutator of two of the vector fields $Z_1, Z_2, \ldots$.

First, some notation. Let $f\in C^\infty(S)$. Then, for each $q\in Q$, f determines a Hamiltonian vector field on the symplectic manifold $\Lambda_q\cap S$. As q varies, these vector fields fit together to form a vector field on S everywhere tangent to the submanifolds $\Lambda_q\cap S$, which will be denoted H_f. If f is extended to a function on M by making it constant along the leaves of D, then, on S, H_f will differ from X_f, the Hamiltonian vector field generated on M by f, by a vector field tangent to D.

LEMMA. *Let* $Y_1, Y_2, Y_3\in V(Q)$ *with* $Y_3 = [Y_1, Y_2]$. *Then* $[Z_1, Z_2] = Z_3 - H_f$ *where* $f = 2\omega(Z_1, Z_2)$.

Proof. It is sufficient to consider the case in which $Y_1 = \partial/\partial x^1$ and $Y_2 = \partial/\partial x^2$ are coordinate vector fields on Q. Then $Y_3 = 0$ and, using (2·14),

$$\begin{aligned}[Z_1, Z_2] \lrcorner \omega_{\alpha\beta}dr^\alpha\wedge dr^\beta &= 2\omega_{\alpha\beta}\left[\frac{\partial X_1^\alpha}{\partial q^2}-\frac{\partial X_2^\alpha}{\partial q^1}+X_1^\gamma\frac{\partial X_2^\alpha}{\partial r^\gamma}-X_2^\gamma\frac{\partial X_1^\alpha}{\partial r^\gamma}\right]dr^\beta\\ &= \frac{\partial}{\partial r^\beta}\left[\frac{\partial A_2}{\partial q^1}-\frac{\partial A_1}{\partial q^2}-2\omega_{\alpha\gamma}X_1^\alpha X_2^\gamma\right]dr^\beta.\end{aligned} \tag{3·19}$$

But

$$2\omega(Z_1, Z_2) = \frac{\partial A_2}{\partial q^1}-\frac{\partial A_1}{\partial q^2}-2\omega_{\alpha\beta}X_1^\alpha X_2^\beta \tag{3·20}$$

and so the lemma follows. ∎

An immediate consequence of this is that K is generally not integrable if S is not coisotropic.

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4. *Prequantization and connections*

Now suppose that (M, ω) is (pre-)quantizable. Then there exists a Hermitian line bundle $L \to M$ and a connection ∇ on L with curvature ω. That is

$$\nabla_X \nabla_Y \phi - \nabla_Y \nabla_X \phi - \nabla_{[X,Y]} \phi = -2i\omega(X, Y)\phi \tag{4·1}$$

for all smooth sections ϕ of L and for all vector fields $X, Y \in V(M)$.

For each $q \in Q = M/E$, L induces a Hermitian line bundle with connection on the symplectic manifold $M_q = \Lambda_q/D$. We shall denote this line bundle L_q. If Δ is a leaf of D in Λ_q, then the fibre of L_q at the corresponding point of M_q is the space of sections of L over Δ which are covariantly constant with respect to ∇. The fibres are one-dimensional since Δ is simply connected and the restriction of ∇ to Δ is flat. The smooth sections of L_q are naturally represented by smooth sections ϕ of L over Λ_q covariantly constant along D. This also gives a convenient representation of the connection on L_q: if A is a vector field on Λ_q which projects onto a vector field B on M_q, then $\nabla_A \phi$ represents the covariant derivative of the corresponding section of L_q along B.

Let W_q denote the space of smooth sections of L_q. Since M_q is compact, there is a well-defined but incomplete L^2 inner product on W_q. Thus, as q varies, the W_q's will fit together to form the fibres of an infinite-dimensional Hermitian vector bundle $W \to Q$.

Any smooth section ϕ of L covariantly constant along D will determine a section w_ϕ of W by restriction to the leaves of E. Let $\Gamma(W)$ denote the subspace of sections of W that arise in this way.

LEMMA. *For each $q \in Q$, there is an open neighbourhood $U \ni q$ in Q and a coisotropic section C of D in $\rho^{-1}(U)$.*

Proof. We have to construct a local coisotropic section of D which is large enough to have the property that, if it intersects a leaf Λ of E, then it also intersects every leaf of D in Λ.

Let S be a global section of D and let Y be a vector field on Q which is non-zero in some neighbourhood V of q. Let Z be the lift of Y to S (as in § 3) and, at each $m' \in M$, put

$$h(m') = 2\omega_m(X, Z), \tag{4·2}$$

where m is the intersection point with S of the leaf Δ of D through m' and X is the position vector of m' relative to m in Δ. Then h restricts to a linear function on each leaf of D and has non-vanishing gradient in $\rho^{-1}(V)$.

In $\rho^{-1}(V)$, the Hamiltonian vector field $X_{h^2} = 2hX_h$ generated by h^2 vanishes only on S. Let σ_t ($t \in \mathbb{R}$) be the flow of X_{h^2} and let N be some open neighbourhood of $S \cap \Lambda_q$ in M. If N is sufficiently small, then $\sigma_1 : N \to M$ will be a well-defined canonical transformation fixing $\Lambda_q \cap S$. Let $C' = \sigma_1(\Lambda_q \cap N)$: C' is a coisotropic submanifold of M intersecting Λ_q transversally along S. Thus C' is transversal to D on $S \cap \Lambda_q$ and hence in some neighbourhood of $S \cap \Lambda_q$. Therefore, since $S \cap \Lambda_q$ is compact, there is a neighbourhood U of q and an open subset C of C' such that C is a section of D in $\rho^{-1}(U)$. ∎

Applying the Proposition 1 with M replaced by $\rho^{-1}(U)$, we have that

$$\rho^{-1}(U) = T^*U \times M_2$$

as a symplectic manifold. The whole of T^*U is included since the leaves of D are geodesically complete; $M_2 = C/K$ is a compact symplectic manifold canonically

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diffeomorphic to each $\Lambda_{q'}/D$, $q' \in U$, the diffeomorphism being given by identifying both M_2 and $\Lambda_{q'}/D$ with $C \cap \Lambda_{q'}$.

This decomposition of $\rho^{-1}(U)$ extends to the prequantum line bundle L and thereby gives a local trivialization of W.

LEMMA. *If U is simply connected, then $L|_{\rho^{-1}(U)} = pr_2^*(L_2)$ where L_2 is a prequantum line bundle on M_2.*

Proof. By parallel propagation along the leaves of D (on which the connection on L is flat) we can identify $L|_{\rho^{-1}(U)}$ with $\tau^*(L|_C)$ where $\tau: \rho^{-1}(U) \to C$ is the projection along D.

Let L_2 be the prequantum bundle induced by $L|_C$ on M_2: at $m_2 \in M_2$, the fibre of L_2 is the space of covariantly constant sections of $L|_C$ over the corresponding leaf of K in C. As before, this is a one-dimensional complex vector space since the leaves of K in C are isotropic and simply connected (they are each diffeomorphic to U). As with the bundles L_q, L_2 inherits a connection from $L|_C$ and it is clear that L_2 is a prequantum bundle over M_2. In fact, if we identify M_2 with $\Lambda_{q'} \cap S$, then we can identify L_2 with $L_{q'}$ by identifying both with $L|_{\Lambda_{q'} \cap S}$: any section of $L|_{\Lambda_{q'} \cap S}$ extends in a unique way both to a section of $L|_C$ covariantly constant along K and to a section of $L|_{\Lambda_{q'}}$ covariantly constant along D. Moreover, it follows from the construction that $L|_C = \sigma^*(L_2)$ and hence that $L = pr_2^*(L_2)$ since $pr_2 = \sigma \circ \tau$. ∎

An immediate corollary is that the restriction of W to $\rho^{-1}(U)$ can be identified with $U \times W_2$ where W_2 is the space of smooth sections of L_2 and that the elements of $\Gamma(W|_U)$ can be identified with smooth sections of $L|_C$ and hence with functions on U taking values in W_2.

The point of this construction is that, if S is any global section of D, then we can use S to define a connection $\hat{\nabla}$ on W. More precisely, we can associate with any $Y \in V(Q)$ an operator $\hat{\nabla}_Y: \Gamma(W) \to \Gamma(W)$; $\hat{\nabla}_Y$ is defined by

$$\hat{\nabla}_Y w_\phi = w_\psi, \tag{4·3}$$

where ψ is the unique section of L equal to $\nabla_Z \phi$ on S and covariantly constant along D. As before, Z is the vector field on S which is tangent to the distribution K on S and which projects onto Y under $S \to Q$.

Let $\{p_a, q^b, r^\alpha\}$ be a local coordinate system on M constructed from C as in §2 and let

$$\theta = p_a dq^a + \theta_\alpha(r) dr^\alpha, \tag{4·4}$$

where $d(\theta_\alpha dr^\alpha) = \omega_{\alpha\beta} dr^\alpha \wedge dr^\beta$. Then $d\theta = \omega$ and we can use θ as a local connection form for ∇, representing the sections of L locally by complex-valued functions of p_a, q^b, and r^α. In particular, the sections of L corresponding to elements of $\Gamma(W)$ will be represented by functions $\phi(q, r)$ independent of the p_a's.

Let $Y_a = \partial/\partial x^a$ and suppose that S is given by $p_a = A_a(q, r)$. Then

$$Z_a = \frac{\partial}{\partial q^a} - X_a^\alpha \frac{\partial}{\partial r^\alpha} + \text{terms in} \frac{\partial}{\partial q_a}, \tag{4·5}$$

with X_a^α as in (3·13). Hence, if $\phi(q, r)$ represents $w_\phi \in \Gamma(W)$, then

$$\nabla_{Z_a} \phi = \frac{\partial \phi}{\partial q_a} - X_a^\alpha \frac{\partial \phi}{\partial r^\alpha} - iA_a \phi + iX_a^\alpha \theta_\alpha \phi \tag{4·6}$$

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on S. But, for fixed q, $X_a^\alpha \partial/\partial r^\alpha$ is simply the Hamiltonian vector field on M_2 generated by the function $A_a(q,r) \in C^\infty(M_2)$. Therefore

$$X_a^\alpha \frac{\partial \phi}{\partial r^\alpha} - iX_a^\alpha \theta_\alpha \phi + iA_a \phi = i\hat{A}_a(\phi), \tag{4·7}$$

where, for fixed q, $\hat{A}_a$ is the prequantum operator generated by A_a on M_2.

To put these pieces together: we can represent W locally as $U \times W_2$ where $U \subset Q$ is some open neighbourhood and W_2 is the space of smooth sections of the prequantum bundle over M_2. The elements of $\Gamma(W)$ are then represented by functions w on U taking values in W_2 and the connection $\hat{\nabla}$ is given by

$$\hat{\nabla}_X w = X(w) - i\widehat{X \lrcorner A}(w); \quad X \in V(Q). \tag{4·8}$$

Here $X(w)$ is simply the derivative of w along X and, as in § 2, $A = A_a dq^a$ is regarded as a 1-form on Q taking values in $C^\infty(M_2)$. The operator $\widehat{X \lrcorner A}$ is obtained by fixing $q \in Q$ and applying the prequantization construction on M_2 to the function $X \lrcorner A$ on M_2.

We see here the analogy between sections of D and connections on vector bundles becoming sharper. The $C^\infty(M_2)$-valued 1-form A associated with S becomes on quantization the Christoffel symbol of a connection on W. In the next section, we shall make the analogy complete by using a polarization to pick out a finite-dimensional sub-bundle of W. First, however, let us look at the curvature of $\hat{\nabla}$.

Let $Y_1, Y_2, Y_3 = [Y_1, Y_2] \in V(Q)$ and let Z_1, Z_2, Z_3 be the corresponding vector fields on S. Let ϕ be a section of L covariantly constant along D. Then

$$\tfrac{1}{2}i(\hat{\nabla}_{Y_1}\hat{\nabla}_{Y_2} - \hat{\nabla}_{Y_2}\hat{\nabla}_{Y_1} - \hat{\nabla}_{[Y_1,Y_2]})\, w_\phi = w_\psi \tag{4·9}$$

where, on S,

$$\begin{aligned} \psi &= \tfrac{1}{2}i(\nabla_{Z_1}\nabla_{Z_2} - \nabla_{Z_2}\nabla_{Z_1} - \nabla_{Z_3})\,\phi \\ &= \tfrac{1}{2}(-i\nabla_{H_f} + f)\,\phi; \quad f = 2\omega(Z_1, Z_2) \end{aligned} \tag{4·10}$$

using the lemma of the last section.

Now the map

$$(Y_1, Y_2) \to \Omega(Y_1, Y_2) = \tfrac{1}{2}f = \omega(Z_1, Z_2) \tag{4·11}$$

defines a 2-form on Q taking values in $C^\infty(M_q)$ at $q \in Q$: to each pair of vector fields Y_1, Y_2 on Q and, to each point $q \in Q$, it assigns a function $\Omega_q(Y_1, Y_2)$ on M_q. The expression (4·9) for ψ shows that the curvature $\hat{\Omega}$ of $\hat{\nabla}$ is given by applying prequantization to Ω on the symplectic manifolds M_q. That is, for each $X_1, X_2 \in V(Q)$ and for each $q \in Q$, $\hat{\Omega}_q(X_1, X_2)$ is the linear operator on W_q obtained by quantizing $\Omega_q(X_1, X_2) \in C^\infty(M_q)$.

5. *Quantization*

A strongly integrable polarization P of M is a distribution that assigns a complex Lagrangian subspace $P_m \subset T_m^{\mathbb{C}} M$ to each $m \in M$, with the properties

(1) P is integrable: in some neighbourhood of each $m \in M$, there exist independent functions $f_1, f_2, \ldots, f_n \in C^\infty_{\mathbb{C}}(M)$ such that $\bar{X}(f_i) = 0$ for every $X \in V_{\mathbb{C}}(M; P)$;

(2) $P \cap \bar{P}$ has constant dimension;

(3) the real part of $P + \bar{P}$ is an integrable real distribution.

Under these conditions, the real part of $P + \bar{P}$ is a coisotropic foliation of M and the real part of $P \cap \bar{P}$ is the corresponding isotropic foliation. We shall suppose that M is

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equipped with a strongly integrable polarization P for which $P+\overline{P}=E^{\mathbb{C}}$ and $P\cap\overline{P}=D^{\mathbb{C}}$ where E and D are as in §2–4. Then each of the compact symplectic manifolds M_q will inherit from P the structure of a Kähler manifold; P is said to be *positive* if the Kähler metric on each M_q is positive definite.

Associated with P, there are the various function spaces introduced by Kostant (7, 8). Of these, we shall be interested in two: the first is the space (or, more properly, sheaf) of (local) P-functions; that is, functions which are constant along $\overline{P}$: $f\in C^\infty_{\mathbb{C}}(M)$ ($C^\infty_{\mathbb{C}}(U)$) is a P-function (local P-function on $U\subset M$) if f is constant along D and if the restriction of f to each leaf Λ_q of E is equal to the pull-back to Λ_q of a (local) holomorphic function on M_q. Alternatively, $f\in C^\infty_{\mathbb{C}}(M)$ is a P-function if and only if X_f is everywhere tangent to $\overline{P}$. Of course in general there will be no global P-functions, but the existence of local P-functions is guaranteed by the integrability of P.

The second is the space of P^1-functions: a function $f\in C^\infty_{\mathbb{C}}(M)$ ($C^\infty_{\mathbb{C}}(U)$) is a P^1-function (local P^1-function on $U\subset M$) if $X_f(g)$ is a local P-function for every local P-function g.

A (local) section S of D is called a *P^1-section* if it is given locally by k independent equations of the form

$$f_a=0;\quad a=1,2,\ldots,k, \tag{5·1}$$

where the f_a's are real local P^1-functions. We say that P is *locally trivial* if, for some neighbourhood U of every $q\in Q$, there exists a coisotropic P^1-section S of D in $\rho^{-1}(U)\subset M$. Essentially, this says that the complex structure on the manifolds M_q does not vary with q.

Examples. (1) Returning to the example of §2, suppose that M_2 is a Kähler manifold. Then $T^*Q\times M_2$ carries a natural strongly integrable polarization P and $Q_0\times M_2$, where Q_0 is the zero-section in T^*Q, is a global coisotropic P^1-section.

(2) Let $z_1(q)$, $z_2(q)$ be two real curves in $\mathbb{C}$ parametrized by $q\in\mathbb{R}$ such that

$$z(q)=z_1(q)/z_2(q), \tag{5·2}$$

is not constant and is not real for any value of q. Let M_2 be the torus obtained by identifying the sides of the unit square $\{0\leqslant r^1\leqslant 1, 0\leqslant r^2\leqslant 1\}$ in $\mathbb{R}^2$ with the symplectic structure

$$\omega_2=dr^1\wedge dr^2. \tag{5·3}$$

Let $M=T^*\mathbb{R}\times M_2$ with coordinates p, q, r^1, r^2 and put

$$f=q,\quad f'=z_1(q)\,r^1+z_2(q)\,r^2. \tag{5·4}$$

Then $\overline{X}_f$ and $\overline{X}_{f'}$ span a complex polarization P for which the leaves of E are given by $q=$ const. and the leaves of D by $q=$ const., $r^1=$ const., $r^2=$ const.

As a complex manifold, M_q ($q\in Q=\mathbb{R}$) is the torus $\mathbb{C}/G$ where G is the subgroup of $\mathbb{C}$ generated by $z_1(q)$ and $z_2(q)$. Note that M_q and $M_{q'}$ are distinct complex manifolds except for special choices of q and q': M_q and $M_{q'}$ are isomorphic as complex manifolds if and only if $z(q)$ and $z(q')$ are related by

$$z(q)=(az(q')+b)/(cz(q')+d) \tag{5·5}$$

for some a, b, c, $d\in\mathbb{Z}$ such that $ad-bc=\pm 1$ (see, for example, Gunning (3)). ∎

We shall see shortly that, if the coisotropic section C in proposition 1 is also a P^1-section, then we can strengthen the statement: M_2 is a Kähler manifold and, when we

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represent M as the product $T^*(M/E) \times M_2$, P is the corresponding polarization on M (the product of the vertical polarization of $T^*(M/E)$ and the Kähler polarization on M_2). By applying this with M replaced by $\rho^{-1}(U)$, it follows that, if P is locally trivial, then for nearby $q, q' \in M/E$, M_q and $M_{q'}$ must be isomorphic complex manifolds (if P is a product polarization, then both M_q and $M_{q'}$ are isomorphic to M_2). Thus the second example above is a polarization that is *not* locally trivial.

It is important to note that, if C is coisotropic, but not P^1, then although $M = T^*(M/E) \times M_2$ as a symplectic manifold, there is no natural Kähler structure on M_2 and P does not arise as in the first example above. Although we get a complex structure on M_2 by identifying M_2 with M_q for some $q \in Q$, this complex structure will vary as q varies, as in the second example.

LEMMA. *A section $S \subset M$ of D is a P^1-section if and only if, in the coordinates of* §2, *S is given by $p_a = A_a(q, r)$ where $g_a = p_a - A_a$ is a P^1-function for $a = 1, 2, \ldots, n$.*

Proof. The 'if' part is trivial. To establish the converse, suppose that $S \subset M$ is a P^1-section of D given locally by (5·1). Since each f_a is a P^1-function and each of the coordinates q^a is a P-function, each

$$X_{f_a}(q^b) = \frac{\partial f_a}{\partial p_b}, \tag{5·6}$$

is also a P-function. Moreover, the matrix with entries $\partial f_a/\partial p_b$ is nonsingular everywhere on S since S cuts the leaves of D transversally.

Now suppose that S is also given by $g_a = p_a - A_a(q, r) = 0$. Then

$$f_a(A(q, r), q, r) = 0, \tag{5·7}$$

identically in q and r. Hence if h is a P-function

$$\frac{\partial f_c}{\partial p_b} X_{g_b}(h) = X_{f_c}(h), \tag{5·8}$$

on S. Thus on S, $X_{g_b}(h)$ is equal to the restriction to S of a P-function. But $X_{g_b}(h)$ is independent of the coordinates p_a. It follows that the $X_{g_b}(h)$'s are P-functions and hence that the g_a's are P^1-functions.

COROLLARY. *Let C be a global coisotropic section of D as in proposition* 1. *Then M is the product of $M_1 = T^*Q$ and $M_2 = C/K$. Moreover, if C is P^1, then there is a natural Kähler structure on M_2 and P is the product of the vertical polarization on T^*Q and the Kähler polarization on M_2.*

Proof. We can proceed as in §2 to obtain the identification between M and $M_1 \times M_2$: we now get the whole of M_1 since we are now assuming that the leaves of D are geodesically complete. For each $q \in Q$, we can identify $M_q = \Lambda_q/D = \Lambda_q \cap C$ (as symplectic manifolds) with $M_2 = C/K = \Lambda_q \cap C$ and hence obtain a complex structure J_q on M_2. All that we have to show is that J_q is independent of q.

In the local coordinates p_a, q^b, r^α associated with C, C is given by $p_a = 0$. Hence, by the lemma above, the p_a's are P^1-functions.

Let $h(q, r)$ be a local P-function. In other words, for each $q \in Q$, h is a complex-valued function on M_2 which is holomorphic with respect to J_q. Then

$$X_{p_a}(h) = \frac{\partial h}{\partial q^a} \tag{5·9}$$

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is also a P-function and hence is holomorphic with respect to J_q for each fixed q. It follows that

$$\frac{\partial}{\partial q^a}(J_q) = 0, \tag{5·10}$$

and therefore that J_q is independent of q. ∎

Let us now suppose that P is locally trivial and that L is a prequantum line bundle over M as in §4. Then each L_q is a holomorphic line bundle with holomorphic sections represented by sections of L over Λ_q satisfying

$$\nabla_{\overline{X}}\phi = 0, \tag{5·11}$$

for every complex vector field X on Λ_q tangent to P. The holomorphic (or polarized) sections form a subspace $V_q \subset W_q$ which is finite-dimensional since M_q is compact.

Thus the quantization construction associates a Hermitian vector bundle V to (M, ω) and P: V is locally trivial since P is. If we pick a neighbourhood $U \subset Q$ such that, in $\rho^{-1}(U)$, D admits a coisotropic P^1-section C, then $\rho^{-1}(U) = T^*U \times M_2$, with P coinciding with the product of the vertical polarization of T^*U and the Kähler polarization of M_2. Moreover, the line bundle $L_2 \to M_2$ is also holomorphic (its holomorphic sections correspond to sections of L over C satisfying (5·11) for every X on C tangent to $P|_C$) and $V|_U = U \times V_2$ where $V_2 \subset W_2$ is the space of polarized sections of L_2 over M_2.

The Hilbert space constructed from (M, ω) and P by quantization is simply the space of square integrable sections of V over Q. Some examples are given in Woodhouse (**14**).

Let $\Gamma(V) \subset \Gamma(W)$ be the space of smooth sections of L satisfying $\nabla_{\overline{Z}}\phi = 0$ for every $Z \in V_{\mathbb{C}}(M, P)$: the elements of $\Gamma(V)$ represent the smooth sections of V over Q. Let $\hat{\nabla}$ be the connection on W associated with a section S of D.

LEMMA. *If S is a P^1-section, then $\hat{\nabla}_X\colon \Gamma(V) \to \Gamma(V)$ for every $X \in V(Q)$.*

Proof. Let $X \in V(Q)$ and let C be a coisotropic P^1-section of D in $\rho^{-1}(U)$ where U is some neighbourhood in Q. Let p_a, q^b, r^α be corresponding coordinates on M. Since C is a P^1-section, the p_a's are P^1-functions. Therefore S is given by $p_a = A_a(q, r)$ where the A_a's are P^1-functions.

The section C splits $\rho^{-1}(U)$ into a product $T^*U \times M_2$, with $\hat{\nabla}$ given by equation (4·8). Since the A_a's are P^1-functions, $X \lrcorner A = X^a A_a(q, r)$ is a P^1-function on M_2 for every $q \in Q$. Thus if $v\colon U \to V_2 \subset W_2$ is a local representative of an element of $\Gamma(V)$, then $\widehat{X \lrcorner A}(v)$ takes values in V_2 since the operators corresponding to P^1-functions on M_2 take polarized sections of L_2 into polarized sections. Therefore $\hat{\nabla}_X v$ also takes values in V_2 and so $\hat{\nabla}_X\colon \Gamma(V) \to \Gamma(V)$. ∎

Finally, therefore, we have:

PROPOSITION 3. *Let P be a locally trivial polarization of M such that $P + \overline{P} = E^{\mathbb{C}}$. Then the quantization construction associates to P a Hermitian vector bundle $V \to Q$ and, to every P^1-section of D, it associates a connection $\hat{\nabla}$ on V. If the section is coisotropic, then $\hat{\nabla}$ is flat.*

In the final section, it will be shown that every Hermitian vector bundle V and every connection on V arises in this way and that there is a close relationship between the Chern classes of V and the obstructions introduced in §3.

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6. *Chern classes*

Let $V \to Q$ be a Hermitian vector bundle of rank $N+1$ over a k-dimensional real manifold Q and let $\hat{\nabla}$ be a connection on V compatible with the Hermitian structure. Then the dual bundle $V^* \to Q$ also carries a Hermitian structure and a compatible connection ∇^*.

Our aim is to construct a polarized symplectic manifold from V^* from which V and $\hat{\nabla}$ can be recovered by quantization, and thus to establish the generality of the construction of §5.

For each $q \in Q$, V_q^* is a $2(N+1)$-dimensional real vector space equipped with a complex structure J_q: $V_q^* \to V_q^*$ and a symplectic form σ_q such that

$$\sigma_q(J_q U, J_q U') = \sigma_q(U, U') \quad \forall U, U' \in V_q^*. \tag{6·1}$$

These are related to the Hermitian inner product $\langle .,. \rangle_q$ on V_q^* by

$$\langle U, U' \rangle_q = \sigma_q(U, J_q U') - i\sigma_q(U, U'). \tag{6·2}$$

On the total space V^*, thought of as a $(2N+k+2)$-dimensional real manifold, let τ be the 1-form defined at $(q, U) \in V^*$ by

$$\tau_{(q,U)}(X) = \sigma_q(U, X_{\text{ver}}); \quad X \in T_{(q,U)} V^*, \tag{6·3}$$

where X_{ver} is the vertical part of X. Then τ is related to the curvature of ∇^* by the following lemma.

LEMMA. *If $X, X' \in T_{(q,U)} V^*$, then*

$$d\tau_{(q,U)}(X, X') = \sigma_q(X_{\text{ver}}, X'_{\text{ver}}) + \sigma_q(U, J_q \hat{\Omega}_q^*(Y, Y') U),$$

where $Y, Y' \in T_q Q$ are the projections into Q of X and X'.

Here $\hat{\Omega}$ is the curvature of $\hat{\nabla}$: to each $Y, Y' \in T_q Q$, it assigns a complex linear transformation $\hat{\Omega}_q(Y, Y')$: $V_q \to V_q$, self-adjoint with respect to $\langle .,. \rangle$. The dual transformation $\hat{\Omega}_q^*(Y, Y')$: $V_q^* \to V_q^*$ is *minus* the curvature of ∇^*. Thus, for a section W of V^*,

$$\tfrac{1}{2} J(\nabla_Y^* \nabla_{Y'}^* - \nabla_{Y'}^* \nabla_Y^* - \nabla_{[Y,Y']}^*) W = -\hat{\Omega}^*(Y, Y') W. \tag{6·4}$$

Proof. First note that, if X and X' are vertical, then $Y = Y' = 0$ and the statement is an immediate consequence of equation (6·3).

Let W: $Q \to V^*$ be a section of V^* and let $Y, Y' \in V(Q)$. Put $\alpha = W^* \tau$. Then

$$\alpha(Y) = \sigma(W, \nabla_Y^* W), \tag{6·5}$$

since if $X = W_* Y$ then $X_{\text{ver}} = \nabla_Y^* W$. Hence $d\alpha = W^* d\tau$ is given by

$$\begin{aligned} 2d\alpha(Y, Y') &= Y(\alpha(Y')) - Y'(\alpha(Y)) - \alpha([Y, Y']) \\ &= Y(\sigma(W, \nabla_{Y'}^* W)) - Y'(\sigma(W, \nabla_Y^* W)) - \sigma(W, [Y, Y']). \end{aligned} \tag{6·6}$$

But $Y(\sigma(W, \nabla_{Y'}^* W)) = \sigma(\nabla_Y^* W, \nabla_{Y'}^* W) + \sigma(W, \nabla_Y^* \nabla_{Y'}^* W)$. Therefore

$$\begin{aligned} d\alpha(Y, Y') &= \sigma(X_{\text{ver}}, X'_{\text{ver}}) + \tfrac{1}{2}\sigma(W, (\nabla_Y^* \nabla_{Y'}^* - \nabla_{Y'}^* \nabla_Y^* - \nabla_{[Y,Y']}^*) W) \\ &= \sigma(X_{\text{ver}}, X'_{\text{ver}}) + \sigma(W, J\Omega^*(Y, Y') W), \end{aligned} \tag{6·7}$$

where $X = W_* Y$ and $X' = W_* Y'$. The lemma now follows since we can pick W to be an arbitrary section through (q, U). ∎

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Now the construction of the symplectic manifold. Consider the Whitney sum of V^* and T^*Q (as real vector bundles). The total space M_0 is a real manifold of dimension $2N+2k+2$, the points of which are represented by triples (p, q, U) in which $q \in Q$, $p \in T_q^*Q$, $U \in V_q^*$. Let ω_0 be the symplectic form on M_0 given, with a slight abuse of notation, by

$$\omega_0 = d(\tau + \theta_C) = d\tau + \omega_C, \tag{6·8}$$

where θ_C and ω_C are the canonical 1- and 2-forms on T^*Q.

Let $h: M_0 \to \mathbb{R}$ be the function $h(p, q, U) = \langle U, U \rangle_q$ and let (M, ω) be the reduction of the constraint $h = 1$. In other words, M is obtained by factoring out the characteristic foliation of $\omega_0|_{h=1}$ on the submanifold $h = 1$; (M, ω) is a symplectic manifold of dimension $2n = 2(N+k)$.

With $q \in Q$ fixed, the reduction of $S_q = \langle U, U \rangle_q$ in (V_q^*, σ_q) is the Kähler manifold PV_q^*, isomorphic to $\mathbb{C}P_N$. Thus M carries a natural strongly integrable complex polarization P for which the leaves of E are given by fixing $q \in Q$ and for which the M_q's are the projective spaces PV_q^*. Moreover, there is a natural section of $D = E^0$ in M on which $p = 0$: it can be identified with the complex projective bundle $PV^* \subset M$. It is claimed that V arises from the quantization of M, that the section PV^* is a P^1-section of D, and that the corresponding connection on V is $\hat{\nabla}$.

The first statement follows from the results of Appendix B.

We can introduce coordinates p_a, q^b, $Z^j = z^j/z^0$ on M by taking the q^a's to be coordinates on Q, the p_a's to be the corresponding momentum components on T^*Q, and the z^j's to be complex linear coordinates on the fibres of V^*. Then, from the lemma above,

$$\omega = dp_a \wedge dq^a + \text{terms in } dZ \wedge d\bar{Z} \text{ and } dq \wedge dq. \tag{6·9}$$

The local P-functions on M are of the form $f = f(q^a, Z^j)$ and, from (6·9), the Hamiltonian vector fields X_{p_a} are of the form

$$X_{p_a} = \frac{\partial}{\partial q^a} + \text{terms in} \frac{\partial}{\partial p_b}. \tag{6·10}$$

Consequently, $X_{p_a}(f)$ is a P-function whenever f is, and so $PV^* = \{p_a = 0\}$ is a P^1-section of D.

Finally, to see that the corresponding connection is $\hat{\nabla}$, note that a section of V can be represented by a function $\phi: V^* \to \mathbb{C}$ which is holomorphic and homogeneous of degree one on each fibre. The covariant derivative along $Y \in V(Q)$ is then represented by $Z_0(\phi)$ where Z_0 is the horizontal lift of Y to V^*.

The sections of the prequantum bundle $L \to M$ can be represented in a similar way. First take the trivial bundle $L_0 = M_0 \times \mathbb{C}$ with the connection

$$\nabla_X \psi_0 = X(\psi_0) - i(X \lrcorner \theta_0)\psi_0; \quad \psi_0 \in C_{\mathbb{C}}^\infty(M_0), \quad X \in V(M_0), \tag{6·11}$$

where $\theta_0 = \tau + \theta_C - (i/2)dh$, with $h(p, q, U) = \langle U, U \rangle_q$. Restricted to $W = V_q^* \subset M_0$, θ_0 is essentially the 1-form used in the quantization construction of Appendix B. Then L_0 induces a prequantum bundle $L \to M$ whose sections are represented by functions $\psi_0 \in C_{\mathbb{C}}^\infty(S_0)$ satisfying

$$T(\psi_0) - i(T \lrcorner \theta_0)\psi_0 = 0, \tag{6·12}$$

where T is the Hamiltonian vector field generated on M_0 by h and $S_0 = \{h = 1\}$. The polarized sections are the restrictions to S_0 of the functions on M_0 which are holo-

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morphic and homogeneous of degree one on the fibres of V^* and independent of p: they are determined by their values on $V^* \subset M_0$. Thus we obtain an explicit correspondence between the sections of V and the polarized sections of the prequantum bundle over M. (It is worth remarking that if Q is not simply connected then L is not the only prequantum bundle: we can obtain others by adding to θ_C a 1-form on Q which is closed but not exact. The quantization construction then gives not V but a tensor product of V with a nontrivial line bundle over Q.)

Let ψ be a polarized section of L over M and let $\psi_0 \in C^\infty_c(S_0)$ be the corresponding function on S_0. Then ψ_0 extends uniquely to a function on M_0 independent of p and holomorphic and homogeneous of degree one on the fibres of V^*. Let $Y \in V(Q)$ and let Z be the lift of Y to $PV^* \subset M$, as in § 3. Let Z_0 be the horizontal lift of Y to $V^* \subset M_0$.

It follows from the lemma above that $\sigma(Z_0, X) = 0$ whenever $X \in T_{m_0} M_0$ is tangent to the submanifold $q = \text{const.}$ through some point $m_0 \in V^* \subset M_0$. Moreover, $[Z_0, T] = 0$ and Z_0 is tangent to $S_0 \cap V^*$ since the connection on V^* preserves both J and $\langle .,. \rangle$. Therefore $Z_0|_{S_0 \cap V^*}$ projects onto Z.

Now, from the construction of L, $\nabla_Z \psi$ is reprseented by $\nabla_{Z_0} \psi_0$. But $Z_0 \lrcorner \theta_0 = 0$. Therefore, on V^*, $\nabla_{Z_0} \psi_0 = Z_0(\psi_0)$, from which it follows that the connection on V associated with the P^1-section (of D) PV^* is $\hat{\nabla}$.

Finally, we come to the relationship between the Chern classes of V and the obstructions to the existence of a coisotropic section of D. In the conventions used here, the Chern classes are the $2r$-forms $\hat{\gamma}_r$ on Q given by

$$\det\left(1 + \frac{1}{2\pi}\hat{\Omega}\right) = 1 + \hat{\gamma}_1 + \hat{\gamma}_2 + \ldots. \tag{6·13}$$

To represent the forms γ_r in a more convenient way, we need another piece of notation. Let α be an m-form on Q taking values at $q \in Q$ in $C^\infty(M_q)$. Then $\mathrm{Int}(\alpha)$ is the real m-form on Q given by

$$(2\pi)^N \mathrm{Int}(\alpha)(Y_1, Y_2, \ldots, Y_m)_q = \int_{M_q} \alpha(Y_1, Y_2, \ldots, Y_m)\, \omega_q^N \tag{6·14}$$

for $Y_1, Y_2, \ldots \in V(Q)$. Using this, we can rewrite equation (3·2) as

$$\gamma_r = \binom{r+N}{r} \mathrm{Int}\left[\left(\frac{1}{2\pi}\Omega\right)^r\right], \tag{6·15}$$

where Ω is the $C^\infty(M_q)$-valued 2-form introduced in equation (4·11) and $N = n - k$.

In the particular example that we are dealing with here,

$$\Omega_q(Y, Y')(m_q) = \sigma_q(U, J_q \hat{\Omega}_q^*(Y, Y')\, U)$$

where $m_q \in M_q = PV_q^*$ is the projection into M_q of $U \in V_q^* \cap S_0$.

Either from the general theory developed in § 4, or from the discussion in Appendix B applied to $W = V_q^*$, it follows that the quantization of $\Omega_q(Y, Y')$ as a function on M_q is the self-adjoint linear transformation $\hat{\Omega}_q(Y, Y')\colon V_q \to V_q$. Therefore, applying (B 20), we have:

Proposition 4. $1 + \hat{\gamma}_1 + \hat{\gamma}_2 + \ldots = [1 - \gamma_1 + \gamma_2 - \ldots]^{-1}$.

The sum on the right-hand side is finite and the inversion is interpreted by a formal application of the binomial theorem. Note that the γ_r's here are integral as a consequence of the quantization condition on M and that a similar expression holds for the

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relationship between the cohomology classes of the γ_r's and the Chern classes (the cohomology classes of the $\hat{\gamma}_r$'s) in the cohomology ring of Q.

Bundles with more general structure groups can be obtained by quantizing symplectic manifolds constructed from T^*Q and bundles of coadjoint orbits over Q. One can obtain generalizations of the proposition by applying Kirillov's character formula (5), which gives a more general relationship of the traces of quantized operators to integrals over coadjoint orbits.

7. *Addendum*

I am grateful to the referee for his criticism of the presentation of §3 in the first draft of this paper. The version given here is somewhat simpler. The referee also drew my attention to an alternative method of constructing the symplectic manifold M in §6, which brings out the connection between some of the ideas in this paper and the earlier work of Sternberg (11), Guillemin and Sternberg (2), and Weinstein (13). It goes as follows: let P be the principal bundle associated to V and let M_1 be the symplectic manifold $T^*P \times \overline{\mathbb{C}P^*_N}$, where the bar indicates a reversal of the symplectic structure. There is a natural symplectic action of the structure group $G = U(N+1)$ on M_1 and a natural moment map $\kappa: M_1 \to \mathscr{G}^*$ (the dual to the Lie algebra of G). The manifold M in §6 is simply the reduction of $\kappa^{-1}(0)$. To see this, one first identifies $\kappa^{-1}(0)$ with $C = \lambda^{-1}(\mathscr{O}) \subset T^*P$ where $\lambda: T^*P \to \mathscr{G}^*$ is the moment map for the action of G on T^*P and $\mathscr{O} \subset \mathscr{G}^*$ is the co-adjoint orbit corresponding to $\mathbb{C}P^*_N$. Restricted to $\kappa^{-1}(0)$, the symplectic form on M_1 coincides with

$$\omega|_C - \lambda^*(\sigma),$$

where ω is the canonical 2-form on T^*P and σ is the symplectic form on $\mathscr{O}$. The reduction of $\kappa^{-1}(0)$ is achieved by factoring out the action of G on C.

When P carries a connection, the fibre of T^*P over $p \in P$ splits into the direct sum of $\mathscr{G}^*$ and T^*_qQ where $q \in Q$ is the projection of p. In this case, C, and hence also $\kappa^{-1}(0)$, becomes a bundle over P with the product $\mathscr{O} \times T^*_qQ$ as fibre at p. On factoring out the action of G, one obtains an obvious identification with the symplectic manifold M.

With this method, one sees that the construction of M itself does not really involve the connection. One can also obtain the prequantum structure on M from the functorial properties of prequantization.

Appendix A. Notation and conventions

The conventions for the exterior calculus and geometric quantization are the same as in Woodhouse (14), except that here $\hbar = 1$. In particular, the prequantum operator generated by $h \in C^\infty(M)$ on a symplectic manifold (M, ω) is locally

$$\hat{h}(\phi) = -iX_h(\phi) - (X_h \lrcorner \theta)\phi + h\phi, \tag{A 1}$$

where $d\theta = \omega$ and X_h, the Hamiltonian vector field generated by h, is defined by $X_h \lrcorner \omega + dh = 0$.

Round brackets are used to denote symmetrization. Thus, if $T^{ab\ldots c}$ has p indices,

$$T^{(ab\ldots c)} = \frac{1}{p!}\sum_\sigma T^{\sigma(a)\sigma(b)\ldots\sigma(c)}, \tag{A 2}$$

where the summation is over all permutations of $a, b, \ldots, c$.

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Differentiation under the integral sign: if S is a p-surface in M and α is a p-form, then, dragging S along a vector field X, we obtain a one-parameter family of p-surfaces S_t and

$$\frac{d}{dt}\left[\int_{S_t}\alpha\right]_{t=0}=\int_S X\lrcorner d\alpha+\int_{\partial S}X\lrcorner\alpha \tag{A 3}$$

(by Stokes' theorem).

The set of real (complex) vector fields on a manifold M is denoted $V(M)$ ($V_{\mathbb{C}}(M)$); the subset tangent to a real (complex) distribution P is denoted $V(M;P)$ ($V_{\mathbb{C}}(M;P)$).

Appendix B. Quantization of $\mathbb{C}P_N$

Let W be a $2(N+1)$-dimensional real vector space with a symplectic form σ and a positive compatible complex structure J. That is,

$$\sigma(JZ,JZ')=\sigma(Z,Z')\quad\forall Z,Z'\in W\quad\text{and}\quad\sigma(Z,JZ)>0\quad\forall\,0\neq Z\in W. \tag{B 1}$$

Then W is also a complex vector space with Hermitian inner product

$$\langle Z,Z'\rangle=\sigma(Z,JZ')-i\sigma(Z,Z'),$$

and it is possible to introduce complex linear coordinates z^a ($a=0,1,\ldots,N$) on W in which

$$\sigma=ig_{ab}\,dz^a\wedge d\bar{z}^b\quad\text{and}\quad J\frac{\partial}{\partial z^a}=i\frac{\partial}{\partial z^a}, \tag{B 2}$$

where the matrix g_{ab} is real, symmetric, and positive definite.

The complex projective space $M=PW$ (regarded as a $2N$-dimensional symplectic manifold with Kähler polarization) can be obtained from W by reducing the constraint

$$\langle Z,Z\rangle=g_{ab}z^a\bar{z}^b=1. \tag{B 3}$$

In other words, by identifying the points Z and $e^{it}Z$ ($t\in\mathbb{R}$) on the $(2N+1)$-sphere $S=\{\langle Z,Z\rangle=1\}$, or, which amounts to the same thing, by factoring out from S the flow of the Hamiltonian vector field

$$T=iz^a\frac{\partial}{\partial z^a}-i\bar{z}^a\frac{\partial}{\partial\bar{z}^a}, \tag{B 4}$$

generated by $g=g_{ab}z^a\bar{z}^b$.

The symplectic structure ω on M is given by $\pi^*(\omega)=\sigma|_S$, where $\pi\colon S\to M$ is the projection, and the polarization is the complex structure obtained by identifying M with $\mathbb{C}P_N$.

Using the 1-form $\theta=-ig_{ab}\bar{z}^b dz^a$ as a potential, we can quantize W by taking the trivial prequantum bundle $L=W\times\mathbb{C}$ with the connection

$$\nabla_X\phi=X(\phi)-i(X\lrcorner\theta)\phi, \tag{B 5}$$

where $\phi\in C^\infty_{\mathbb{C}}(W)$. This leads to a representation of the sections of the unique prequantum bundle on M as functions $\phi\colon S\to\mathbb{C}$ satisfying

$$\nabla_T\phi=T(\phi)-i(T\lrcorner\theta)\phi=iz^a\frac{\partial\phi}{\partial z^a}-i\bar{z}^a\frac{\partial\phi}{\partial\bar{z}^a}-i\phi=0 \tag{B 6}$$

with the inner product given by

$$\langle\phi,\phi'\rangle=\int_M\phi\bar{\phi}'\omega^N \tag{B 7}$$

(note that $\phi\bar{\phi}'$ is well defined as a function on M).

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The polarized sections are represented by holomorphic functions on W (restricted to S) satisfying

$$z^a \frac{\partial \phi}{\partial z^a} - \phi = 0; \tag{B 8}$$

in other words, by functions of the form $\phi = V_a z^a$. It follows that the Hilbert space associated to M by quantization is simply the dual space W^* (with an appropriate scaling of the inner product).

We shall need an integration lemma.

LEMMA. *Let f: $W \to \mathbb{C}$ be holomorphic and homogeneous of degree m:*

$$f(Z) = f_{ab\ldots c} z^a z^b \ldots z^c;$$

and let k: $M \to \mathbb{R}$ be the function obtained by restricting $f\bar{f}$ to S. Then

$$\binom{N+m}{m} \int_M k\omega^N = (2\pi)^N g^{ac} \ldots g^{bd} f_{a\ldots b} \bar{f}_{c\ldots d}, \tag{B 9}$$

with the usual conventions: $g^{ab} g_{bc} = \delta^a_c$.

Proof. Put

$$R = z^a \frac{\partial}{\partial z^a} + \bar{z}^a \frac{\partial}{\partial \bar{z}^a} \quad \text{and} \quad \tau = \frac{i}{2} g_{ab} (z^a d\bar{z}^b - \bar{z}^b dz^a) \tag{B 10}$$

and $z^a = x^a + iy^a$. Then $\sigma = d\tau$ and

$$R \lrcorner \sigma = 2\tau, \quad R \lrcorner \tau = 0, \quad T \lrcorner \tau|_S = 1, \quad T \lrcorner \sigma|_S = 0. \tag{B 11}$$

The integral curves of T on S are circles of length 2π. Therefore it follows from Stokes' theorem that, if D is the interior of S, then

$$\int_M k\omega^N = \frac{1}{2\pi} \int_S f\bar{f} \tau \wedge \sigma^N = \frac{1}{2\pi} \int_D (d(f\bar{f}) \wedge \tau \wedge \sigma^N + f\bar{f} \sigma^{N+1}). \tag{B 12}$$

But $R \lrcorner (\tau \wedge \sigma^N) = 0$, so that

$$\begin{aligned} \int_D d(f\bar{f}) \wedge \tau \wedge \sigma^N &= \int_0^1 \left[\int_{S_r} R(f\bar{f}) \tau \wedge \sigma^N \right] dr \\ &= 2m \int_0^1 r^{2N+2m+1} \left[\int_S f\bar{f} \tau \wedge \sigma^N \right] dr \\ &= \frac{m}{N+m+1} \int_S f\bar{f} \tau \wedge \sigma^N, \end{aligned} \tag{B 13}$$

since $f\bar{f}$ is homogeneous of degree $2m$ in the real sense; here $S_r = \{g_{ab} z^a \bar{z}^b = r\}$. Therefore

$$2\pi \int_M k\omega^N = \frac{N+m+1}{N+1} \int_D f\bar{f} \sigma^{N+1}. \tag{B 14}$$

Alternate applications of Euler's theorem and Stokes' theorem yield

$$\begin{aligned} \int_D f\bar{f} \sigma^{N+1} &= (4m(N+m+1))^{-1} \int_D \nabla^2 (f\bar{f}) \sigma^{N+1} \qquad \text{(B 15)} \\ &= \frac{(N+1)!}{4^m m! (N+m+1)!} (\nabla^2)^m (f\bar{f}) \int_D \sigma^{N+1} \\ &= \frac{(N+1)! m!}{(N+m+1)!} 2^{N+1} \pi^{N+1} g^{ac} \ldots g^{bd} f_{a\ldots b} \bar{f}_{c\ldots d} \qquad \text{(B 16)} \end{aligned}$$

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since, with appropriate orientation, σ^{N+1} is $2^{N+1}(N+1)!$ times the volume element $d^{N+1}x\,d^{N+1}y$ on W and the volume of D is $\pi^{N+1}/\Gamma(N+2)$. The lemma now follows. ∎

Consider the function $h = g_{ab}A^a{}_c\bar{z}^b z^c$, where $A^a{}_c$ is the Hermitian matrix of a self-adjoint complex linear transformation $A: W \to W$. Note that the restriction to S projects onto a well-defined real function on M and that $h(Z) = \sigma(Z, JAZ)$.

The flow of the Hamiltonian vector field

$$X_h = iA^a{}_c z^c \frac{\partial}{\partial z^a} - i\bar{A}^a{}_c \bar{z}^c \frac{\partial}{\partial \bar{z}^c} \tag{B 17}$$

coincides with the action of the one-parameter unitary group generated by iA and the action of the operator $\hat{h}$ on W^* is given by

$$\hat{h}(\phi) = -i(X_h(\phi) - i(X_h \lrcorner \theta)\,\phi) + h\phi = A^a{}_c z^c V_a,$$

where $\phi = V_a z^a$; that is, $\hat{h} = A^*$ is the dual transformation on W^*.

Applying the lemma, we have

$$\binom{N+m}{m} \int_M h^m \omega^N = (2\pi)^N A^a{}_{(a} A^b{}_b \dots A^c{}_{c)}, \tag{B 18}$$

which relates the integrals of the powers of h over M to the traces of A acting on the symmetric tensor products of W.

But

$$1 + \sum_{k=1}^{\infty} A^a{}_{(a} A^b{}_b \dots A^c{}_{c)} = \frac{1}{\det(1-A)}, \tag{B 19}$$

where k is the number of A's in each term. Hence

$$\det(1-A) = (2\pi)^N \left[\sum_{m=0}^{\infty} \int_M \binom{N+m}{m} h^m \omega^N \right]^{-1}. \tag{B 20}$$

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