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Improvement of the generalized quantal Bohr–Sommerfeld quantization condition

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Abstract

The quantal quartic oscillator, characterized by two real and two complex conjugate transition points (simple zeros), is studied by means of the phase-integral method developed by Fröman and Fröman, and various quantization conditions are obtained. The main results, obtained in §3·3 and §4, are summarized below.

A correction to the generalized Bohr–Sommerfeld quantization condition, due to the complex conjugate transition points, is obtained from estimates of the F-matrix for a path passing above the complex conjugate transition points. This quantization condition is closely related to the quantization condition for a double-oscillator. The correction is shown to be rigorously valid when the distance between the real transition points is large compared to the distance between the complex conjugate ones, which may lie at an unspecified distance from each other.

At sufficiently large distances from the cluster of four transition points the solutions of the Schrödinger equation can be characterized by the Stokes constants between anti-Stokes lines emerging from the cluster in six directions towards infinity. There are initially six non-trivial such Stokes constants in the problem under consideration, but, from the results in previous papers by N. Fröman and the present authors, these Stokes constants are linked by a number of algebraic relations, so that one or two Stokes constants suffice to describe the solutions far away from the cluster of transition points. The quantization condition is expressed in terms of these Stokes constants. Approximate values of the relevant Stokes constants are given in the three limiting cases when all four transition points coalesce (and the first-order phase-integral approximation is used), when the distance between the real transition points is sufficiently large (extreme double-oscillator), and in the Bohr–Sommerfeld limit.

1. *Introduction*

For the excited states of the quartic oscillator, Balian, Parisi and Voros[1] have indicated physical reasons, supported by a numerical check, for a small correction to the quantal Bohr–Sommerfeld quantization condition, due to the presence of those two transition points which lie away from the real axis. The derivation in that paper of the correction is, however, more intuitive than rigorous, and the final formula is valid only for the case of a symmetric potential well. The same authors have

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continued and extended the investigation of the quartic oscillator in several publications which are quoted in a comprehensive paper by Voros [**16**].

In the present paper we discuss corrections to the quantal Bohr–Sommerfeld quantization condition, when there are, besides the two real turning points, two transition points off the real axis, on the basis of the phase-integral method developed by Fröman and Fröman [**3**, **4**, **5**, **7**], [**8**], pp. 126–131, [**9**]. A summary of that method which is convenient as a background to the present paper can be found in a recent paper by N. Fröman and the present authors [**10**].

We are thus in this paper considering the quantization condition for a quantal particle in a one-dimensional real potential well with quartic anharmonicity. The problem is described by the Schrödinger equation

$$\frac{d^2\psi}{dz^2} + R(z)\psi = 0, \tag{1·1}$$

where z is a complex variable and the function $R(z)$, defined by

$$R(z) = \frac{2m}{\hbar^2}(E - V(z)) \tag{1·2}$$

with obvious notation, is real on the real axis and has two real zeros (classical turning points) on this axis and two complex conjugate simple zeros not on the real axis of the complex z-plane.

The approximate solutions of arbitrary order which are used in the phase-integral method for treating the differential equation (1·1) are generated from an unspecified base function $Q(z)$ which can be chosen conveniently: see [**7**], [**8**], pp. 126–131, [**9**]. This base function is in general chosen either to be equal to $R^{\frac{1}{2}}(z)$ or to be approximately equal to $R^{\frac{1}{2}}(z)$ except close to poles of $Q^2(z)$; for the choice $Q(z) = R^{\frac{1}{2}}(z)$ the approximation would fail close to such a pole. The zeros and poles of $Q^2(z)$, which are called transition points [**10**], appear in a decisive way in the phase-integral calculations. For the quartic oscillator we denote the real transition points by t_1 and t_2 ($> t_1$) and the complex conjugate transition points by t_0 and t_0^* where $\operatorname{Im} t_0 > 0$, and we assume that $t_1 < \operatorname{Re} t_0 < t_2$. The Stokes and anti-Stokes lines (see [**4**], [**5**], p. 60) emerging from these transition points (when the first-order phase-integral approximation is used) are shown in Fig. 1.

For the definitions of some important quantities, we now assume the complex z-plane to be cut according to Fig. 2 (*a*). The phase of the quantity $q^{\frac{1}{2}}(z)$, where $q(z)$ is defined by equations (2·3–6) in [**10**], is also indicated in the same figure. We denote real values of the complex variable z by x. For a cut along the real axis through x, we let $x + i0$ denote a point immediately above the cut, while $x - i0$ denotes a point immediately below the cut.

We shall use a convenient notation, introduced in §2.1 of [**10**], for a contour integral along a certain contour on a Riemann sheet for $q(z)$; see the discussion in connection with (2·7–13) in [**10**]. In analogy to (2·8) in [**10**] we thus introduce the function

$$w(z) = \int_{(t_1)}^{z} q(z)\,dz. \tag{1·3}$$

When the phase-integral approximation is used in arbitrary order, the integral in (1·3) is, by definition, half of the integral along the contour $\Gamma_{t_1}(z)$, shown in Fig. 2 (*a*)

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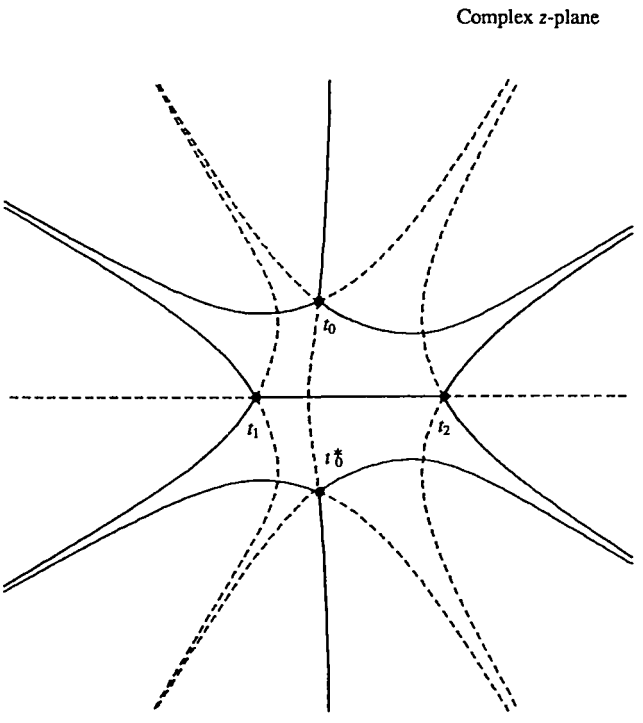


Fig. 1. For the quartic oscillator this figure shows the first-order Stokes lines (dashed) and anti-Stokes lines (full-drawn) emerging from the transition points, i.e. the zeros of $Q^2(z)$.

or Fig. 3(a), which encircles the transition point t_1 on a Riemann sheet: cf. (2·7) in [**10**]. When the first-order approximation is used, the integral in (1·3) can be expressed as an ordinary integral from t_1 to z; cf. (2·2) in [**10**].

We shall also introduce some quantities corresponding to those defined by (1a–c) and (2a, f) in [**13**]; cf. also Fig. 2 in [**13**]. We thus define (cf. the contours in our Figs. 2(b) and 2(c))

$$L_1 = \tfrac{1}{2}\,\mathrm{Re}\int_{\Gamma_{L_1}} q(z)\,dz = \mathrm{Re}\int_{(t_1)}^{(t_0^*)} q(z)\,dz, \tag{1·4a}$$

$$L_2 = \tfrac{1}{2}\,\mathrm{Re}\int_{\Gamma_{L_2}} q(z)\,dz = \mathrm{Re}\int_{(t_0)}^{(t_2)} q(z)\,dz, \tag{1·4b}$$

$$K = \tfrac{1}{2}i\int_{\Gamma_K} q(z)\,dz = i\int_{(t_0^*)}^{(t_0)} q(z)\,dz, \tag{1·4c}$$

$$w_1(x_1) = \frac{1}{2}\int_{\Gamma_{t_1}(x_1)} q(z)\,dz = \int_{(t_1)}^{x_1} q(z)\,dz \quad (x_1 < t_1), \tag{1·4d}$$

$$w_2(x_2) = \frac{1}{2}\int_{\Gamma_{t_2}(x_2)} q(z)\,dz = \int_{(t_2)}^{x_2} q(z)\,dz \quad (x_2 > t_2), \tag{1·4e}$$

where x_1 and x_2 stand for x_1+i0 and x_2+i0, respectively. When the first-order approximation is used, the integrals in (1·4a–e) can, as already mentioned, be written

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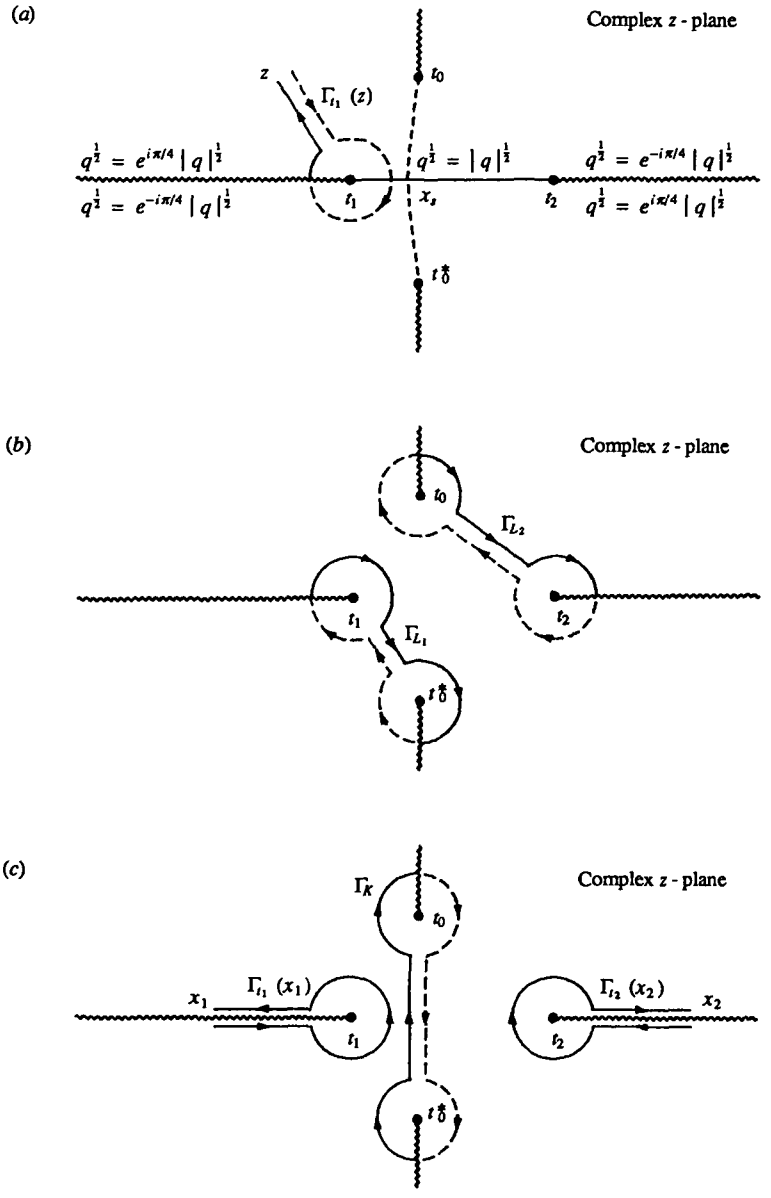


Fig. 2*a–c*. For legend see opposite.

as ordinary integrals between the limits of integration. The quantities L_1 and L_2 are positive, while K is negative. The quantities $w_1(x_1)$ and $w_2(x_2)$ are purely imaginary, $\operatorname{Im} w_1(x_1)$ and $\operatorname{Im} w_2(x_2)$ being both negative. We also note that (1·4*a*, *b*) can be written as follows

$$L_1 = \int_{(t_1)}^{x_s} q(z)\,dz, \quad L_2 = \int_{x_s}^{(t_2)} q(z)\,dz, \qquad (1\cdot4'a, b)$$

where x_s is the point on the real axis where the complex conjugate integrals $\int_{(t_0)}^{x_s} q(z)\,dz$ and $\int_{(t_0^*)}^{x_s} q(z)\,dz$ are purely imaginary. When the first-order approximation is used, x_s is thus the point where the Stokes line joining t_0 and t_0^* crosses the real axis: cf. Fig. 2(*a*).

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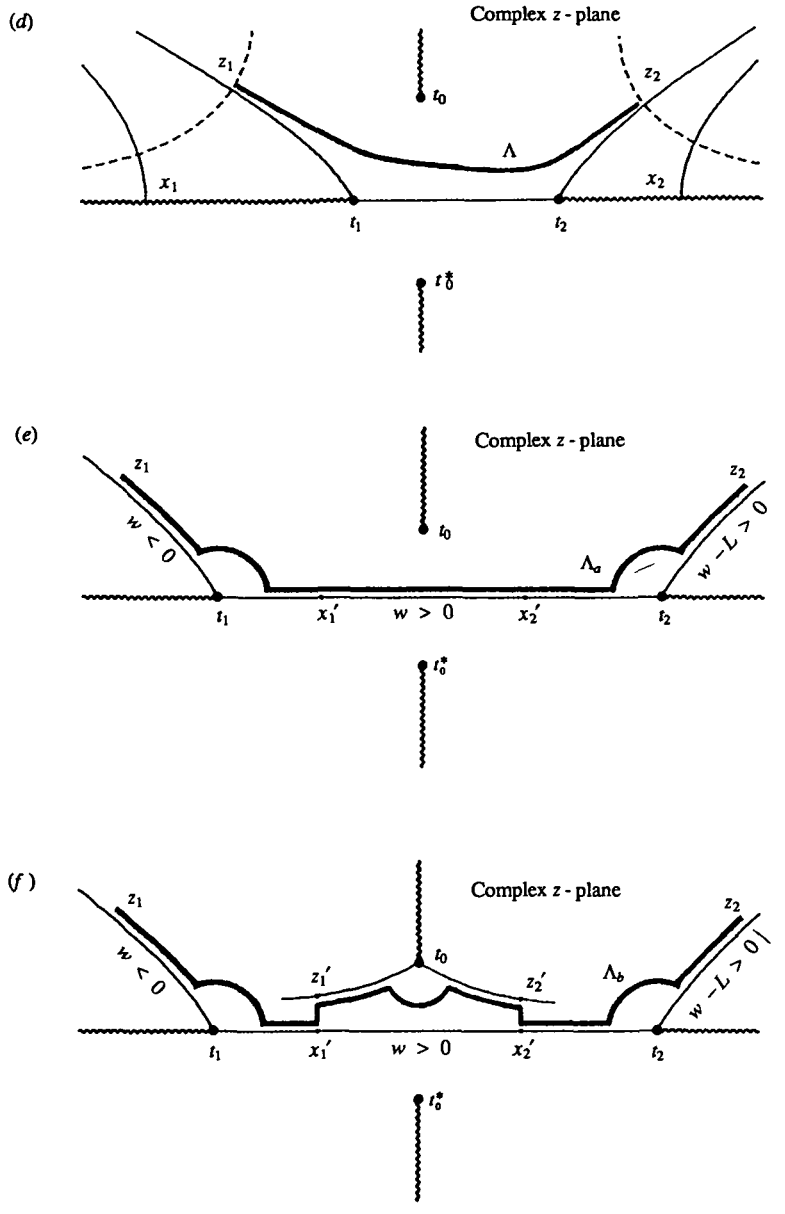


Fig. 2. This figure refers to the treatment of the quartic oscillator in §§1, 2 and 3. (*a*) Cuts from the transition points of $Q^2(z)$, used to make $q(z)$ and $w(z)$ single-valued, are shown (wavy lines) together with the phase chosen for $q^{\frac{1}{2}}(z)$. The contour of integration $\Gamma_{t_1}(z)$ is used for obtaining the phase-integral $w(z)$ according to (1·3). The dashed part of this contour lies on a Riemann sheet adjacent to the complex z-plane under consideration. The dashed line $t_0 x_s t_0^*$ is the Stokes line joining t_0 and t_0^*, when the first-order phase-integral approximation is used. (*b*), (*c*) Contours of integration used for obtaining the phase-integral quantities according to (1·4*a–e*). The dashed parts of the contours lie on Riemann sheets adjacent to the complex z-plane under consideration. (*d*) This figure is used in the derivation of the quantization condition (2·4). (*e*) This figure shows the path Λ_a used in §3·1 to derive the generalized Bohr–Sommerfeld quantization condition (3·5). (*f*) This figure shows the path Λ_b used in §3·2 to derive the quantization condition (3·12). It is convenient to assume z_1' and z_2' to lie on the Stokes lines through x_1' and x_2', respectively.

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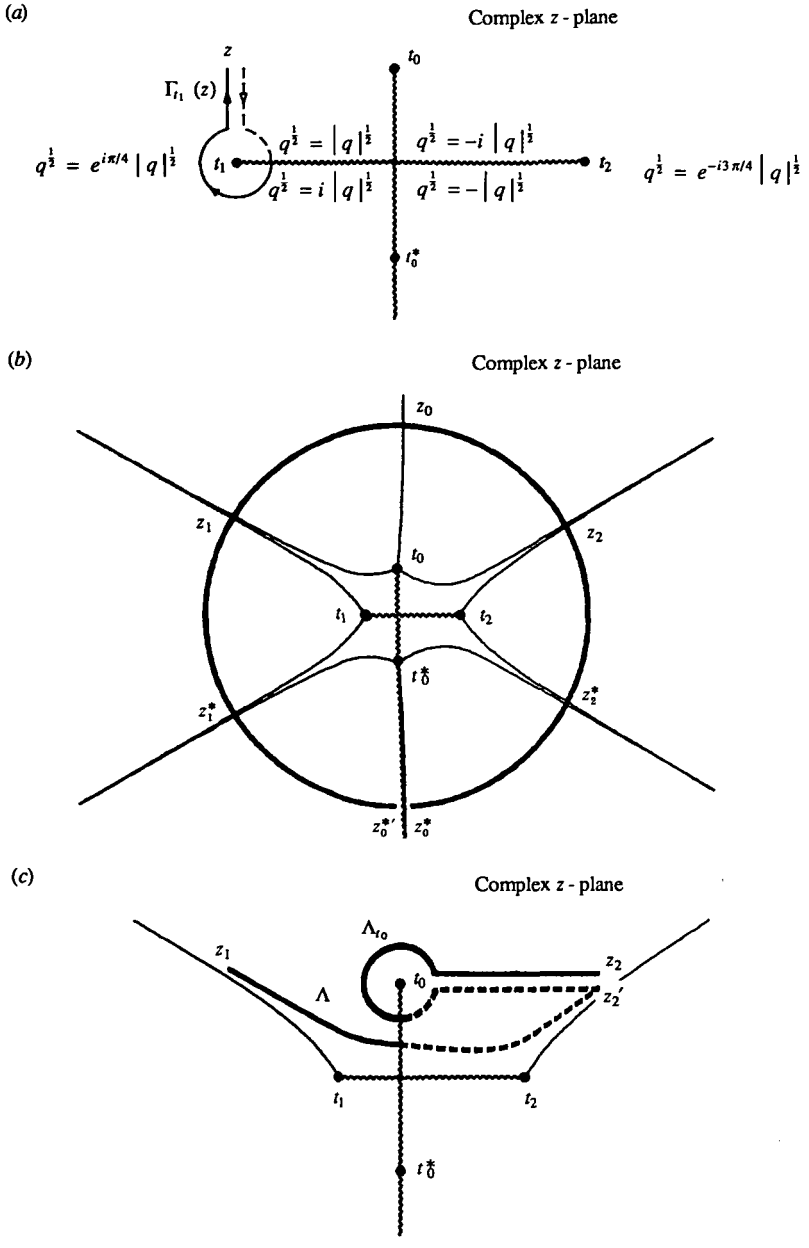


Fig. 3. This figure refers to the treatment of the quartic oscillator in §4. (*a*) Cuts from the transition points of $Q^2(z)$, used to make $q(z)$ and $w(z)$ single-valued, are shown together with the phase chosen for $q^{\frac{1}{2}}(z)$. Note that, for the symmetry relations to be valid, one must not consider the F-matrix along a curve which, after reflection with respect to the real axis or the origin, crosses a cut. The contour of integration $\Gamma_{t_1}(z)$ is used for obtaining $w(z)$ according to (1·3). (*b*) This figure shows first-order anti-Stokes lines (full-drawn) emerging from the transition points towards infinity in six directions. The figure also shows a path (heavy line) encircling the cluster at a large distance. The division into sub-paths running between adjacent anti-Stokes lines is indicated by the notation for the points written out along the path. (*c*) This figure is used in the derivation of the quantization condition (2·5) and the matrices (4·19*a–d*) and (4·24*a–d*). The path Λ, which joins z_1 with the point z_2', which one arrives at by moving from z_2 in the positive sense one full turn around t_0, is the same as the path Λ in Fig. 2(*d*). The path Λ_{t_0} joins z_2 with z_2'. The combined path $\Lambda + \Lambda_{t_0}$ joins z_1 with z_2 when the complex z-plane is cut as shown in Fig. 3(*a*).

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Furthermore we introduce the quantities

$$L = L_1 + L_2, \tag{1·5a}$$

$$\lambda = L_1 - L_2. \tag{1·5b}$$

From (1·4*a*, *b*) and (1·5*a*) it follows that L can be expressed as

$$L = \frac{1}{2}\int_{\Gamma_L} q(z)\,dz = \int_{(t_1)}^{(t_2)} q(z)\,dz, \tag{1·6}$$

where Γ_L is a closed contour which encircles t_1 and t_2, just as Γ_{L_1} encircles t_1 and t_0^* and Γ_{L_2} encircles t_0 and t_2 in Fig. 2(*b*). Thus we have an expression for L in which the transition points t_0 and t_0^* need not be considered at all. The quantity λ is a measure of the asymmetry of the potential; for a symmetric well we have $\lambda = 0$.

In §3 we shall use cuts in the complex z-plane according to Fig. 2(*a*), while in §4 we shall use the alternative cuts shown in Fig. 3(*a*). We shall, however, make use of the quantities defined by (1·4*a–c*) and (1·5*a–b*), with the contours in Fig. 2(*b*, *c*), also in §4.

2. *Exact quantization conditions*

Assuming still the complex z-plane to be cut according to Fig. 2(*a*), we shall now in particular deal with the upper half of the z-plane. At the points $x_1 = x_1 + i0$ and $x_2 = x_2 + i0$, which according to Fig. 2(*c*) lie immediately above the real axis, we have by (1·3), (1·4*d*, *e*), (1·6) and Fig. 2(*a*)

$$w(x_1) = w_1(x_1) = -i\,|w_1(x_1)| \qquad (x_1 < t_1), \tag{2·1a}$$

$$w(x_2) = L + w_2(x_2) = L - i|w_2(x_2)| \quad (x_2 > t_2). \tag{2·1b}$$

From these formulae it is seen that $\exp[iw(x+i0)]$ increases monotonically as $x \to \pm\infty$. We thus have the situation of chapter 10 in [**5**], in which the bound-state solutions of (1·1) in the limits $x \to \pm\infty$ are proportional to the phase-integral function $f_2(x+i0)$, defined by (2·1*b*) in [**10**]. According to (10·13) in [**5**] we then have the exact quantization condition

$$F_{12}(-\infty + i0, +\infty + i0) = 0: \tag{2·2}$$

cf. §2·2 in [**10**] for an outline of the F-matrix technique.

The F-matrix entry in the quantization condition (2·2) refers to the points $-\infty + i0$ and $+\infty + i0$. It is useful to consider also a quantization condition which follows from (2·2) and in which there appears an F-matrix entry referring to infinitely remote points on two anti-Stokes lines. To obtain this quantization condition we consider the point x_1 in Fig. 2(*d*), which lies just above the real axis to the left of t_1, and the point z_1, which lies on a suitable anti-Stokes line emerging from the cluster into the upper left half of the complex z-plane: cf. Fig. 2(*d*) which shows the situation in the first-order approximation. We have a certain freedom in the choice of this anti-Stokes line: cf. §§3, 5 in [**10**]. The points x_1 and z_1 can be joined by a path running along one anti-Stokes and one Stokes line: see Fig. 2(*d*). When one moves from z_1 to x_1 along this path, the quantity $|\exp[i\,w(z)]|$ increases monotonically (in the non-strict sense). When x_1 and z_1 move away from t_1 towards infinity, $|\exp\{i[w(x_1) - w(z_1)]\}|$ tends to infinity, while the μ-integral for the path of monotonicity joining x_1

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and z_1 tends to zero: see equation (2·35) in [**10**]. Using the basic estimates, given by (2·34*a–d*) in [**10**], in the direction from z_1 to x_1, and inverting the result by means of (2·23) in [**10**], we therefore obtain

$$\lim_{\substack{x_1\to-\infty \\ z_1\to\infty}} F_{12}(z_1, x_1) = 0, \tag{2·3a}$$

$$\lim_{\substack{x_1\to-\infty \\ z_1\to\infty}} F_{22}(z_1, x_1) = 1, \tag{2·3b}$$

whereas the other two entries of $\mathbf{F}(z_1, x_1)$ remain undetermined. Because of (2·14*a*), (2·16) and (2·17) in [**10**] and (2·3*a*, *b*) the solution corresponding to a bound state, which on the negative part of the real axis is proportional to $f_2(x_1)$ as $x_1\to-\infty$, is on the chosen anti-Stokes line emerging from the cluster into the upper left half of the complex z-plane proportional to $f_2(z_1)$ as $z_1\to\infty$. Considering the points x_2 and z_2 in Fig. 2(*d*) and arguing in a similar way, one finds that the solution corresponding to a bound state is proportional to $f_2(z_2)$ as $z_2\to\infty$ on a suitable anti-Stokes line emerging from the cluster into the upper right half of the complex z-plane. Since the limit of $\mathbf{F}(z_1, z_2)$ as $z_1\to\infty$ and $z_2\to\infty$ exists and is finite, we therefore obtain the exact quantization condition

$$\lim_{\substack{z_1\to\infty \\ z_2\to\infty}} F_{12}(z_1, z_2) = 0, \tag{2·4}$$

where z_1 and z_2 tend to infinity along suitable anti-Stokes lines emerging from the cluster to the left and to the right, respectively, into the upper half of the complex z-plane: cf. Fig. 2(*d*).

The quantization conditions (2·2) and (2·4) apply when the complex z-plane is cut as in Fig. 2(*a*) and the phase of $q(z)$ is chosen as shown there. Below we shall, for later use, give the quantization condition corresponding to (2·4) for the case when the complex z-plane is cut as in Fig. 3(*a*) and the phase of $q(z)$ is chosen as shown there.

The path Λ which joins z_1 and z_2 in Fig. 2(*d*) crosses in Fig. 3(*c*) the cut between t_0 and t_0^*. To obtain the matrix $\mathbf{F}(z_1, z_2)$ of Fig. 3 we therefore multiply the matrix $\mathbf{F}(z_1, z_2)$ of Fig. 2 on the right by the F-matrix for the loop Λ_{t_0} shown in Fig. 3(*c*), i.e. by the matrix given by the exact formula (4·11*a*) in [**10**] with $m = 1$. In the latter matrix the diagonal entries are equal to zero, while the non-diagonal entries are different from zero. The matrix entry $F_{12}(z_1, z_2)$ of Fig. 2 is therefore proportional to the matrix entry $F_{11}(z_1, z_2)$ of Fig. 3, the proportionality factor being finite and different from zero. Hence we obtain from the exact quantization condition (2·4) of Fig. 2 the exact quantization condition

$$\lim_{\substack{z_1\to\infty \\ z_2\to\infty}} F_{11}(z_1, z_2) = 0, \tag{2·5}$$

which pertains when the complex z-plane is cut as in Fig. 3(*a*) and the phase of $q(z)$ is chosen as shown there.

3. *Approximate quantization conditions obtained by F-matrix estimates through regions within the cluster of transition points*

In this section we assume the complex z-plane to be cut as in Fig. 2 and the phase of $q(z)$ to be chosen as shown there, and we derive various approximations to the

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quantization condition (2·4). We achieve this by using F-matrix estimates along paths going rather close to the transition points, and therefore the μ-integrals (cf. equation (4·2) in [**5**] or equation (2·35) in [**10**]), appearing as measures of the errors, cannot be made arbitrarily small.

Throughout this section we assume the transition points t_1 and t_2 to be well separated both from each other and from the other two transition points, so that $\mathrm{Re}\,(t_0)-t_1$ and $t_2-\mathrm{Re}\,(t_0)$ are positive and sufficiently large. The case when t_0 and t_0^* lie remote and have no influence is treated in §3·1. The case when t_0 and t_0^* lie not too far away from the real axis, so that the passage along the classically allowed (in the generalized sense) region of the real axis across the region in the neighbourhood of t_0 and t_0^* can be treated as a super-barrier transmission problem, is treated in §§3·2, 3·3.

3·1. *The points t_0 and t_0^* are so remote that their influence can be neglected (the Bohr–Sommerfeld limit)*

Considering the path Λ_a in Fig. 2(*e*) which joins z_1 and z_2 via the intermediate points x_1' and x_2' on the real axis between t_1 and t_2, we have

$$\mathbf{F}(z_1, z_2) = \mathbf{F}(z_1, x_1')\,\mathbf{F}(x_1', x_2')\,\mathbf{F}(x_2', z_2). \tag{3·1}$$

Since we neglect the presence of the points t_0 and t_0^*, we have

$$\mathbf{F}(z_1, x_1') \approx \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \tag{3·2a}$$

according to equation (7·5*a*) in [**5**],

$$\mathbf{F}(x_1', x_2') \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3·2b}$$

according to (6·1*a*, *b*) and (6·3*a*, *b*) in [**5**] or (2·34*a*–*d*) in [**10**], and

$$\mathbf{F}(x_2', z_2) \approx \begin{pmatrix} 1 & i\exp(-2iL) \\ 0 & 1 \end{pmatrix}, \tag{3·2c}$$

according to (7·5*a*) in [**5**] and the rule for the change of the F-matrix when a constant is added to $w(z)$: see p. 21 in [**5**] or (2·27) in [**10**]. The absolute errors of the entries of the matrices (3·2*a*–*c*) are of the order of the appropriate μ-integral. Inserting (3·2*a*–*c*) into (3·1), we obtain

$$\mathbf{F}(z_1, z_2) \approx \begin{pmatrix} 1 & i[1+\exp(-2iL)] \\ 0 & 1 \end{pmatrix}, \tag{3·3}$$

where the absolute error of every matrix entry is $O(\mu_a)$, μ_a being the μ-integral for the path Λ_a in Fig. 2(*e*).

With the aid of (3·3) we obtain from (2·4) the quantization condition

$$1+\exp(-2iL)+O(\mu_a) = 0, \quad \text{i.e.} \quad \cos L = O(\mu_a), \tag{3·4}$$

from which one easily obtains

$$L = (k+\tfrac{1}{2})\pi + O(\mu_a) \quad (k = 0, 1, 2, \ldots), \tag{3·5}$$

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L being defined by (1·6). The μ-integral in (3·4) and (3·5) is to be taken along the path Λ_a as z_1 and z_2 tend to infinity along appropriate anti-Stokes lines: cf. Fig. 2(*e*). The complex conjugate transition points t_0 and t_0^* are obviously not taken into account at all in the quantization condition (3·5). That this quantization condition, with the term $O(\mu_a)$ neglected, nevertheless can give extremely accurate eigenvalues for the anharmonic oscillator, particularly with the use of phase-integral approximations of higher order, is clear from the results in a paper by Lakshmanan *et al.* [**14**].

Since (3·5) applies to the phase-integral approximation of arbitrary order generated from an unspecified base function, we call it, when $O(\mu_a)$ is neglected, the generalized Bohr–Sommerfeld quantization condition. When $Q(z)$ is chosen to be equal to $R^{\frac{1}{2}}(z)$ and the first-order approximation is used, it reduces to the usual quantal Bohr–Sommerfeld quantization condition.

3·2. *The distance between the points t_1 and t_2 is large compared to the distance between the points t_0 and t_0^* which are well separated from each other*

We shall now derive a correction to the generalized Bohr–Sommerfeld quantization condition of §3·1 by deriving an approximate expression for the matrix $\mathbf{F}(x_1', x_2')$ in (3·1) according to the method described in §3 of [**15**]. To this end we use, instead of the path Λ_a of Fig. 2(*e*), the path Λ_b of Fig. 2(*f*) which joins z_1 with z_2 via the intermediate points x_1', z_1', z_2' and x_2', getting

$$\mathbf{F}(x_1', x_2') = \mathbf{F}(x_1', z_1')\,\mathbf{F}(z_1', z_2')\,\mathbf{F}(z_2', x_2') = \mathbf{F}(x_1', z_1')\,\mathbf{F}(z_1', z_2')\,\mathbf{F}^{-1}(x_2', z_2'). \tag{3·6}$$

The matrix $\mathbf{F}(z_1', z_2')$ here, which can be obtained from equation (49) in [**15**], is

$$\mathbf{F}(z_1', z_2') = \begin{pmatrix} 1+O(\mu) & O(\mu)\exp|K| \\ -i[1+O(\mu)]\exp(-|K|+2iL_1) & 1+O(\mu) \end{pmatrix}, \tag{3·7}$$

with the μ-integral taken along the path from z_1' to z_2'. The remaining two matrices in the right-hand side of (3·6) are obtained from the basic estimates (2·34*a–d*) in [**10**]. Noting the direction of increasing absolute value of $\exp[i\,w(z)]$ along the Stokes lines $x_1'z_1'$ and $x_2'z_2'$, we get

$$\mathbf{F}(x_i', z_i') = \begin{pmatrix} 1+O(\mu_s) & O(\mu_s)\exp|K| \\ O(\mu_s) & 1+O(\mu_s)+O(\mu_s^2)\exp|K| \end{pmatrix} \quad (i = 1, 2), \tag{3·8}$$

where μ_s is the μ-integral along the Stokes line $x_i'z_i'$.

In [**15**] the transition points t_0 and t_0^* were assumed to be sufficiently well isolated from the other transition points, so that x_i' and z_i' can be removed from the vicinity of t_0 as far as is necessary to make the integral μ_s along the Stokes line $x_i'z_i'$ fulfil the condition $\mu_s \ll \exp(-|K|)$. In the present situation the distances $\mathrm{Re}\,(t_0)-t_1$ and $t_2-\mathrm{Re}\,(t_0)$ put a limit on how small μ_s can actually be made. Hence we must require that those distances be so large that $\mu_s \ll \exp(-|K|)$ for both paths $x_1'z_1'$ and $x_2'z_2'$; otherwise some error terms in (3·8) may be large. Assuming these requirements to be fulfilled, i.e. that we are treating energy levels above the top of a rather extreme double-oscillator, we can neglect the quantities $O(\mu_s)\exp|K|$ while performing the multiplication of the matrices in the right-hand side of (3·6). Thus, inserting (3·7) and (3·8), with these quantities neglected, into (3·6), and using the symmetry relation

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(6·1*b*) in [5] for obtaining $F_{12}(x_1', x_2')$ from $F_{21}(x_1', x_2')$, we arrive at the approximate formula

$$\mathbf{F}(x_1', x_2') \approx \begin{pmatrix} 1 & i\exp(-|K|-2iL_1) \\ -i\exp(-|K|+2iL_1) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & i\exp[-|K|-i(L+\lambda)] \\ -i\exp[-|K|+i(L+\lambda)] & 1 \end{pmatrix}, \quad (3{\cdot}9)$$

where, to obtain the last matrix, we have also used (1·5*a*, *b*).

The improvement of the treatment in the present subsection, compared to that in §3·1, is due to the replacement of (3·2*b*) by (3·9). Inserting (3·2*a*, *c*) and (3·9) into (3·1), we obtain

$$\mathbf{F}(z_1, z_2) \approx \begin{pmatrix} 1+\exp[-|K|+i(L+\lambda)] & 2i\exp(-iL)[\cos L+\exp(-|K|)\cos\lambda] \\ -i\exp[-|K|+i(L+\lambda)] & 1+\exp[-|K|-i(L-\lambda)] \end{pmatrix}. \quad (3{\cdot}10)$$

From (2·4) we therefore obtain the approximate quantization condition

$$\cos L+\exp(-|K|)\cos\lambda = 0, \quad (3{\cdot}11)$$

from which we obtain

$$L = (k+\tfrac{1}{2})\pi+(-1)^k \arcsin[\exp(-|K|)\cos\lambda] \quad (k = 0, 1, 2, \dots). \quad (3{\cdot}12)$$

To conclude this subsection, we shall briefly return to the error terms, neglected when we derived (3·9) and hence (3·12). By μ_b we understand the μ-integral along the entire path Λ_b used in joining z_1 and z_2 in Fig. 2(*f*), while μ_s still denotes the μ-integral along the Stokes lines in the estimate (3·8). In the extreme double-oscillator limit μ_s may become arbitrarily small as $|t_1-t_2|$ grows, while μ_b is larger than some positive constant, since $|t_0-t_0^*|$ is constant. Including the error terms in the derivation of (3·12), one obtains this quantization condition with the absolute error $O(\mu_b)+O(\mu_b\mu_s^2)\exp|K|$. We remark that the determinants of the expressions (3·9) and (3·10) for $\mathbf{F}(x_1', x_2')$ and $\mathbf{F}(z_1, z_2)$, respectively, are different from unity. Therefore those expressions do not fulfil the condition (2·22) in [**10**]. The deviation from unity is however only $\exp(-2|K|)$, which is very small compared to the uncertainty in the matrix entries themselves since $\exp(-2|K|) \ll 1$. One can see this by comparing (3·9) with (3·2*b*). Since the error in each non-diagonal entry of (3·2*b*) is $O(\mu)$, where μ denotes the μ-integral along the real axis from x_1' to x_2', one realizes that $\exp(-|K|) = O(\mu)$ and hence $\exp(-|K|) \ll 1$ when $\mu \ll 1$.

3·3. *The distance between the points t_1 and t_2 is large compared to the distance between the points t_0 and t_0^* which may lie at an unspecified distance from each other (extreme double-oscillator)*

When, for fixed t_0, the quantities $\mathrm{Re}(t_0)-t_1$ and $t_2-\mathrm{Re}(t_0)$ are both (positive and) sufficiently large, the potential studied in the present paper is that of a double-oscillator [**2**, **12**, **13**]. In the case to be considered now one can therefore obtain the matrix $\mathbf{F}(x_1', x_2')$ from a general treatment of the super-barrier transmission problem [**6**] by considering the situation when there is no cut which crosses the real axis: see Fig. 2. We obtain the matrix $\mathbf{F}(x_1', x_2')$ needed here by proceeding in two steps from the matrix $\mathbf{F}(x_1, x_2)$, as given by (22*a*, *b*) and (43*a*, *b*) in [**6**], which we shall here call

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$\bar{\mathbf{F}}(x_1', \bar{x}_2')$, where $\bar{x}_2'$ is the point obtained by moving from x_2' in our Fig. 2 one full turn in the negative sense around t_0. First $\mathbf{F}(x_1', \bar{x}_2')$ is obtained by transforming the lower limit in the w-integral in the matrix $\bar{\mathbf{F}}(x_1', \bar{x}_2')$ from $\bar{Z} = x_s$ to $Z = t_1$ by means of (2·27) in [**10**] with $\omega(\bar{Z}) = L_1$: cf. equation (2·26) in [**10**] and (1·4′a) in the present paper. Then $\mathbf{F}(x_1', x_2')$ is obtained by multiplying $\mathbf{F}(x_1', \bar{x}_2')$ on the right by $\mathbf{F}(\bar{x}_2', x_2')$, i.e. by the F-matrix over one full turn in the negative sense around t_0, so by the inverse of the matrix in equation (4·11a) in [**10**] with $m = 1$ and $\Omega = 2L_1 + i|K|$; cf. equation (4·3) in [**10**]. Thus, omitting the error estimates and using (1·5a,b), we get

$$\mathbf{F}(x_1', x_2') \approx \begin{pmatrix} [1+\exp(-2|K|)]^{\frac{1}{2}} \exp(+2i\sigma) & i\exp[-|K|-i(L+\lambda)] \\ -i\exp[-|K|+i(L+\lambda)] & [1+\exp(-2|K|)]^{\frac{1}{2}} \exp(-2i\sigma) \end{pmatrix}. \quad (3·13)$$

The quantity σ is given by equations (10) and (10a–c) in [**12**] and is small except for energies rather close to the top of the barrier in the double-well potential. According to equation (51) in [**6**] we have $\sigma = O(\mu)$ for sufficiently large $|K|$, with the μ-integral taken along the real axis between the points x_1' and x_2' in Fig. 2(e) in the present paper.

Comparing (3·13) to (3·9) in §3·2, we find that the off-diagonal entries in the right-hand sides are identical, while the diagonal entries differ by the phase $\pm 2\sigma$ and a quantity $O[\exp(-2|K|)]$. Since $\exp(-|K|) = O(\mu)$ and $\sigma = O(\mu)$ when $|K|$ is large, the discrepancy is then only $O(\mu)$. This quantity is not greater than the expected error in the diagonal entries of (3·13); see (3·2b) with the error $O(\mu)$ included in each entry. However (3·13) can also be used when $|K|$ is small. In contrast to the determinant of the matrix (3·9), the determinant of the matrix (3·13) is equal to unity and thus fulfils the F-matrix determinant condition exactly.

Inserting (3·2a,c) and (3·13) into (3·1), we obtain

$$F_{11}(z_1, z_2) \approx [1+\exp(-2|K|)]^{\frac{1}{2}} \exp(+2i\sigma) + \exp[-|K|+i(L+\lambda)], \quad (3·14a)$$

$$F_{12}(z_1, z_2) \approx 2i\exp(-iL)\{[1+\exp(-2|K|)]^{\frac{1}{2}} \cos(L-2\sigma) + \exp(-|K|)\cos\lambda\}, \quad (3·14b)$$

$$F_{21}(z_1, z_2) \approx -i\exp[-|K|+i(L+\lambda)], \quad (3·14c)$$

$$F_{22}(z_1, z_2) \approx [1+\exp(-2|K|)]^{\frac{1}{2}} \exp(-2i\sigma). \quad (3·14d)$$

From (2·4) and (3·14b) we obtain the approximate quantization condition

$$\cos(L-2\sigma) = -\frac{\cos\lambda}{[1+\exp(2|K|)]^{\frac{1}{2}}}, \quad (3·15)$$

which can be written in the more explicit form

$$L-2\sigma = (k+\tfrac{1}{2})\pi + (-1)^k \arcsin\frac{\cos\lambda}{[1+\exp(2|K|)]^{\frac{1}{2}}}, \quad (3·16)$$

where k is an unspecified non-negative integer.

When $R(z) = R(-z)$ and $Q^2(z) = Q^2(-z)$ we have $q(z) = q(-z)$ and hence $\lambda = 0$ by (1·4′a,b) and (1·5b). Therefore after some further trigonometry we get instead of (3·16) the simplified quantization condition

$$L-2\sigma = (k+\tfrac{1}{2})\pi + (-1)^k \operatorname{arctg}[\exp(-|K|)]. \quad (3·17)$$

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For the pure quartic oscillator we have $L = |K|$, and since the barrier has then disappeared, σ should be neglected; then (3·17) agrees with the quantization condition given by Balian *et al.* as equation (7) in [1].

For (3·13) to be valid, it is in general necessary that $|K|$ be not too large, since from the error terms in (43*a*) in [**6**] it is clear that the off-diagonal entries of (3·13) are given with the relative error $O(\mu)\exp|K|$, which may be large if $|K|$ is large; μ is here the μ-integral along an arc which joins x_1' and x_2' by running above the transition point t_0. It can be shown that the relative error $O(\mu)\exp|K|$ also pertains to the correction term in the quantization condition (3·16), which means that the absolute error in (3·16) is $O(\mu)$, but now μ is the μ-integral along a path which joins z_1 to z_2 by running via x_1' and x_2' just as the paths Λ_a and Λ_b in Figs. 2(*e*) and 2(*f*), but by running from x_1' to x_2' in a large arc above the transition point t_0. Hence we conclude that the correction term in (3·16) is in general significant only when $\mu \ll \exp(-|K|)$. This is the situation in the limiting case of a double-oscillator, when $|t_1 - t_2|$ is allowed to grow while $|t_0 - t_0^*|$ is kept constant; the path can then be allowed to run arbitrarily far away from all transition points, and then μ will tend to zero. In the general case, however, $|t_1 - t_2|$ is not much larger than $|t_0 - t_0^*|$, and there may not be space enough to choose the path such that $\mu \ll \exp(-|K|)$. Thus we cannot prove the significance of the correction term in the quantization condition (3·16) for the general quartic oscillator.

Therefore, when $|K|$ is large and μ is not small compared to $\exp(-|K|)$, we cannot prove that the correction term in (3·16) is significant (i.e. large compared to the error of the formula), but we can still keep this correction term in (3·16), since this does not essentially increase the absolute error in (3·16), which is $O(\mu)$. In fact, when $|K|$ is large but $|t_1 - t_2|$ is not large compared to $|t_0 - t_0^*|$, (3·16) need not be less accurate than the quantization condition (3·5) with $O(\mu_a)$ neglected: numerical calculations using a computer program for the evaluation of phase-integrals indicate that the correction is significant for the quartic oscillator when $t_0 = \frac{1}{2}(t_1 + t_2) + \frac{1}{2}i(t_2 - t_1)$, if the phase-integral approximation is used in the optimal order (F. Karlsson, unpublished). These results suggest that (3·16) is also a rather satisfactory quantization condition for the general quartic oscillator, which need not correspond to a double-oscillator, in which case σ is to be left out in (3·16).

The quantization condition (3·16) is similar to the quantization condition (3·12) derived in §3·2, and differs from it only by the presence of σ and the square root in the correction term. In fact, when $|K|$ is large, the leading terms in the corrections are identical and, because of the previously mentioned relation $\exp(-|K|) = O(\mu)$, the inverse of the square root in (3·16) is equal to $(\exp(-|K|))(1 + O(\mu^2))$. This difference is certainly insignificant compared with the absolute errors in the quantization conditions, which are $O(\mu)$ in the case of (3·16). In practical calculations with large $|K|$, therefore, we may expect (3·12) to have about the same accuracy as (3·16) for the case under consideration here.

In §3·2 we found that (3·12) contains the absolute error $O(\mu_b) + O(\mu_b \mu_s^2)\exp|K|$. If $\mu_s \to 0$, this error resembles the error involved in (3·16), which is $O(\mu)$, but in general the error estimate involved in (3·12) is a rather awkward one, since $\exp|K|$ may be large. For these and other reasons, the quantization condition (3·16), based on the double-oscillator formula, is the more satisfactory one and it can also be used when $|K|$ is small, when the inclusion of σ is important. When $|K|$ is large, (3·16) can be used

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instead of the Bohr–Sommerfeld formula (3·5) with $O(\mu_a)$ neglected, even if the correction term cannot be proved to be significant.

Since the case now treated is that of a double-oscillator when the energy lies above that of the barrier top, the quantization condition (3·16) can alternatively be obtained from that for the double-oscillator, which is given by equation (6) in [**12**]. With our notation and for over-barrier energies ($K < 0$) it is given by

$$\operatorname{tg}(L_1-\pi/2-\sigma)\operatorname{tg}(L_2-\pi/2-\sigma)=\frac{[1+\exp(2|K|)]^{\frac{1}{2}}-1}{[1+\exp(2|K|)]^{\frac{1}{2}}+1}. \tag{3·18}$$

Using (1·5a,b), we can rewrite (3·18) in the form

$$\frac{\cos\lambda+\cos(L-2\sigma)}{\cos\lambda-\cos(L-2\sigma)}=\frac{[1+\exp(2|K|)]^{\frac{1}{2}}-1}{[1+\exp(2|K|)]^{\frac{1}{2}}+1}. \tag{3·19}$$

Solving (3·19) with respect to $\cos(L-2\sigma)$, we obtain (3·15) and hence (3·16), which is thus, as already mentioned, equivalent to the quantization condition for a double-oscillator when the energy lies above the top of the barrier [**2**, **12**, **13**]. It remains valid when the complex conjugate transition points lie close to each other.

4. *Quantization condition expressed in terms of the Stokes constants associated with suitable anti-Stokes lines emerging from the cluster towards infinity*

An interesting application of the results in Fröman, Fröman and Lundborg [**10**, **11**] concerning the Stokes constants for a cluster of transition points [**10**] and the symmetry relations for connection matrices in the phase-integral method [**11**], is the eigenvalue problem for the general quartic anharmonic oscillator. In that problem, with a cluster of the degree $m = 4$, one can choose six suitable anti-Stokes lines emerging from the cluster towards infinity, with which there are associated six complex Stokes constants which (for all energies, i.e. not only for the bound-state energies) determine the wave-function at large distance from the cluster. According to [**10**] these Stokes constants are connected by three algebraic relations. The symmetry relations derived in [**11**] further restrict the Stokes constants. Thus, as we shall see, the number of unknown complex Stokes constants is reduced from six to *two* (restricted by one condition on the real and imaginary parts of these two Stokes constants) if the only symmetry is that $R(z)$ and $Q^2(z)$ be real on the real z-axis, but to *one* if also $R(-z) = R(z)$ and $Q^2(-z) = Q^2(z)$. In this section we shall show how the exact quantization condition can be expressed in terms of the above-mentioned remaining Stokes constant(s). In particular limiting cases we shall also obtain approximate expressions for these Stokes constants.

We cut the complex z-plane and choose the phase of $q^{\frac{1}{2}}(z)$ according to Fig. 3(a). The cuts as well as the first-order anti-Stokes lines emerging from the transition points towards infinity are shown in Fig. 3(b).

Since no other transition points than t_1, t_2, t_0 and t_0^* are assumed to be present in the problem, the error quantity μ for the F-matrices over the remote sub-paths z_0^*, $z_2^*, \ldots, z_1^*, z_0^{*\prime}$ shown in Fig. 3(b) can be made arbitrarily small by moving these paths sufficiently far away from the cluster consisting of the four transition points. All calculations in §§4·1, 4·2 are therefore, in principle, exact. In §4·3 we give approximate results in three limiting cases.

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4·1. *Exact relations for the Stokes constants in the general case*

By the general case we here understand the case when $R(x)$ and $Q^2(x)$ are real, while no other restrictions are imposed on the positions of the transition points t_0, t_0^*, t_1 and t_2. Because of the analyticity properties assumed for $R(z)$ and $Q^2(z)$, the reality of these functions on the real axis implies that $R(z^*) = [R(z)]^*$ and $Q^2(z^*) = [Q^2(z)]^*$, and hence $q^2(z^*) = [q^2(z)]^*$. The cuts in the complex z-plane shown in Fig. 3 furthermore imply that

$$q(z^*) = -[q(z)]^*, \quad \text{i.e.} \quad iq(z^*) = [iq(z)]^*, \tag{4·1}$$

i.e. the case of (2·3b) in [**11**]. Thus symmetry relations derived in §2 of that paper apply here and we shall make use of them later in this subsection.

First we shall introduce the general relations connecting the Stokes constants for a cluster of fourth degree that are given in §5 of [**10**]. From the choice of phase for $q(z)$, the Stokes constants $a_{-1}, b_0, \ldots$ for the sectors between the anti-Stokes lines in Fig. 3(b) appear in the sequence complementary to the one given in (5·1″) in [**10**] with $m' = 4$, i.e. from the explanation above (5·11′) in [**10**] we have

$$a_{-1} = F_{21}(z_2^*, z_0^*), \quad b_0 = F_{12}(z_2, z_2^*), \quad a_1 = F_{21}(z_0, z_2), \tag{4·2a–c}$$

$$b_2 = F_{12}(z_1, z_0), \quad a_3 = F_{21}(z_1^*, z_1), \quad b_4 = F_{12}(z_0^{*\prime}, z_1^*), \tag{4·2d–f}$$

where the points in the arguments of the F-matrix entries are located in the complex z-plane as shown in Fig. 3(b). In particular, the points $z_0^{*\prime}$ and z_0^* lie on opposite lips of the cut along the anti-Stokes line emerging downwards from t_0^*.

The F-matrix over a full turn around the cluster, from z_0^* to $z_0^{*\prime}$, is given by (4·11b) in [**10**] with $m = 4$ and Ω given by (4·3) in [**10**], which in the present case is equivalent to an integral over an arbitrary closed contour running in the positive sense around the cluster in the complex z-plane, cut as in Fig. 3(b). Using (1·4′a,b) and (1·5 b), and considering the present choice of phase for $q(z)$, we can then express Ω as follows:

$$\Omega = -2\lambda. \tag{4·3}$$

The relations between the Stokes constants are obtained from a modification of (5·27$_1$–27$_3$) in [**10**]. Due to the sequence (see the explanation of this concept below (5·10b_3) in [**10**]) of the Stokes constants (4·2a–f) we obtain the correct relations by changing the sign of Ω in (5·27$_1$–27$_3$) of [**10**], while the definition of Ω, given by (4·3) in [**10**], is left unchanged: see the simple rule formulated below (5·14) in [**10**]. Using (5·27$_1$–27$_3$) in [**10**], with Ω replaced by $-\Omega$, and using (4·3), we hence obtain

$$b_2 = -\exp(2i\lambda)\, b_0 - \exp(-2i\lambda)\,(1 + a_{-1} b_0)\, b_4, \tag{4·4a}$$

$$a_3 = -\exp(-2i\lambda)\,(a_{-1} + a_1 + a_{-1} b_0 a_1), \tag{4·4b}$$

$$(a_{-1} + a_1 + a_{-1} b_0 a_1)\, b_4 = \exp(2i\lambda) - \exp(4i\lambda)\,(1 + b_0 a_1). \tag{4·4c}$$

Now let us introduce the symmetry resulting from (4·1). According to (2·13b_1–b_4) in [**11**] this symmetry links the entries of the matrix $\mathbf{F}(z, z_0)$ to those of the matrix $\mathbf{F}(z^*, z_0^*)$. In the symmetry relations there appears the integral (cf. (2·26) in [**10**] with $Z = t_1$)

$$\omega(t_1^*) = \int_{(t_1)}^{(t_1^*)} q(z)\, dz \tag{4·5}$$

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on the Riemann surface for $q(z)$, where t_1^* is reached by passing from t_1 across the same interval of the real axis that one crosses when z moves to z^* and z_0 moves to z_0^*. When we pass across the negative real axis, to the left of t_1, we have $t_1^* = t_1$, so the integral (4·5) is obviously then equal to zero. The symmetry relation between $F_{12}(z_1, z_0)$ and $F_{12}(z_1^*, z_0^{*\prime})$, where the points are lying as shown in Fig. 3(*b*), then yields according to (2·13b_2) in [11]

$$b_4 = -b_2^*. \tag{4·6a}$$

Here, as for the following symmetry relations, we have also used the inversion formula (2·23) in [**10**] and (4·2*a–f*) in the present paper. The symmetry relation (2·13b_3) in [11], relating $F_{21}(z_1^*, z_1)$ to $F_{21}(z_1, z_1^*)$, gives

$$\operatorname{Re} a_3 = 0, \quad \text{i.e. } a_3 = \pm i\,|a_3|. \tag{4·6b}$$

When we pass instead across the positive real axis to the right of t_2, when we are joining z to z^* and z_0 to z_0^*, we reach t_1^* by moving from t_1 one full turn around the cluster in the negative sense. Hence the integral in (4·5) is to be taken over a contour corresponding to the one used for defining Ω, but running in the opposite direction: see (4·3) in [**10**]. Thus we find with the aid of (4·3) in the present paper that now $\omega(t_1^*) = 2\lambda$. With the points involved lying as shown in Fig. 3(*b*), equations (2·13b_3) and (2·13b_2) in [11] give

$$a_{-1} = -a_1^* \exp(4i\lambda) \tag{4·6c}$$

and

$$\operatorname{Re}[b_0 \exp(2i\lambda)] = 0, \quad \text{i.e.} \quad b_0 = \pm i\,|b_0| \exp(-2i\lambda). \tag{4·6d}$$

Inserting (4·6*a,c*) into (4·4*a–c*) we obtain after some calculations

$$b_0 \exp(2i\lambda) = -\frac{2i \operatorname{Im}[b_2 \exp(i\lambda)]}{[(1+a_1 b_2)\exp(-i\lambda)]^*} = -\frac{2i \operatorname{Im}[b_2 \exp(i\lambda)]}{(1+a_1 b_2)\exp(-i\lambda)}, \tag{4·7a}$$

$$a_3 = -\frac{2i \operatorname{Im}[a_1 \exp(-i\lambda)]}{[(1+a_1 b_2)\exp(-i\lambda)]^*} = -\frac{2i \operatorname{Im}[a_1 \exp(-i\lambda)]}{(1+a_1 b_2)\exp(-i\lambda)}, \tag{4·7b}$$

$$\operatorname{Im}[(1+a_1 b_2)\exp(-i\lambda)] = 0. \tag{4·7c}$$

It is easily seen that (4·6*b,d*) follow from (4·7*a–c*). Hence (4·6*b,d*) are consequences of (4·4*a–c*) and (4·6*a,c*) which were the only equations used in the derivation of (4·7*a–c*).

We now consider a_1 and b_2 as quantities restricted only by the condition (4·7*c*). The other four Stokes constants are then expressed in terms of a_1, b_2 and λ by (4·6*a,c*) and (4·7*a,b*). This is all that can be obtained from the utilization of the relations (4·4*a–c*) and the complex conjugation symmetry.

Finally we shall express the quantization condition in terms of the two Stokes constants a_1 and b_2. With the cuts in the complex z-plane and the phase of $q^{\frac{1}{2}}(z)$ as shown in Fig. 3(*a*), the quantization condition is given by (2·5). From that equation and

$$\mathbf{F}(z_1, z_2) = \mathbf{F}(z_1, z_0)\,\mathbf{F}(z_0, z_2) = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} = \begin{pmatrix} 1+a_1 b_2 & b_2 \\ a_1 & 1 \end{pmatrix} \tag{4·8}$$

we readily obtain

$$1 + a_1 b_2 = 0 \tag{4·9}$$

as the exact quantization condition selecting the energies of the bound states.

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We note that $1+a_1 b_2$ also appears in the denominators of (4·7 a,b). The Stokes constants are finite, and hence the numerators of (4·7 a,b) must also be equal to zero for the energies of the bound states, which means that the conditions

$$\text{Im}\,[a_1 \exp(-i\lambda)] = 0, \quad \text{i.e. } a_1 = \pm|a_1| \exp(+i\lambda) \tag{4·10a}$$

and

$$\text{Im}\,[b_2 \exp(+i\lambda)] = 0, \quad \text{i.e. } b_2 = \pm|b_2| \exp(-i\lambda) \tag{4·10b}$$

are fulfilled when (4·9) is fulfilled. On the other hand, (4·9) is not a consequence of (4·10 a,b).

4·2. *Exact relations for the Stokes constants when the potential is symmetric*

If the potential is symmetric with respect to $z = 0$, i.e. if $R(-z) = R(z)$, and if also $Q^2(-z) = Q^2(z)$, we have $q^2(-z) = q^2(z)$. From (1·5 b) and (1·4′ a,b) we then get $\lambda = 0$, and all results of §4·1 still apply. This new symmetry implies, however, further restriction on the Stokes constants, as we shall now see.

Assuming this symmetry, we find, with the choice of phase for $q^{\frac{1}{2}}(z)$ shown in Fig. 3(a), that (3·1 a) in [11] applies, i.e. that

$$q(z) = q(-z) \tag{4·11}$$

in the entire z-plane. The quantity $w(z)+w(-z)$, which is constant by (3·3 a) in [11], can be obtained from (1·3), (4·11) and (1·4 c) with due regard to the phase of $q(z)$ obtained from Fig. 3(a). Thus we obtain

$$w(z)+w(-z) = i|K|. \tag{4·12}$$

The symmetry relations are now given by (3·9 a_1–a_4) in [11] and (4·12), and, with $z_2^* = -z_1$, $z_0^* = -z_0$ and $z_0^{*\prime} = -z_0$ in (4·2 a–f), they link the Stokes constants that are oppositely located around the cluster in Fig. 3 to each other as follows:

$$b_2 = a_{-1} \exp(2|K|), \quad a_3 = b_0 \exp(-2|K|), \quad b_4 = a_1 \exp(2|K|). \tag{4·13a–c}$$

If we introduce also the symmetry relations (4·6 a,c), resulting from the potential being real on the real axis, and note that $\lambda = 0$, we obtain from (4·13 a–c) the equations

$$b_2 = -a_1^* \exp(2|K|), \quad a_3 = b_0 \exp(-2|K|), \quad -b_2^* = a_1 \exp(2|K|). \tag{4·14a–c}$$

Clearly (4·14 a) and (4·14 c) are identical and make the condition (4·7 c) with $\lambda = 0$ automatically fulfilled. Treating a_1 as an independent quantity it is now possible to express all other Stokes constants in terms of a_1 and $|K|$ only. From (4·6 c) and (4·14 a) we obtain a_{-1} and b_2, then b_4 is obtained from (4·6 a), and finally b_0 and a_3 are obtained from (4·7 a,b), still with $\lambda = 0$. Collecting the resulting formulae, we have

$$a_{-1} = -a_1^*, \tag{4·15a}$$

$$b_0 = -\frac{2i\,\text{Im}\,a_1 \exp(2|K|)}{1-|a_1|^2 \exp(2|K|)}, \tag{4·15b}$$

$$a_1 = (\text{unknown complex quantity}), \tag{4·15c}$$

$$b_2 = -a_1^* \exp(2|K|), \tag{4·15d}$$

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$$a_3 = -\frac{2i \operatorname{Im} a_1}{1-|a_1|^2 \exp(2|K|)}, \tag{4·15e}$$

$$b_4 = a_1 \exp(2|K|). \tag{4·15f}$$

The conditions (4·7*c*) with $\lambda = 0$ and (4·14*b*) are obviously automatically satisfied by (4·15*d*) and (4·15*b*, *e*), respectively.

Having utilized the two symmetries of §§4·1, 4·2, we are thus left with only one complex degree of freedom, i.e. two real degrees of freedom, for the Stokes constants. The unknown Stokes constant a_1 in (4·15*a*–*f*) cannot at present be determined within the phase-integral method.

The exact quantization condition (4·9) now takes the following simple form

$$1-|a_1|^2 \exp(2|K|) = 0, \quad \text{i.e.} \quad |a_1| = \exp(-|K|), \tag{4·16}$$

and the equations (4·10*a*, *b*) with $\lambda = 0$ become

$$\operatorname{Im} a_1 = 0, \quad \text{i.e.} \quad a_1 = \pm |a_1|, \tag{4·17}$$

which is fulfilled if (4·16) is fulfilled. On the other hand, when (4·17) is fulfilled, (4·16) need not be fulfilled.

4·3. *Limiting values for the Stokes constants*

We shall consider the Stokes constants in three limiting cases, firstly the case when all four transition points coalesce, secondly the case of the extreme double-oscillator, which was the subject of §3·3, and thirdly the Bohr–Sommerfeld limit, which was the subject of §3·1.

The case when all four transition points coalesce

In this case we obtain one fourth-order zero of $Q^2(z)$, and when we use the first-order phase-integral approximation the quantities L_1, L_2 and K then tend to zero by (1·4*a*–*c*). The connection problem around a well isolated zero of order n is treated in the first-order approximation in §7·3 of [5], the result being expressed by the formulae (7·13*a*, *b*) in [5]. According to these formulae with $n = 4$ all Stokes constants tend to $i\sqrt{3}$ as the zeros are coalescing. Putting $\lambda = 0$ and $|K| = 0$, it is easily verified that $a_j = b_k = i\sqrt{3}$ indeed satisfies all algebraic relations in §§4·1, 4·2, except, of course, the quantization conditions.

The case of the extreme double-oscillator

We obtain this case by letting $\operatorname{Re}(t_0) - t_1$ (> 0) and $t_2 - \operatorname{Re}(t_0)$ (> 0) grow sufficiently large while t_0 is kept constant; the results of §3·3 are then rigorously valid. Let us consider the matrix $\mathbf{F}(z_1, z_2)$, corresponding to the cuts in Fig. 3 and expressed by (4·8), between the points z_1 and z_2 in Fig. 3(*b*). We can obtain an approximate expression for this matrix by using (3·14*a*–*d*) for the matrix $\mathbf{F}(z_1, z_2)$ corresponding to the cuts in Fig. 2, which in the present context we write as $\mathbf{F}(z_1, z_2')$, where the point z_2' corresponds to z_2 in Fig. 3(*b*), but lies in the complex z-plane of Fig. 2 instead of Fig. 3. This means that one obtains z_2' by starting from z_2 in Fig. 3(*b*) and moving one full turn around t_0 in the positive sense; see Fig. 3(*c*). Hence,

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by multiplying $\mathbf{F}(z_1, z_2')$ on the right by the matrix $\mathbf{F}(z_2', z_2)$, obtained from (4·11*a*) in [**10**] with $m = 1$ and Ω in that formula given by

$$\Omega = 2L_1 + i|K| = L + \lambda + i|K| \tag{4·18}$$

(see (4·3) in [**10**] and (1·4*a*, *c*) with (1·5*a*, *b*) in the present paper), we find

$$F_{11}(z_1, z_2) \approx -2\exp(-|K| + i\lambda)[1+\exp(-2|K|)]^{\frac{1}{2}} \times \left(\cos(L-2\sigma) + \frac{\cos\lambda}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right), \tag{4·19a}$$

$$F_{12}(z_1, z_2) \approx i\exp(|K| - i\lambda)[1+\exp(-2|K|)]^{\frac{1}{2}} \times \left(\exp[-i(L-2\sigma)] + \frac{\exp(i\lambda)}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right), \tag{4·19b}$$

$$F_{21}(z_1, z_2) \approx i\exp(-|K| + i\lambda)[1+\exp(-2|K|)]^{\frac{1}{2}} \times \left(\exp[i(L-2\sigma)] + \frac{\exp(i\lambda)}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right), \tag{4·19c}$$

$$F_{22}(z_1, z_2) \approx 1. \tag{4·19d}$$

The matrix $\mathbf{F}(z_1, z_2)$ thus obtained can now be identified with (4·8). The non-diagonal entries of these expressions give the following approximate values for two of the Stokes constants:

$$a_1 \approx i\exp(-|K| + i\lambda)[1+\exp(-2|K|)]^{\frac{1}{2}}\left(\exp[i(L-2\sigma)] + \frac{\exp(i\lambda)}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right), \tag{4·20a}$$

$$b_2 \approx i\exp(|K| - i\lambda)[1+\exp(-2|K|)]^{\frac{1}{2}}\left(\exp[-i(L-2\sigma)] + \frac{\exp(i\lambda)}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right), \tag{4·20b}$$

which according to (4·21) below are also consistent with the (1, 1)-entry of (4·8) and (4·19*a*), and which with the aid of (4·6*a*, *c*) give a_{-1} and b_4. From (4·20*a*, *b*) we obtain

$$(1 + a_1 b_2)\exp(-i\lambda) \approx -2\exp(-|K|)[1+\exp(-2|K|)]^{\frac{1}{2}} \times \left(\cos(L-2\sigma) + \frac{\cos\lambda}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right), \tag{4·21}$$

and thus the Stokes constants a_1 and b_2 given by (4·20*a*, *b*) fulfil the condition (4·7*c*). From (4·20*a*, *b*) we also obtain

$$\mathrm{Im}[a_1\exp(-i\lambda)] \approx \exp(-|K|)[1+\exp(-2|K|)]^{\frac{1}{2}} \times \left(\cos(L-2\sigma) + \frac{\cos\lambda}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right), \tag{4·22a}$$

$$\mathrm{Im}[b_2\exp(+i\lambda)] \approx \exp|K|[1+\exp(-2|K|)]^{\frac{1}{2}} \times \left(\cos(L-2\sigma) + \frac{\cos\lambda}{[1+\exp(2|K|)]^{\frac{1}{2}}}\right). \tag{4·22b}$$

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It is seen that in the right-hand sides of (4·21) and (4·22a, b) there appears a factor which is equal to zero when the quantization condition (4·9) is fulfilled: cf. (4·10a, b). Inserting (4·21) and (4·22a, b) into (4·7a, b), when (3·15) is not fulfilled, we obtain

$$b_0 \approx i \exp(2|K| - 2i\lambda), \quad a_3 \approx i. \tag{4·23a, b}$$

The formulae for the symmetric case of §4·2 are obtained when one puts $\lambda = 0$ in the above formulae. It is easy to verify that the Stokes constants given by (4·6a, c), (4·20a, b) and (4·23a, b) with $\lambda = 0$ automatically fulfil the symmetry relations (4·13a–c).

The Bohr–Sommerfeld limit

Finally we shall now consider the Stokes constants in the Bohr–Sommerfeld limit considered in §3·1. This amounts to attempting to express the matrix $\mathbf{F}(z_1, z_2')$, defined in the discussion above (4·18), by means of (3·3) instead of (3·14a–d) and proceeding as in the treatment of the extreme double-oscillator. Thus one obtains instead of (4·19a–d)

$$F_{11}(z_1, z_2) \approx -2 \exp(-|K| + i\lambda) \cos L, \quad F_{12}(z_1, z_2) \approx i \exp[|K| - i(L + \lambda)], \tag{4·24a, b}$$

$$F_{21}(z_1, z_2) \approx i \exp[-|K| + i(L + \lambda)], \qquad F_{22}(z_1, z_2) \approx 0. \tag{4·24c, d}$$

It is easily seen that (4·24a–c) agree with the leading terms of (4·19a–c) considered as expansions in powers of $\exp(-|K|)$ and with σ neglected, while (4·24d) disagrees completely with (4·19d). Identifying the non-diagonal entries of (4·8) and (4·24a–d), we find

$$a_1 \approx i \exp[-|K| + i(L + \lambda)], \quad b_2 \approx i \exp[|K| - i(L + \lambda)], \tag{4·25a, b}$$

which are the leading terms of (4·20a, b) considered as expansions in powers of $\exp(-|K|)$ and with σ neglected. The quantization condition (2·5) with (4·24a) gives the result (3·4), obtained in §3·1, but with the error term $O(\mu_a)$ neglected. However, when the expressions (4·25a, b) are inserted into the (1, 1)-entry of (4·8), this entry becomes zero, in complete disagreement with (4·24a). Furthermore, with a_1 and b_2 given by (4·25a, b), the quantization condition (4·9) becomes fulfilled for all energies, and by (4·7a, b) a_3 and b_0 become infinite except for $L = (k + \frac{1}{2})\pi$ where k is an integer. These inconsistencies arise because of the μ-quantities neglected in (3·3), which are here important, the terms included in (3·3) being unable to yield the structure of (4·8) when $a_1 \neq 0$. In fact, the matrix in the right-hand side of (3·3), which was used in the derivation (4·24a–d), has the structure of an F-matrix over one sector between adjacent anti-Stokes lines, while the corresponding exact matrix (4·8) is an F-matrix over two such neighbouring sectors. Thus we conclude that the F-matrix estimates giving the Bohr–Sommerfeld limit cannot be consistently used in the scheme of the Stokes constants for a fourth-degree cluster.

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